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A MODIFIED HALPERN-TYPE ITERATION ALGORITHM FOR QUASI-ϕ-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND APPLICATIONS

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ABSTRACT. The purpose of this article is to modify the Halpern-type iteration algorithm for quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of Banach spaces. The results presented in the paper improve and extend the corresponding results of Qin et al. [*Convergence of a modified Halpern-type iterative algorithm for quasi-ϕ-nonexpansive mappings*, Appl. Math. Lett. **22** (2009), 1051–1055], Wang et al. [*A modified Halpern-type iteration algorithm for a family of hemi-relative nonexpansive mappings and systems of equilibrium problems in Banach spaces*, J. Comput. Appl. Math. **235** (2011), 2364–2371], Su et al. [*Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications*, Nonlinear Anal. **73** (2010), 3890–3906], Nartinez-Yanes et al. [*Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), 2400–2411], and others.

1. Introduction

Throughout this paper we assume that E is a real Banach space with the dual E^* and $J\colon E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \qquad x \in E.$$

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In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathscr{R}$ to denote the set of all real numbers. Recall that a mapping $T\colon C \to C$ is *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

One classical way to study nonexpansive mappings is to use contraction to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t\colon C \to C$ by

$$T_t x = tu + (1-t)Tx \qquad \text{for all} \quad x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, for the case of T having a fixed point, Browder [1] proved that if H is a Hilbert space, then x_t converges strongly to a fixed point of T which is the nearest to u.

Motivated by Browder's results, Halpern [2] considered the following explicit iteration:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \text{for all } \; n \geq 0, \tag{1.1}$$

where T is nonexpansive. He proved the strong convergence of $\{x_n\}$ to a fixed point of T provided that $\alpha_n = n^{-\theta}$, where $\theta \in (0,1)$.

Recently, many authors improved the result of Halpern [2] and studied the restrictions imposed on the control sequence $\{\alpha_n\}$ in iteration algorithm (1.1). In 2006, Martinez-Yanes and Xu [3] proposed the following modification of the Halpern iteration for a single nonexpansive mapping T in a Hilbert space and proved the following theorem:

Theorem MYX. ([3]) *Let H be a real Hilbert space, C be a closed and convex subset of H and $T\colon C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \big\{z \in C : \; \|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n\big(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\big)\big\}, \\ Q_n = \big\{z \in C : \; \langle x_0 - x_n, x_n - z\rangle \geq 0\big\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 1. \end{cases} \tag{1.2}$$

converges strongly to $P_{F(T)}x_0$.

Very recently Qin et al. [4, 5] and Wang et al. [6] improved the result of Martinez-Yanes and Xu [3] from Hilbert spaces to Banach spaces for relatively nonexpansive mappings [7, 8], quasi-ϕ-nonexpansive mappings and a family of quasi-ϕ-nonexpansive mappings and under suitable conditions some strong convergence theorems are proved.

The purpose of this paper is to consider a hybrid projection algorithm for modifying the iterative process (1.1) to have strong convergence for quasi-ϕ-asymptotically nonexpansive mappings which contains relatively nonexpansive mappings, quasi-ϕ-nonexpansive mappings (or hemi-relatively nonexpansive mappings) as its special cases [9] in the framework of Banach spaces. The results presented in the paper extend and improve the corresponding results of Qin et al. [4, 5], Wang, Su et al. [6], Martinez-Yanes and Xu [3] and others.

2. Preliminaries

In the sequel, we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. In what follows, we always use $\phi\colon E \times E \to \mathscr{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x,\, Jy\rangle + \|y\|^2 \qquad \text{for all} \quad x, y \in E. \tag{2.1}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \qquad \text{for all} \quad x, y \in E. \tag{2.2}$$

Following Alber [10], the *generalized projection* $\Pi_C\colon E \to C$ is defined by

$$\Pi_C(x) = \arg\inf_{y\in C} \phi(y, x) \qquad \text{for all} \quad x \in E.$$

Lemma 2.1. ([10]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$ for all $x \in C$ and $y \in E$;*

(b) *If $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \iff (\forall y \in C)\big(\langle z - y,\, Jx - Jz\rangle \ge 0\big);$$

(c) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$;*

Remark 1. If E is a real Hilbert space H, then $\phi(x, y) = \|x-y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Recall that a point $p \in C$ is said to be an *asymptotic fixed point* of T if, there exists a sequence $\{x_n\} \subset C$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if, there exists a sequence $\{x_n\} \subset C$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Denote the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 2.1.

(1) A mapping $T\colon C \to C$ is said to be *relatively nonexpansive* [8, 11], if $F(T) \neq \emptyset$, $F(T) = \hat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$, $p \in F(T)$.

(2) A mapping $T\colon C \to C$ is said to be *weakly relatively nonexpansive* [12], if $F(T) \neq \emptyset$, $F(T) = \tilde{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ ll $x \in C$, $p \in F(T)$.

(3) A mapping $T\colon C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

From the definitions, it is easy to see that each relatively nonexpansive mapping T is a weak relatively nonexpansive mapping and

$$\hat{F}(T) \subset \tilde{F}(T). \tag{2.3}$$

If T is continuous, then a strong asymptotic fixed point of T coincides with a fixed point of T.

Next we give an example which is a weak relatively nonexpansive mapping but it is not a relatively nonexpansive mapping in Banach space.

Example 1. ([12]) Let $E = l^2$. It is wellknown that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\}$ be a sequence in l^2 defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots) \\ x_1 &= (1, 1, 0, 0, \dots) \\ x_2 &= (1, 0, 1, 0, 0, \dots) \\ x_3 &= (1, 0, 0, 1, 0, 0, \dots) \\ &\ \vdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \dots, \xi_{n,k}, \dots), \end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1, & \text{if } k = 1, n+1 \\ 0, & \text{if } k \neq 1,\ k \neq n+1 \end{cases}$$

for all $n \geq 1$. Define a mapping $T\colon l^2 \to l^2$ as follows:

$$Tx = \begin{cases} \frac{n}{n+1} x_n, & \text{if } x = x_n \text{ (there exists } n \geq 1), \\ -x, & \text{if } x \neq x_n \text{ (for all } n \geq 1). \end{cases}$$

It was proved that (see, [12]),

(1) $\{x_n\}$ converges weakly to x_0;

(2) $\{x_n\}$ is not a Cauchy sequence;

(3) T has a unique fixed point 0, i.e., $F(T) = \{0\}$;

(4) x_0 is an asymptotic fixed point of T;

(5) T has a unique strong asymptotic fixed point 0, so that $F(T) = \tilde{F}(T)$;

(6) T is a weak relatively nonexpansive mapping;

(7) T is not a relatively nonexpansive mapping.

Summing up the above arguments, and noting (2.3), we obtain that $\hat{F}(T) \subset \tilde{F}(T)$.

Definition 2.2.

(1) A mapping $T\colon C \to C$ is said to be *quasi-ϕ-nonexpansive* (or *hemi-relatively nonexpansive*) if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x) \qquad \text{for all} \quad x \in C, \quad p \in F(T).$$

(2) A mapping $T\colon C \to C$ is said to be *quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x) \qquad \text{for all} \quad n \geq 1, \quad x \in C, \quad p \in F(T).$$

Remark 2. From the definitions, it is obvious that weak relatively nonexpansive mappings is a quasi-ϕ-nonexpansive (hemi-relatively nonexpansive) mapping and a quasi-ϕ-nonexpansive mapping is a quasi-ϕ-asymptotically nonexpansive mapping. However, the converse is not true.

Example 2. ([9]) Let E be a uniformly smooth and strictly convex Banach space and $A\colon E \to E^*$ be a maximal monotone mapping such that $A^{-1}0 \neq \emptyset$, then $J_r = (J + rA)^{-1} J$ is closed and quasi-ϕ-nonexpansive from E onto $D(A)$;

Example 3. ([9]) Let Π_C be the generalized projection from a smooth, reflexive and strictly convex Banach space E onto a nonempty closed convex subset C of E, then Π_C is a closed and quasi-ϕ-nonexpansive from E onto C.

Example 4. ([14]) Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty closed and convex subset of E and $f\colon C \times C \to$ be a bifunction satisfying the conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4) for each given $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The "so-called" *equilibrium problem for f* is to find a $x^* \in C$ such that $f(x^*, y) \geq 0$, for all $y \in C$. The set of its solutions is denoted by $EP(f)$.

Let $r > 0$, $x \in E$ and define a mapping $T_r\colon E \to C$ as follows:

$$\big(\forall x \in E\big)\big[T_r(x) = \big\{z \in C : \ f(z, y) + \tfrac{1}{r}\langle y - z, \, Jz - Jx\rangle \geq 0 \ \text{ for all } \ y \in C\big\}\big], \tag{2.4}$$

then

(1) T_r is single-valued, and so $z = T_r(x)$;

(2) T_r is a relatively nonexpansive mapping, therefore it is a closed quasi-ϕ-nonexpansive mapping;

(3) $F(T_r) = EP(f)$.

Lemma 2.2. *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3. ([9]) *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and C be a nonempty closed subset of E. Let $T\colon C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping, then $F(T)$ is a closed convex subset of C.*

Definition 2.3.

(1) A countable family of mappings $\{T_i\}\colon C \to C$ is said to be *uniformly quasi-ϕ-asymptotically nonexpansive*, if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist real sequences $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ such that for each $i \geq 1$,

$$\phi(p, T_i^n x) \leq k_n \phi(p, x) \qquad \text{for all} \quad x \in C, \quad p \in \bigcap_{i=1}^{\infty} F(T_i) \tag{2.5}$$

(2) A mapping $T\colon C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$, for all $n \geq 1$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space E and $T_i\colon C \to C$, $i = 1, 2, \ldots,$ be a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$, and for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E \ \text{ chosen arbitrarily;} \quad C_1 = C, \\ y_{n,m} = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T_m^n x_n], \quad m \geq 1, \\ C_{n+1} = \Big\{ z \in C_n : \ \sup_{m\geq 1} \phi(z, y_{n,m}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n \Big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \text{for all} \ \ n \geq 1, \end{cases} \tag{3.1}$$

where $\xi_n = (k_n - 1) \sup_{p \in \mathscr{F}} \phi(p, x_n)$, $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1}. *If* $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i)$ *is bounded in* C, *then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathscr{F}} x_1$.

Proof.

(I) First we prove that $\mathscr{F}$ and C_n, $n \geq 1$ all are closed and convex subsets in C.

In fact, it follows from Lemma 2.3 that $F(T_i)$, $i \geq 1$, is a closed and convex subset of C. Therefore $\mathscr{F}$ is closed and convex in C.

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have that

$$\begin{aligned} C_{n+1} &= \Big\{ z \in C_n : \sup_{m \geq 1} \phi(z, y_{n,m}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n \Big\} \\ &= \bigcap_{m \geq 1} \big\{ z \in C : \phi(z, y_{n,m}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n \big\} \cap C_n \\ &= \bigcap_{m \geq 1} \big\{ z \in C : 2\alpha_n \langle z, Jx_1 \rangle + 2(1 - \alpha_n)\langle z, Jx_n \rangle - 2\langle z, Jy_{n,m} \rangle \\ &\qquad\qquad \leq \alpha_n \|x_1\|^2 + (1 - \alpha_n)\|x_n\|^2 - \|y_{n,m}\|^2 \big\} \cap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex. The conclusion is proved.

(II) Now we prove that $\mathscr{F} \subset C_n$, for all $n \geq 1$.

In fact, it is obvious that $\mathscr{F} \subset C_1 = C$. Suppose that $\mathscr{F} \subset C_n$ for some $n \geq 2$. Hence for any $u \in \mathscr{F} \subset C_n$ we have

$$\begin{aligned} &\phi(u, y_{n,m}) = \phi(u, J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n) JT_m^n x_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_1 + (1 - \alpha_n) JT_m^n x_n \rangle + \|\alpha_n Jx_1 + (1 - \alpha_n) JT_m^n x_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_1 \rangle - 2(1 - \alpha_n)\langle u, JT_m^n x_n \rangle + \alpha_n \|x_1\|^2 + (1 - \alpha_n)\|T_m^n x_n\|^2 \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, T_m^n x_n) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) k_n \phi(u, x_n) \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n)[\phi(u, x_n) + (k_n - 1)\phi(u, x_n)] \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, x_n) + (k_n - 1) \sup_{p \in \mathscr{F}} \phi(p, x_n) \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, x_n) + \xi_n \qquad \text{for all} \quad m \geq 1, \end{aligned} \tag{3.2}$$

Therefore we have

$$\sup_{m \geq 1} \phi(u, y_{n,m}) \leq \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, x_n) + \xi_n. \tag{3.3}$$

This shows that $u \in C_{n+1}$, and so $\mathscr{F} \subset C_{n+1}$. The conclusion is proved.

(III) Next we prove that $\{x_n\}$ is a Cauchy sequence in C.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 2.1(b) we have

$$\langle x_n - y, \, Jx_1 - Jx_n \rangle \geq 0 \qquad \text{for all} \quad y \in C_n.$$

Again since $\mathscr{F} \subset C_n$ for all $n \geq 1$, we have

$$\langle x_n - u, \, Jx_1 - Jx_n \rangle \geq 0 \qquad \text{for all} \quad u \in \mathscr{F}.$$

It follows from Lemma 2.1(a) that for each $u \in \mathscr{F}$ and for each $n \geq 1$

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{3.4}$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, for all $n \geq 1$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of $\{C_n\}$, for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$. This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \to 0 \qquad \text{as} \quad n, m \to \infty.$$

It follows from Lemma 2.2 that $\lim_{n,m\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is complete, without loss of generality, we can assume that $x_n \to p^*$ (some point in C).

By the assumption, it is easy to see that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} (k_n - 1) \sup_{p \in \mathscr{F}} \phi(p, x_n) = 0. \tag{3.5}$$

(IV) Now we prove that $p^* \in \mathscr{F}$.

In fact, since $x_{n+1} \in C_{n+1}$, it follows from (3.1) and (3.5) that

$$\sup_{m\geq 1} \phi(x_{n+1}, y_{n,m}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \xi_n \to 0 \quad (\text{as } n \to \infty).$$

Since $x_n \to p^*$, by virtue of Lemma 2.2 for each $m \geq 1$

$$\lim_{n\to\infty} y_{n,m} = p^*. \tag{3.6}$$

Since $\{x_n\}$ is bounded and $\{T_m\}_{m=1}^{\infty}$ is uniformly quasi-ϕ-asymptotically nonexpansive, $\{T_m^n x_n\}$ is uniformly bounded. Since $\alpha_n \to 0$, from (3.1) we have

$$\lim_{n\to\infty} \|Jy_{n,m} - JT_m^n x_n\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - JT_m^n x_n\| = 0 \qquad \text{for each} \quad m \geq 1. \tag{3.7}$$

Since J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} T_m^n x_n = p^* \qquad \text{for each} \quad m \geq 1. \tag{3.8}$$

Again by the assumptions that for each $m \geq 1$, T_m is uniformly L_m-Lipschitz continuous, thus we have

$$\begin{aligned}\|T_m^{n+1}x_n - T_m^n x_n\| &\leq \|T_m^{n+1}x_n - T_m^{n+1}x_{n+1}\| + \|T_m^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T_m^n x_n\| \\ &\leq (L_m + 1)\|x_{n+1} - x_n\| + \|T_m^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_n - T_m^n x_n\|.\end{aligned} \tag{3.9}$$

It follows from (3.8) and $x_n \to p^*$ that $\lim_{n\to\infty} \|T_m^{n+1}x_n - T_m^n x_n\| = 0$ and $\lim_{n\to\infty} T_m^{n+1}x_n = p^*$, i.e, $\lim_{n\to\infty} T_m T_m^n x_n = p^*$. In view of the closeness of T_m, it yields that $T_m p^* = p^*$, i.e., $p^* \in F(T_m)$. By the arbitrariness of $m \geq 1$, we have $p^* \in \mathscr{F}$.

(V) Finally we prove that $x_n \to p^* = \Pi_{\mathscr{F}} x_1$. Let $w = \Pi_{\mathscr{F}} x_1$. Since $w \in \mathscr{F} \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$ for all $n \geq 1$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{3.10}$$

In view of the definition of $\Pi_{\mathscr{F}} x_1$, from (3.10) we have $p^* = w$. Therefore, $x_n \to p^* = \Pi_{\mathscr{F}} x_1$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let E, C be the same as in* Theorem 3.1. *Let $\{T_i\}_{i=1}^{\infty} \colon C \to C$ be a countable families of closed and quasi-ϕ-nonexpansive mappings such that $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E \quad \text{chosen arbitrarily;} \quad C_1 = C, \\ y_{n,m} = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T_m^n x_n], \\ C_{n+1} = \Big\{ z \in C_n : \sup\limits_{m\geq 1} \phi(z, y_{n,m}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \Big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \text{for all } \ n \geq 1. \end{cases} \tag{3.11}$$

where $\{\alpha_n\}$ are sequences in [0, 1]. If $\alpha_n \to 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathscr{F}} x_1$.

P r o o f. Since $\{T_i\}_{i=1}^{\infty}$ is a countable family of closed quasi-ϕ-nonexpansive mappings, by Remark 2, it is a countable family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_n = 1\}$. Hence $\xi_n = (k_n - 1) \sup_{u\in\mathscr{F}} \phi(u, x_n) = 0$. Therefore the conditions appearing in Theorem 3.1: "$\mathscr{F}$ is a bounded subset in C" and "for each $i \geq 1$, T_i is uniformly L_i-Lipschitz" are no use here. Therefore all conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. □

Remark 3. Theorems 3.1 and 3.2 improve and extend the corresponding results of Qin et al. [5,6], Wang, Su et al. [7], Martinez-Yanes and Xu [4], Kang et al. [15] and others.

4. Application to a system of equilibrium problems

In this section we shall utilize Theorem 3.2 to study a modified Halpern-type iterative algorithm for a system of equilibrium problems. We have the following result.

Theorem 4.1. *Let C, E and $\{\alpha_n\}$ be the same as in* Theorem 3.2. *Let $\{f_i\colon C\times C\to\mathscr{R}\}$ be a countable family of bifunctions satisfying conditions* (A1)–(A4) *as given in* Example 4. *Let $\{T_{r,i}\colon E\to C\}$ be the family of mappings defined by* (2.4) *for f_i, i.e., $T_{r,i}(x)=\big\{z\in C:\ f_i(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\geq 0$ for all $y\in C\big\}$, $x\in E$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1\in E \ \ \textit{chosen arbitrarily;} \quad C_1=C,\\ f_m(u_n,y)+\dfrac{1}{r}\langle y-u_n,Ju_n-Jx_n\rangle\geq 0, \quad \textit{for all}\ \ y\in C,\ \ r>0,\ \ m\geq 1;\\ y_{n,m}=J^{-1}[\alpha_nJx_1+(1-\alpha_n)Ju_{n,m}],\\ C_{n+1}=\Big\{z\in C_n:\ \sup\limits_{m\geq 1}\phi(z,y_{n,m})\leq\alpha_n\phi(z,x_1)+(1-\alpha_n)\phi(z,x_n)\Big\}\\ x_{n+1}=\Pi_{C_{n+1}}x_1 \quad \textit{for all}\ \ n\geq 1, \end{cases} \tag{4.1}$$

If $\mathscr{F}:=\bigcap\limits_{i=1}^{\infty}F(T_{r,i})\neq\emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{\mathscr{F}}x_1$.

Proof. In Example 4 we have pointed out that $u_{n,m}=T_{r,m}(x_n)$, $F(T_{r,m})=EP(f_m)$ for all $m\geq 1$ and $T_{r,m}$ is a countable family of closed quasi-ϕ-nonexpansive mappings. Hence (4.1) can be rewritten as follows:

$$\begin{cases} x_1\in E \ \ \text{chosen arbitrarily;} \quad C_1=C,\\ y_{n,m}=J^{-1}[\alpha_nJx_1+(1-\alpha_n)JT_{r,m}x_n],\\ C_{n+1}=\Big\{z\in C_n:\ \sup\limits_{m\geq 1}\phi(z,y_{n,m})\leq\alpha_n\phi(z,x_1)+(1-\alpha_n)\phi(z,x_n)\Big\}\\ x_{n+1}=\Pi_{C_{n+1}}x_1 \quad \text{for all}\ \ n\geq 1. \end{cases} \tag{4.2}$$

Therefore the conclusion of Theorem 4.1 can be obtained from Theorem 3.2. □

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