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FIXED POINT THEOREMS FOR BLOCK OPERATOR MATRIX AND AN APPLICATION TO A STRUCTURED PROBLEM UNDER BOUNDARY CONDITIONS OF ROTENBERG'S MODEL TYPE

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ABSTRACT. In this manuscript, we introduce and study the existence of solutions for a coupled system of differential equations under abstract boundary conditions of Rotenberg's model type, this last arises in growing cell populations. The entries of block operator matrix associated to this system are nonlinear and act on the Banach space $X_p := L_p([0,1] \times [a,b]; \mathrm{d}\mu\, \mathrm{d}v)$, where $0 \le a < b < \infty$; $1 < p < \infty$.

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1. Introduction

Many problems arising in mathematical physics, biology, etc. may be described, in a first formulation, using systems of partial or ordinary differential equations. The theory of block operator matrices opens up a new line of attack of theses problems. During the past years, several papers are devoted to the investigation of linear operator matrices defined by 2×2 block operator matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1}$$

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the entries of which are not necessary bounded in Banach spaces. The paper does not claim to be complete in any respect, e.g. it does not touch upon the vast literature about block operator matrices in spectral theory (see e.g. the book of C. Tretter [32] and the papers [2–4, 6, 10, 11, 14, 24, 25, 29]) nor on semigroup theory (see e.g. [19]). In our paper, our assumptions are as follows: the operators occurring in the representation (1.1) are nonlinear, A maps a closed convex non-empty subset Ω of a Banach space X into X, B from another closed convex non-empty subset Ω' of a Banach space Y into X, C from Ω into Y and D from Ω' into Y.

Fixed point theorems are very important in mathematical analysis, they are an interesting way to show that some thing exists without setting it out. Which sometimes is very hard, or even impossible to do. Fixed point theorems give the conditions under which maps have solutions. The first result in the field was the Schauder's fixed point theorem, proved in 1930 by Juliusz Schauder [31]. Quite a number of further results followed. One way in which fixed point theorems of this kind have had a larger influence on mathematics as a whole has been that one approach is to try carry over methods of algebraic topology. Many problems arising from the most diverse areas of natural science, when modeled under the mathematical point of view, involve the study of solutions of nonlinear equations of the form

$$Au + Bu = u, \qquad u \in \mathcal{M}$$

where $\mathcal{M}$ is a closed and convex subset of a Banach space X, see for example [5, 8, 9, 16–18]. Motivated by the observation that the inversion of a perturbed differential operator could yield a sum of a contraction and a compact operator Krasnosel'skii proved in [26] a fixed point theorem, called the Krasnosel'skii's fixed point theorem which appeared as a prototype for solving equations of the previous type. We discuss the two major fixed point theorems, which are based on a notion of compactness, and we have to generalize them for the matrix case.

In 1983, M. Rotenberg proposed the singular partial differential equation.

$$\begin{aligned} v\frac{\partial\psi}{\partial\mu}(\mu,v) + \sigma(\mu,v)\psi(\mu,v) + \lambda\ \psi(\mu,v) \\ - \int_a^b r(\mu,v,v',\psi(\mu,v'))dv' = 0, \end{aligned} \tag{1.2}$$

which models the evolution of a cell population. Each cell is distinguished by two parameters, the degree of maturity μ and the velocity v.

The boundary conditions are modeled by

$$\psi_{|\Gamma_0} = K\left(\psi_{|\Gamma_1}\right). \tag{1.3}$$

In [30] M. Rotenberg studied essentially the Fokker-Plank approximation of Equation (1.2) for which he obtained numerical solutions. Using eigenfunction expansion technique, C. Van der Mee and P. Zweifel [33] obtained analytical solution for a variety of linear boundary conditions. A. Jeribi [21–23] obtained several existence results for the boundary value problem (1.2)–(1.3) in L_p spaces $1 < p < \infty$. The analysis started in [27] is based essentially on compactness results established only for $1 < p < \infty$, and use the Schauder and Krasnosels'kii fixed point theorems. Recently, A. Ben Amar, A. Jeribi and M. Mnif in [7] estabish some results regarding the existence of solution on L_1 spaces to the boundary problem (1.2)–(1.3).

In the investigation of cell population dynamics it is important to consider the structure of the population with respect to individual properties such as age, degree of maturation, or other physical characteristics. In many cases, not all cells are progressing to mitosis, but some are in a quiescent state for an extended period of time. Motivated by the work of O. Arino, E. Sánchez and G.F. Webb in [1] where they have analyzed a linear model of cell population dynamics structured by age with two interacting compartments: proliferating cells and quiescent cells. Proliferating cells grow, divide, and transit to the quiescent compartment, whereas quiescent cells do not grow and can only transit back and forth to proliferation, the central purpose of this paper is to give some existence results for a structured problem on L_p-spaces $(1 < p < \infty)$ under abstract boundary conditions of Rotenberg's model type [30]. This problem is formulated by:

$$\begin{pmatrix} -v\frac{\partial}{\partial\mu} - \sigma_1(\mu, v, \cdot) & R_{12} \\ R_{21} & -v\frac{\partial}{\partial\mu} - \sigma_2(\mu, v, \cdot) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1.4}$$

$$\psi_{i_{|\Gamma_0}} = K_i\left(\psi_{i_{|\Gamma_1}}\right), \qquad i = 1, 2 \tag{1.5}$$

where $R_{ij}\psi_j(\mu, v) = \int_a^b r_{ij}(\mu, v, v', \psi_j(\mu, v'))dv'$, $(i, j) \in \{(1, 2), (2, 1)\}$, $\mu \in [0, 1]$, $v, v' \in [a, b]$ with $0 \leq a < b < \infty$, $\sigma_i(\cdot, \cdot, \cdot)$, $i = 1, 2$, $r_{ij}(\cdot, \cdot, \cdot, \cdot)$ are nonlinear operators, λ is a complex number, $\Gamma_0 = \{0\} \times [a, b]$ and $\Gamma_1 = \{1\} \times [a, b]$. We denote by $\psi_{i_{|\Gamma_0}}$ (resp. $\psi_{i_{|\Gamma_1}}$) the restriction of ψ_i to Γ_0 (resp. Γ_1) while K_i are nonlinear operators from a suitable function space on Γ_1 to a similar one on Γ_0. The main point in the equation (1.4) of the proposed model is the non-linear dependence of the functions $r_{ij}(\mu, v, v', \psi_j(\mu, v'))$ on ψ_j. More specifically, we suppose that

$$r_{ij}(\mu, v, v', \psi(\mu, v')) = k_{ij}(\mu, v, v')f(\mu, v', \psi(\mu, v')); \qquad (i, j) \in \{(1, 2), (2, 1)\}$$

where f is a measurable function defined by

$$f\colon [0,1]\times[a,b]\times\mathbb{C}\to\mathbb{C},\quad (\mu,v,u)\mapsto f(\mu,v,u)$$

with $k_{ij}(\cdot,\cdot,\cdot)$, $(i,j)\in\{(1,2),(2,1)\}$ are measurable functions from $[0,1]\times[a,b]\times\mathbb{C}$ to $\mathbb{C}$.

Using the previous work we will establish some existence results of the bidimensional mixed boundary problem (1.4)–(1.5) based on new generalized Schauder and Krasnoselskii fixed point theorems for a 2×2 block operator matrix.

The content of this paper is organized in four sections. Section 2 deals with some fixed point results for 2×2 block operator matrices, which consist of nonlinear operators acting on non empty closed convex sets in Banach spaces. These results are based on the Schauder and the Krasnoselskii fixed point theorems. In Section 3, we present existence results for a particular boundary value problem. In Section 4, we are interested with the general mixed model (1.4)–(1.5). In fact we allow the transition rates and the total cross section to depend on the densities of populations. The existence solutions is discussed in Theorem 4.1.

2. Fixed point results

Definition 2.1. A mapping between two metric spaces $f\colon X\to Y$ is called compact, if f maps bounded sets into relatively compact sets. It is called completely continuous, if it is continuous and for every bounded $M\subset X$, $f(M)$ is contained in a compact set in the range.

Let us notice that for mapping defined on bounded sets the two above notions coincide.

Definition 2.2. Let (X,d) be a metric space. We say that $f\colon X\to X$ is a separate contraction if there exist two functions $\Theta_1,\Theta_2\colon\mathbb{R}_+\to\mathbb{R}_+$ satisfying:

(i) $\Theta_2(0)=0$, Θ_2 is strictly increasing,

(ii) $d(f(x),f(y))\le\Theta_1(d(x,y))$,

(iii) $\Theta_1(t)+\Theta_2(t)\le t$ for $t>0$.

Remark 1. It is easy to see that every contraction is a separate contraction.

Definition 2.3. Let (X,d) be a metric space and M be a subset of X. The mapping $T\colon M\to X$ is said to be weakly-expansive if there exist $\Phi\colon X\times X\to\mathbb{R}_+$ with $d(T(x),T(y))\ge\Phi(x,y)$, $x,y\in M$ satisfying

(i) $\Phi(x,y)=0 \iff x=y$,

(ii) $\Phi(x,y)=\Phi(y,x)$,

(iii) $\Phi(x_n,x)\to 0 \implies x_n\to x$.

Remark 2. Obviously, every expansive mapping is weakly-expansive.

Theorem 2.1. ([31]) *Let Ω be a non-empty closed convex subset of a Banach space X. If T is a completely continuous mapping from Ω into Ω, then T has a fixed point in Ω.*

Let Ω and Ω' be closed convex non-empty subsets of two Banach spaces X and Y. We consider the 2×2 block operator matrix

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

in the space $X \times Y$, that is, the nonlinear operator A maps Ω into X, B from Ω' into X, C from Ω into Y and D from Ω' into Y.

Our aim is to develop a general matrix fixed point theory which allows to treat the biological application described in the introduction. In the following we discuss the existence of fixed points for the block operator matrix (2.1) by imposing some conditions on the entries, which are in general nonlinear operators. This discussion is based on the invertibility or not of the diagonal terms of $I - \mathcal{L}$.

<u>Case 1</u>: $I - A$ and $I - D$ are invertible.

Assume that:

($\mathcal{H}1$) The operator $I - A$ is invertible and $(I - A)^{-1}B(\Omega') \subset \Omega$.

($\mathcal{H}2$) $S := C(I - A)^{-1}B$ is an operator with closed graph and the subset $S(\Omega')$ is relatively compact in Y.

($\mathcal{H}3$) The operator $I-D$ is invertible and $(I-D)^{-1}$ is continuous on $(I-D)(\Omega')$.

($\mathcal{H}4$) $(I - D)^{-1}S(\Omega') \subset \Omega'$.

Our purpose here is to establish some fixed point results required in the sequel.

Lemma 2.1. *Let X and Y be two metric spaces. Assume $J\colon X \to Y$ has a closed graph and $\overline{J(X)}$ is a compact set of Y. Then J is continuous.*

Proof. See [20: Theorem 4]. □

Theorem 2.2. *Let $\mathcal{K}$ be a closed convex non-empty subset of a Banach space X. Suppose that J map $\mathcal{K}$ into $\mathcal{K}$ and that*

(i) *J has a closed graph.*

(ii) *$\overline{J(\mathcal{K})}$ is compact.*

Then J has a fixed point in $\mathcal{K}$.

Proof. The proof follows from Lemma 2.1 and the Schauder fixed point theorem. □

Theorem 2.3. *Under assumptions* $(\mathcal{H}1)$–$(\mathcal{H}4)$ *the block matrix operator* (2.1) *has a fixed point in* $\Omega \times \Omega'$.

P r o o f. Since S has a closed graph and $S(\Omega')$ is relatively compact in Y, it follows from Lemma 2.1 that S is continuous on Ω'. Now, let M be a bounded subset of Ω'. Obviously, from $(\mathcal{H}2)$ the set $S(M)$ is relatively compact. Then by hypothesis $(\mathcal{H}3)$, $(I-D)^{-1}S(M)$ is relatively compact. By the Schauder fixed point theorem there exists $y_0 \in \Omega'$ such that

$$(I-D)^{-1}Sy_0 = y_0.$$

Let $x_0 := (I-A)^{-1}By_0$, hence

$$\mathcal{L}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ □

Corollary 2.3.1. *If assumptions* $(\mathcal{H}1)$, $(\mathcal{H}2)$, *and* $(\mathcal{H}4)$ *holds and* D *is a separate contraction mapping satisfying* $C(\Omega) \subset (I-D)(\Omega')$, *then the block matrix operator* (2.1) *has a fixed point in* $\Omega \times \Omega'$.

P r o o f. In [28: Lemma 1.2], it was shown that $I-D$ is an homeomorphism from Ω' onto $(I-D)(\Omega')$, then $(\mathcal{H}3)$ is satisfied and the result follows from Theorem 2.2. □

Corollary 2.3.2. *If assumptions* $(\mathcal{H}1)$, $(\mathcal{H}2)$, *and* $(\mathcal{H}4)$ *holds and if* $I-D$ *is a weakly-expansive mapping satisfying* $C(\Omega) \subset (I-D)(\Omega')$, *then the block matrix operator* (2.1) *has a fixed point in* $\Omega \times \Omega'$.

P r o o f. It is clear from Definition 2.2(i) that $I-D$ is one-to-one. We now show that $(I-D)^{-1}\colon (I-D)(\Omega') \to \Omega'$ is continuous. Let $(y_n)_{n\in\mathbb{N}}$ and y in $(I-D)(\Omega')$ such that $y_n \to y$. Then there exist $\alpha_n, \alpha \in \Omega'$ such that $y_n = (I-D)(\alpha_n)$ and $y = (I-D)(\alpha)$. Now,

$$\begin{aligned}\|y_n - y\| &= \|(I-D)(\alpha_n) - (I-D)(\alpha)\| \\ &\geq \Phi(\alpha_n, \alpha).\end{aligned}$$

It follows that $\Phi(\alpha_n, \alpha) \to 0$ and so by Definition 2.2(iii) $\alpha_n \to \alpha$. Hence $(\mathcal{H}3)$ is satisfied and the result follows from Theorem 2.2. □

Case 2: $I-A$ or $I-D$ is invertible.
We shall treat only the case of invertibility of $I-A$, the other case is similar just simply exchanging the roles of A and D and B and C.

Assume that:

$(\mathcal{H}1)$ The operator $I-A$ is invertible and $(I-A)^{-1}B(\Omega') \subset \Omega$.

$(\mathcal{H}2)$ $S := C(I-A)^{-1}B$ is a contraction map.

($\mathcal{H}3$) The operator D is completely continuous.

($\mathcal{H}4$) $Sy + Dy' \in \Omega'$ for every y, y' in Ω'.

Theorem 2.4. ([26]) *Let Ω be a non-empty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that $M, N\colon \Omega \to X$ are two operators fulfilling the following hypotheses:*

(i) *There exists $L \in [0,1)$ such that $\|Mx - My\| \le L\|x-y\|$ for every $x, y \in \Omega$;*

(ii) *N is continuous and $N(\Omega)$ is relatively compact in X;*

(iii) *$Mx + Ny \in \Omega$ for every $x, y \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = Mx + Nx$.

An immediate application of the Krasnoselskii's theorem for the operator $S + D$, we obtain the following result:

Theorem 2.5. *Under assumptions* ($\mathcal{H}1$)–($\mathcal{H}4$) *the block matrix operator* (2.1) *has a fixed point in $\Omega \times \Omega'$.*

<u>*Case* 3</u>: neither $I - A$ or $I - D$ is invertible.

Here, we discuss the existence of fixed points for the following perturbed block operator matrix by imposing some conditions on the entries, using the Krasnoselskii Theorem.

$$\widetilde{\mathcal{L}} = \begin{pmatrix} A_1 & B \\ C & D_1 \end{pmatrix} + \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{2.2}$$

Assume that the nonlinear operators A_1 and P_1 maps Ω into X, B from Ω' into X, C from Ω into Y and D and P_2 from Ω' into Y Suppose that (2.2) fulfills the following assumptions:

(a) The operator $I - A_1$ resp. $I - D_1$ is invertible from Ω into X, resp. from Ω' into Y .

(b) $(I - A_1)^{-1}B$ and $(I - D_1)^{-1}C$ are completely continuous maps.

(c) $(I - A_1)^{-1}P_1$ and $(I - D_1)^{-1}P_2$ are contractions maps

(d) For every $x_1, x_2 \in \Omega$ and $y_1, y_2 \in \Omega'$, $(I - A_1)^{-1}P_1x_1 + (I - A_1)^{-1}By_2 \in \Omega$ and $(I - D_1)^{-1}Cx_2 + (I - D_1)^{-1}P_2y_1 \in \Omega'$.

Theorem 2.6. *Under assumptions* (a)–(d) *the block matrix operator* (2.2) *has a fixed point in $\Omega \times \Omega'$.*

Proof. Using assumption (a) the following equation

$$\begin{pmatrix} A_1 + P_1 & B \\ C & D_1 + P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

may be transformed into

$$\mathcal{Z}_1\begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{Z}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

where

$$\mathcal{Z}_1 = \begin{pmatrix} (I - A_1)^{-1}P_1 & 0 \\ 0 & (I - D_1)^{-1}P_2 \end{pmatrix}$$

and

$$\mathcal{Z}_2 = \begin{pmatrix} 0 & (I - A_1)^{-1}B \\ (I - D_1)^{-1}C & 0 \end{pmatrix}.$$

Obviously, from (b) the operator matrix $\mathcal{Z}_2$ is completely continuous and from (c) the map $\mathcal{Z}_1$ is a contraction. It follows with (d) and the the Krasnoselskii's theorem that the operator matrix (2.2) has at least a fixed point in $\Omega \times \Omega'$. $\square$

3. Existence solutions for a boundary coupled system

In this section, we consider a particular version of (1.4)–(1.5) where each σ_i does not depends on the density of the population i.

$$\begin{pmatrix} -v\frac{\partial}{\partial\mu} - \sigma_1(\mu, v)I & R_{12} \\ R_{21} & -v\frac{\partial}{\partial\mu} - \sigma_2(\mu, v)I \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.1}$$

$$\psi_{i_{|\Gamma_0}} = K_i\left(\psi_{i_{|\Gamma_1}}\right), \qquad i = 1, 2 \tag{3.2}$$

where $\sigma_i(\cdot,\cdot) \in L^\infty([0,1] \times [a,b])$, λ is a complex number, $\Gamma_0 = \{0\} \times [a,b]$ and $\Gamma_1 = \{1\} \times [a,b]$. $\psi_{i_{|\Gamma_0}}$ (resp. $\psi_{i_{|\Gamma_1}}$) denotes the restriction of ψ_i to Γ_0 (resp. Γ_1) while K_i are nonlinear operators from a suitable function space on Γ_1 to a similar one on Γ_0.

Our interest concentrates on the existence results for the last boundary value problem (3.1)–(3.2). For this purpose, we use the preparatory results presented above. Let

$$X_p := L_p([0,1] \times [a,b]; \mathrm{d}\mu\mathrm{d}v)$$

where $0 \le a < b < \infty$; $1 < p < \infty$. We denote by X_p^0 and X_p^1 the following boundary spaces

$$X_p^0 := L_p(\{0\} \times [a,b]; v\mathrm{d}v),$$
$$X_p^1 := L_p(\{1\} \times [a,b]; v\mathrm{d}v)$$

endowed with their natural norms. Let $\mathcal{W}_p$ be the space defined by

$$\mathcal{W}_p = \Big\{\psi \in X_p \ \mid \ v\frac{\partial\psi}{\partial\mu} \in X_p\Big\}.$$

It is well known (see [12, 13] or [15]) that any ψ in $\mathcal{W}_p$ has traces on the spatial boundary $\{0\}$ and $\{1\}$ which belong respectively to the spaces X_p^0 and X_p^1.

We define the free streaming operator S_{K_i}, $i = 1, 2$ by

$$\begin{aligned} &S_{K_i\colon \mathcal{D}(S_{K_i})\subseteq X_p \to X_p},\\ &\psi \mapsto S_{K_i}\psi(\mu, v) = -v\frac{\partial \psi}{\partial \mu}(\mu, v) - \sigma_i(\mu, v)\psi(\mu, v)\\ &\mathcal{D}(S_{K_i}) = \big\{\psi \in \mathcal{W}_p \mid \psi^0 = K_i(\psi^1)\big\}\end{aligned}$$

where $\psi^0 = \psi_{|\Gamma_0}$, $\psi^1 = \psi_{|\Gamma_1}$ and K_i, $i = 1, 2$, are the following nonlinear boundary operators

$$\begin{aligned} &K_i\colon X_p^1 \to X_p^0,\\ &u \mapsto K_i u\end{aligned}$$

satisfying the following conditions

(A1) there exists $\alpha_i > 0$ such that

$$\|K_i\varphi_1 - K_i\varphi_2\| \le \alpha_i\|\varphi_1 - \varphi_2\| \qquad (\varphi_1,\ \varphi_2 \in X_p^1, \quad i = 1, 2).$$

An immediate consequences of (A1) we have the continuity of the operator K_i from X_p^1 into X_p^0 and $\|K_i\varphi\| \le \alpha_i\|\varphi\| + \|K_i(0)\|$ for all $\varphi \in X_p^1$. Let us consider the equation

$$(\lambda - S_{K_i})\psi_i = g.$$

Our objective is to determine a solution $\psi_i \in \mathcal{D}(S_{K_i})$ where g is given in X_p and $\lambda \in \mathbb{C}$. Let $\underline{\sigma}$ the real defined by

$$\underline{\sigma} := \text{ess-inf}\big\{\sigma_i(\mu, v) \mid (\mu, v) \in [0, 1] \times [a, b], \quad i = 1, 2\big\}.$$

For $\operatorname{Re}\lambda > -\underline{\sigma}$, the solution of is formally given by:

$$\begin{aligned}\psi_i(\mu, v) = {}& \psi_i(0, v)\,\mathrm{e}^{-\frac{1}{v}\int_0^{\mu}(\lambda+\sigma_i(\mu', v))\mathrm{d}\mu'}\\ &+ \frac{1}{v}\int_0^{\mu} \mathrm{e}^{-\frac{1}{v}\int_{\mu'}^{\mu}(\lambda+\sigma_i(\tau, v))\mathrm{d}\tau} g(\mu', v)\ \mathrm{d}\mu'.\end{aligned}$$

Accordingly, for $\mu = 1$, we get

$$\begin{aligned}\psi_i(1, v) = {}& \psi_i(0, v)\,\mathrm{e}^{-\frac{1}{v}\int_0^{1}(\lambda+\sigma_i(\mu', v))\mathrm{d}\mu'}\\ &+ \frac{1}{v}\int_0^{1} \mathrm{e}^{-\frac{1}{v}\int_{\mu'}^{1}(\lambda+\sigma_i(\tau, v))\mathrm{d}\tau} g(\mu', v)\mathrm{d}\mu'.\end{aligned} \tag{3.3}$$

Let the following operators

$$P_{i,\lambda}\colon X_p^0 \to X_p^1, \qquad u \mapsto (P_{i,\lambda}u)(1,v) := u(0,v)\,\mathrm{e}^{-\frac{1}{v}\int\limits_0^1(\lambda+\sigma_i(\mu',v))\,\mathrm{d}\mu'};$$

$$Q_{i,\lambda}\colon X_p^0 \to X_p \qquad u \mapsto (Q_{i,\lambda}u)(\mu,v) := u(0,v)\,\mathrm{e}^{-\frac{1}{v}\int\limits_0^\mu(\lambda+\sigma_i(\mu',v))\,\mathrm{d}\mu'};$$

$$\Pi_{i,\lambda}\colon X_p \to X_p^1 \qquad u \mapsto (\Pi_{i,\lambda}u)(1,v) := \frac{1}{v}\int\limits_0^1 \mathrm{e}^{-\frac{1}{v}\int\limits_{\mu'}^1(\lambda+\sigma_i(\tau,v))\,\mathrm{d}\tau} u(\mu',v)\;\mathrm{d}\mu';$$

and finally

$$R_{i,\lambda}\colon X_p \to X_p, \qquad u \mapsto (R_{i,\lambda}u)(\mu,v) := \frac{1}{v}\int\limits_0^\mu \mathrm{e}^{-\frac{1}{v}\int\limits_{\mu'}^\mu(\lambda+\sigma_i(\tau,v))\,\mathrm{d}\tau} u(\mu',v)\;\mathrm{d}\mu'.$$

Clearly, for λ satisfying $\operatorname{Re}\lambda > -\underline{\sigma}$, the operators $P_{i,\lambda}$, $Q_{i,\lambda}$, $\Pi_{i,\lambda}$ and $R_{i,\lambda}$ are bounded. It's not difficult to check that

$$\|P_{i,\lambda}\| \le \mathrm{e}^{-\frac{1}{b}(\operatorname{Re}\lambda+\underline{\sigma})} \tag{3.4}$$

and

$$\|Q_{i,\lambda}\| \le (p(\operatorname{Re}\lambda+\underline{\sigma}))^{-\frac{1}{p}}. \tag{3.5}$$

Moreover, simple calculations using the Hölder inequality show that

$$\|\Pi_{i,\lambda}\| \le (\operatorname{Re}\lambda+\underline{\sigma})^{-\frac{1}{q}}, \tag{3.6}$$

and

$$\|R_{i,\lambda}\| \le (\operatorname{Re}\lambda+\underline{\sigma})^{-1}. \tag{3.7}$$

Thus, Equation (3.3) may be written abstractly as

$$\psi_i^1 = P_{i,\lambda}\psi_i^0 + \Pi_{i,\lambda}g.$$

On the other hand, ψ_i must satisfy the boundary condition (3.2), thus we obtain

$$\psi_i^1 = P_{i,\lambda}K_i\psi_i^1 + \Pi_{i,\lambda}g. \tag{3.8}$$

Observe that the operator $P_{i,\lambda}K_i$ in Equation (3.8) is defined from X_p^1 into X_p^1. Let $\varphi_1, \varphi_2 \in X_p^1$, from (A1) and Equation (3.4) we have

$$\|P_{i,\lambda}K_i\varphi_1 - P_{i,\lambda}K_i\varphi_2\| \le \alpha_i\,\mathrm{e}^{-\frac{\operatorname{Re}\lambda+\underline{\sigma}}{b}}\|\varphi_1-\varphi_2\| \tag{3.9}$$

for all $\varphi_1, \varphi_2 \in X_p^1$.

Consider now the equation

$$u = P_{i,\lambda} K_i u + \varphi, \qquad \varphi \in X_p^1 \tag{3.10}$$

where u is the unknown function and define the operator $A_{(i,\lambda,\varphi)}$ on X_p^1 by

$$A_{(i,\lambda,\varphi)} \colon X_p^1 \to X_p^1, \quad u \mapsto (A_{(i,\lambda,\varphi)} u)(1, v) := P_{i,\lambda} K_i u + \varphi.$$

It follows from Equation (3.9) that

$$\begin{aligned} \|A_{(i,\lambda,\varphi)}\varphi_1 - A_{(i,\lambda,\varphi)}\varphi_2\| &= \|P_{i,\lambda} K_i \varphi_1 - P_{i,\lambda} K_i \varphi_2\| \\ &\le \alpha_i \mathrm{e}^{-\frac{\operatorname{Re}\lambda + \underline{\sigma}}{b}} \|\varphi_1 - \varphi_2\|. \end{aligned}$$

Consequently, for $\operatorname{Re}\lambda > -\underline{\sigma} + b\log(\alpha_i)$ the operator $A_{(i,\lambda,\varphi)}$ is a contraction mapping and therefore the problem (3.10) has a unique solution

$$u_{(i,\lambda,\varphi)} = u_i.$$

Let $W_{i,\lambda}$ the nonlinear operator defined by

$$W_{i,\lambda}\varphi = u_i \tag{3.11}$$

where u_i is the solution of (3.10). Arguing as the proof of Lemma 2.1 and [27: Proposition 2.1], we have the following result:

Lemma 3.1. *Assume that* (A1) *holds. Then,*

(i) *for every λ satisfying $\operatorname{Re}\lambda > -\underline{\sigma} + b\log(\alpha_i)$, $i = 1, 2$, the operator $W_{i,\lambda}$ is continuous and maps bounded sets into bounded ones and satisfying the following estimate*

$$\|W_{i,\lambda}\varphi_1 - W_\lambda \varphi_2\| \le \Big(1 - \alpha_i \mathrm{e}^{-(\frac{\operatorname{Re}\lambda + \underline{\sigma}}{b})}\Big)^{-1} \|\varphi_1 - \varphi_2\|;$$

where $\varphi_1, \varphi_2 \in X_p^1$, $i = 1, 2$.

(ii) *if $\operatorname{Re}\lambda > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha_i))$, then the operators $(\lambda - S_{K_i})$ are invertible and $(\lambda - S_{K_i})^{-1}$ is given by*

$$(\lambda - S_{K_i})^{-1} = Q_{i,\lambda} K_i W_{i,\lambda} \Pi_{i,\lambda} + R_{i,\lambda}.$$

Moreover, $(\lambda - S_{K_i})^{-1}$ are continuous on X_p and maps bounded sets into bounded ones.

In what follows and for our subsequent analysis, we need the following hypothesis:

(A2) $r_{ij}(\mu, v, v', \psi(\mu, v')) = k_{ij}(\mu, v, v') f(\mu, v', \psi(\mu, v')); \quad (i, j) \in \{(1, 2), (2, 1)\}$ where f is a measurable function defined by

$$f\colon [0,1] \times [a,b] \times \mathbb{C} \to \mathbb{C}, \quad (\mu, v, u) \mapsto f(\mu, v, u)$$

with $k_{ij}(\cdot,\cdot,\cdot)$, $(i,j) \in \{(1,2),(2,1)\}$ are measurable functions from $[0,1]\times [a,b]\times\mathbb{C}$ to $\mathbb{C}$ which defines a bounded linear operator B_{ij} by

$$B_{ij}\colon X_p \to X_p, \quad \psi \mapsto \int_a^b k_{ij}(\mu, v, v')\psi(\mu, v')\mathrm{d}v'. \tag{3.12}$$

Notice that the operators B_{ij}, $(i,j) = (1,2),(2,1)$ act only on the velocity v, so μ may be seen simply as a parameter in $[0,1]$. Then, we will consider B_{ij} as a function

$$B_{ij}(\cdot)\colon \mu \in [0,1] \to B_{ij}(\mu) \in \mathcal{L}(L_p([a,b]; \mathrm{d}v)).$$

In the following, we will make the assumptions:

(A3)
- the function $B_{ij}(\cdot)$ is measurable, i.e., if $\mathcal{O}$ is an open subset of $\mathcal{L}(L_p([a,b]; \mathrm{d}v))$, then $\big\{\mu \in [0,1] : \ B_{ij}(\mu) \in \mathcal{O}\big\}$ is measurable,
- there exists a compact subset $\mathcal{C} \subseteq \mathcal{L}(L_p([a,b]; \mathrm{d}v))$ such that $B_{ij}(\mu) \in \mathcal{C}$ a.e. on $[0,1]$,
- $B_{ij}(\mu) \in \mathcal{K}(L_p([a,b]; \mathrm{d}v))$ a.e. on $[0,1]$.

where $\mathcal{K}(L_p([a,b], \mathrm{d}v))$ stands for the class of compact operators on $L_p([a,b], \mathrm{d}v)$.

Lemma 3.2. ([27: Lemma 3.1]) *Let $p \in (1,\infty)$ and assume that* (A1) *holds. If B_{ij}, $(i,j) \in \{(1,2),(2,1)\}$ satisfies* (A3)*, then for any $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha_i))$, the operators $(\lambda - S_K)^{-1}B_{ij}$ are completely continuous on X_p.*

Now we recall some facts concerning superposition operators required below. Recall $f\colon [0,1] \times [a,b] \times \mathbb{C} \to \mathbb{C}$ is a Carathéodory function if the following conditions are satisfied

$$(\mu, v) \mapsto f(\mu, v, u) \text{ is measurable on } [0,1] \times [a,b] \text{ for all } u \in \mathbb{C}$$

$$u \mapsto f(\mu, v, u) \text{ is continuous on } \mathbb{C} \text{ a.e. } (\mu, v) \in [0,1] \times [a,b].$$

Observe that if f is a Carathéodory function, then we can define the operator $\mathcal{N}_f$ on the set of function $\psi\colon [0,1] \times [a,b] \to \mathbb{C}$ by $(\mathcal{N}_f\psi)(\mu, v) = f(\mu, v, \psi(\mu, v))$ for every $(\mu, v) \in [0,1] \times [a,b]$.

We assume that

(A4) f is a Carathéodory map satisfying

$$|f(\mu, v, u_1) - f(\mu, v, u_2)| \le |h(\mu, v)||u_1 - u_2|$$

where $h \in L^\infty([0,1] \times [a,b], \mathrm{d}\mu\mathrm{d}v)$.

We are now ready to state our first result in this section.

Theorem 3.1. *Assume that* (A1) *and* (A2) *hold. If* B_{ij}, $(i,j) \in \{(1,2),(2,1)\}$ *satisfy* (A3) *on* X_p, *then for each* $r > 0$, *there is* $\lambda_r > 0$ *such that for each* λ *satisfying* $\operatorname{Re}(\lambda) > \lambda_r$ *the problem* (3.1)–(3.2) *has at least one solution in* $B_r \times B_r$.

Proof. Let λ be a complex number such that $\operatorname{Re}\lambda > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha))$ with $\alpha = \max(\alpha_1, \alpha_2)$. Then according to Lemma 3.1 *(ii)*, $\lambda - S_{K_i}$ is invertible and therefore the problem (3.1)–(3.2) may be transformed into

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathcal{L}(\lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \psi_i^0 = K_i \psi_i^1, \quad i = 1,2$$

where

$$\mathcal{L}(\lambda) = \begin{pmatrix} S_{K_1} - (\lambda - 1)I & B_{12}\mathcal{N}_f \\ B_{21}\mathcal{N}_f & S_{K_2} - (\lambda - 1)I \end{pmatrix}.$$

Let $r > 0$. We first check that, for suitable λ, $\Upsilon(\lambda) := (\lambda - S_{K_1})^{-1} B_{12}\mathcal{N}_f$ leaves B_r invariant. Let $\|\psi_2\| \le r$, therefore we have from Lemma 3.1 and the equations (3.4)–(3.7)

$$\begin{aligned} \|\Upsilon(\lambda)(\psi_2)\| &\le \|Q_{1,\lambda} K_1 W_{1,\lambda} \Pi_{1,\lambda} B_{12}\mathcal{N}_f(\psi_2) + R_{1,\lambda} B_{12}\mathcal{N}_f(\psi_2)\| \\ &\le \left[\frac{1}{\operatorname{Re}\lambda + \underline{\sigma}} + \frac{1}{p^{\frac{1}{p}}} \frac{\alpha_1}{1 - \alpha_1 \mathrm{e}^{-\frac{\operatorname{Re}\lambda + \underline{\sigma}}{b}}} \right] \|B_{12}\| M(r) + \Sigma(\operatorname{Re}\lambda) \end{aligned}$$

where $M(r)$ is the upper-bound of $\mathcal{N}_f$ on B_r and $\Sigma(\operatorname{Re}\lambda) = \left[\frac{\alpha_1 \|W_{1,\lambda}(0)\| + \|K_1(0)\|}{(\operatorname{Re}\lambda + \underline{\sigma})^{\frac{1}{p}} p^{\frac{1}{p}}} \right]$.
Let $\varepsilon > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha))$. For $\operatorname{Re}\lambda > \varepsilon$ we have

$$(1 - \alpha_1 \mathrm{e}^{-\frac{\operatorname{Re}\lambda + \underline{\sigma}}{b}})^{-1} \le (1 - \alpha_1 \mathrm{e}^{-\frac{\varepsilon + \underline{\sigma}}{b}})^{-1}.$$

Therefore

$$\|\Upsilon(\lambda)(\psi_2)\| \le \left[\frac{1}{\operatorname{Re}\lambda + \underline{\sigma}} + \frac{1}{p^{\frac{1}{p}}} \frac{\alpha_1}{1 - \alpha_1 \mathrm{e}^{-\frac{\varepsilon + \underline{\sigma}}{b}}} \right] \|B_{12}\| M(r) + \Sigma(\operatorname{Re}\lambda).$$

Using Equation (3.11) we have

$$P_{1,\lambda} K_1 W_{1,\lambda}(0) = W_{1,\lambda}(0).$$

Let $0 < \delta < \frac{1}{\alpha_1}$, from Equation (3.5) there exists λ_r such that for any λ satisfying $\operatorname{Re}\lambda > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha), \lambda_r)$, we have $\|P_{1,\lambda}\| \le \delta$, then using (A1) we obtain

$$\begin{aligned} \|W_{1,\lambda}(0)\| &\le \|P_{1,\lambda}\| \|K_1 W_{1,\lambda}(0)\| \\ &\le \delta(\alpha_1 \|W_{1,\lambda}(0)\| + \|K_1(0)\|). \end{aligned}$$

It follows that

$$\|W_{1,\lambda}(0)\| \leq \frac{\delta\|K_1(0)\|}{1-\delta\alpha_1}.$$

Therefore

$$\begin{aligned}\|\Upsilon(\lambda)(\psi_2)\| &\leq \left[\frac{1}{\operatorname{Re}\lambda+\underline{\sigma}}+\frac{1}{p^{\frac{1}{p}}}\frac{\alpha_1}{1-\alpha_1 \mathrm{e}^{-\frac{\varepsilon+\underline{\sigma}}{b}}}\right]\|B_{12}\|M(r)+\widetilde{\Sigma}(\operatorname{Re}\lambda).\\ &\leq Q(\operatorname{Re}\lambda)\end{aligned}$$

where

$$Q(t)=\left[\frac{1}{t+\underline{\sigma}}+\frac{1}{p^{\frac{1}{p}}}\frac{\alpha_1}{1-\alpha_1 \mathrm{e}^{-\frac{\varepsilon+\underline{\sigma}}{b}}}\right]\|B_{12}\|M(r)+\left[\frac{\left(\frac{\alpha_1\delta}{1-\delta\alpha_1}+1\right)\|K_1(0)\|}{(t+\underline{\sigma})^{\frac{1}{p}}p^{\frac{1}{p}}}\right]$$

and

$$\widetilde{\Sigma}(\operatorname{Re}\lambda)=\left[\frac{\left(\frac{\alpha_1\delta}{1-\delta\alpha_1}+1\right)\|K_1(0)\|}{(\operatorname{Re}\lambda+\underline{\sigma})^{\frac{1}{p}}p^{\frac{1}{p}}}\right].$$

Clearly, $Q(\cdot)$ is continuous strictly decreasing in $t>0$ and satisfies $\lim_{t\to+\infty} Q(t)=0$. Hence there exists λ_r', such that $Q(\lambda_r')\leq r$. Obviously, if $\operatorname{Re}\lambda\geq\lambda_r'$, then $(\lambda-S_{K_1})^{-1}B_{12}\mathcal{N}_f$ maps B_r into itself and $(\mathcal{H}1)$ is satisfied. Obviously, from Lemma 3.2 the operator $S(\lambda)=B_{21}\mathcal{N}_f(\lambda-S_{K_1})^{-1}B_{12}\mathcal{N}_f$ is continuous and so has a closed graph. Now we claim that $S(\lambda)(B_r)$ is relatively compact. Indeed, $\mathcal{N}_f(B_r)$ is a bounded subset of X_p, it follows from Lemma 3.2 that $O_r:=(\lambda-S_{K_1})^{-1}B_{12}\mathcal{N}_f(B_r)$ is relatively compact. Since $B_{21}\mathcal{N}_f$ is continuous, $B_{12}\mathcal{N}_f(\overline{O_r})$ is compact and so $S(\lambda)(B_r)$ is relatively compact. For $\operatorname{Re}\lambda\geq\max(\lambda_r,\lambda_r')$ and $\|\psi_2\|\leq r$, we have

$$\|S(\lambda)(\psi_2)\|\leq r.$$

By Lemma 3.1 there exists $\lambda_r''\geq\max(\lambda_r,\lambda_r')$ such that for any $\operatorname{Re}\lambda\geq\lambda_r''$, we have $\Delta(\lambda)\psi_2:=(\lambda-S_{K_2})^{-1}S(\lambda)\psi_2\in(B_r)$. The result follows from Theorem 2.1. □

Theorem 3.2. *Assume that* (A1), (A2) *and* (A4) *hold. If* B_{ij}, $(i,j)\in\{(1,2),(2,1)\}$ *satisfy* (A3) *on* X_p, *then for each* $r>0$, *there is* $\lambda_r>0$ *such that for each* λ *satisfying* $\operatorname{Re}(\lambda)>\lambda_r$ *the problem* (3.1)–(3.2) *has a unique solution in* $B_r\times B_r$.

P r o o f. Let λ be a complex number such that $\operatorname{Re}\lambda>\max(-\underline{\sigma},-\underline{\sigma}+b\log(\alpha))$ with $\alpha=\max(\alpha_1,\alpha_2)$ and let $\psi_1,\psi_2\in X_p$. Using the same notations as the proof above we have

$$\|S(\lambda)\psi_1-S(\lambda)\psi_2\|\leq\|B_{21}\|\|B_{12}\|\|h\|_\infty^2\mathcal{F}_1(\operatorname{Re}\lambda)\|\psi_1-\psi_2\|$$

where $\mathcal{F}_1(\operatorname{Re}\lambda) = \left[\frac{1}{\operatorname{Re}\lambda+\underline{\sigma}} + \frac{1}{p^{\frac{1}{p}}}\frac{1}{1-\alpha_1 \mathrm{e}^{-\frac{\operatorname{Re}\lambda+\underline{\sigma}}{b}}}\right]$ is continuous positive and strictly decreasing function on $]0,+\infty[$ satisfying $\lim_{t\to+\infty} \mathcal{F}_1(t) = 0$. By the same way we have

$$\begin{aligned}\|\Delta(\lambda)\psi_1 - \Delta(\lambda)\psi_2\| &\leq \left[\frac{1}{\operatorname{Re}\lambda+\underline{\sigma}} + \frac{1}{p^{\frac{1}{p}}}\frac{1}{1-\alpha_2 \mathrm{e}^{-\frac{\operatorname{Re}\lambda+\underline{\sigma}}{b}}}\right]\|S(\lambda)\psi_1 - S(\lambda)\psi_2\| \\ &\leq \mathcal{F}_2(\operatorname{Re}\lambda)\|S(\lambda)\psi_1 - S(\lambda)\psi_2\|\end{aligned}$$

where $\mathcal{F}_1(\cdot)$ is continuous positive and strictly decreasing function on $]0,+\infty[$ satisfying $\lim_{t\to+\infty} \mathcal{F}_1(t) = 0$. Then

$$\begin{aligned}&\|\Delta(\lambda)\psi_1 - \Delta(\lambda)\psi_2\| \\ &\leq \|B_{21}\|\|B_{12}\|\|h\|_\infty^2 \mathcal{F}_1(\operatorname{Re}\lambda)\mathcal{F}_2(\operatorname{Re}\lambda)\|\psi_1 - \psi_2\|.\end{aligned} \tag{3.13}$$

The function $\mathcal{F}_1\mathcal{F}_2(\cdot)$ satisfy the same properties as $\mathcal{F}_i$, $i = 1,2$. One concludes that there exist λ_1 be a complex number such that $\lambda_1 > \max(-\underline{\sigma}, -\underline{\sigma} + b\log(\alpha))$ and

$$\|B_{21}\|\|B_{12}\|\|h\|_\infty^2 \mathcal{F}_1(\operatorname{Re}\lambda)\mathcal{F}_2(\operatorname{Re}\lambda) < 1 \qquad \text{for any} \quad \operatorname{Re}\lambda \geq \lambda_1. \tag{3.14}$$

Evidently from inequalities (3.13)–(3.14), for $\operatorname{Re}\lambda \geq \lambda_1$ the operator $\Delta(\lambda)$ is a contraction mapping on B_r and maps B_r into itself. Hence the use of the Banach fixed point theorem concludes that there exists a unique ψ_2 in B_r such that $\Delta(\lambda)\psi_2 = \psi_2$. Take $\psi_1 := \Upsilon(\lambda)\psi_2$. By the same argument as the precedent proof ψ_1 lie in B_r and so $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ is a unique fixed point for the problem (3.1)–(3.2) in $B_r \times B_r$. $\square$

4. General existence solutions

Now we discuss the existence of solutions for the more general nonlinear boundary problem (1.4)–(1.5). When dealing with this problem some technical difficulties arise. So, we need the following assumption:

(A5) $K_i \in \mathcal{L}(X_p^1, X_p^0)$ and for some $r > 0$,

$$|\sigma_i(\mu, v, \psi_1) - \sigma_i(\mu, v, \psi_2)| \leq |\omega_i(\mu, v)||\psi_1 - \psi_2| \qquad i = 1,2 \quad (\psi_1, \psi_2 \in X_p)$$

where $\mathcal{L}(X_p^1, X_p^0)$ denotes the set of all bounded linear operators from X_p^1 into X_p^0, $\omega_i(\cdot,\cdot) \in L^\infty([0,1]\times[a,b], \mathrm{d}\mu\mathrm{d}v)$ and $\mathcal{N}_{\sigma_i}$ acts from B_r into B_r.

Define the free streaming operator $\widehat{S}_{K_i}$ by

$$\begin{aligned}&\widehat{S}_{K_i}\colon \mathcal{D}(\widehat{S}_{K_i}) \subseteq X_p \to X_p,\\ &\psi \mapsto \widehat{S}_{K_i}\psi(\mu, v) = -v\frac{\partial \psi}{\partial \mu}(\mu, v)\\ &\mathcal{D}(\widehat{S}_{K_i}) = \big\{\psi \in \mathcal{W}_p \mid \psi^0 = K_i(\psi^1)\big\}.\end{aligned}$$

Theorem 4.1. *Assume that* (A2), (A4) *and* (A5) *hold. If* B_{ij}, $(i,j) \in \{(1,2), (2,1)\}$ *satisfy* (A3) *on* X_p, *then for each* $r > 0$, *there is* $\lambda_r > 0$ *such that for each* λ *satisfying* $\mathrm{Re}(\lambda) > \lambda_r$ *the problem* (1.4)–(1.5) *has at least one solution in* $B_r \times B_r$.

Proof. Since K_i, $i = 1, 2$, are linear (by (A5)), the operators $\widehat{S}_{K_i}$ are linear, too. Using Lemma 3.1, $\big\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda > \max\big(0, b\log\|K_1\|, b\log\|K_2\|\big)\big\} \subset \rho(\widehat{S}_{K_i})$ where $\rho(\widehat{S}_{K_i})$ denotes the resolvent set of $\widehat{S}_{K_i}$. Let λ be a complex number such that $\mathrm{Re}\,\lambda > \max(0, b\log\|K_1\|, b\log\|K_2\|)$. Then, by linearity of the operator $(\lambda - \widehat{S}_{K_i})^{-1}$, the problem (1.4)–(1.5) written in the form

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \widehat{\mathcal{L}}(\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \psi_i \in \mathcal{D}(\widehat{S}_{K_i}), \quad i = 1, 2$$

where

$$\widehat{\mathcal{L}}(\lambda) = \begin{pmatrix}\widehat{S}_{K_1} - (\lambda - 1)I + \mathcal{N}_{\sigma_1} & B_{12}\mathcal{N}_f\\ B_{21}\mathcal{N}_f & \widehat{S}_{K_2} - (\lambda - 1)I + \mathcal{N}_{\sigma_2}\end{pmatrix},$$

may be transformed into the form

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathcal{G}_1(\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \mathcal{G}_2(\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \psi_i \in \mathcal{D}(\widehat{S}_{K_i}), \quad i = 1, 2 \tag{4.1}$$

where

$$\mathcal{G}_1(\lambda) = \begin{pmatrix}(\lambda - \widehat{S}_{K_1})^{-1}\mathcal{N}_{\sigma_1} & 0\\ 0 & (\lambda - \widehat{S}_{K_2})^{-1}\mathcal{N}_{\sigma_2}\end{pmatrix}$$

and

$$\mathcal{G}_2(\lambda) = \begin{pmatrix}0 & (\lambda - \widehat{S}_{K_1})^{-1}B_{12}\mathcal{N}_f\\ (\lambda - \widehat{S}_{K_2})^{-1}B_{21}\mathcal{N}_f & 0\end{pmatrix}.$$

Check that, for suitable λ, the operator $\mathcal{G}_1(\lambda)$ is a contraction mapping. Indeed, let $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in X_p$

$$\|(\lambda - \widehat{S}_{K_i})^{-1}(\mathcal{N}_{\sigma_i}\varphi_i - \mathcal{N}_{\sigma_i}\psi_i)\| \le \|(\lambda - \widehat{S}_{K_i})^{-1}\|\|\mathcal{N}_{\sigma_i}\varphi_i - \mathcal{N}_{\sigma_i}\psi_i\|, \quad i = 1, 2.$$

A simple calculation using the estimates (3.4)–(3.7) leads to

$$\|(\lambda - \widehat{S}_{K_i})^{-1}\| \leq \frac{1}{\operatorname{Re}\lambda}\left[1 + \frac{\gamma}{1 - \gamma \mathrm{e}^{-\frac{\operatorname{Re}\lambda}{b}}}\right], \qquad i = 1,2 \tag{4.2}$$

where $\gamma = \max(\|K_1\|, \|K_2\|)$. Moreover, taking into account the assumption on $\sigma_i(\cdot,\cdot,\cdot)$ we get

$$\|\mathcal{N}_{\sigma_i}\varphi_i - \mathcal{N}_{\sigma_i}\psi_i\| \leq \|\omega\|_\infty \|\varphi_i - \psi_i\|$$

where $\|\omega\|_\infty = \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty)$. Using the relations (4.1) and (4.2), we have for $V = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ and $W = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$\begin{aligned} \|\mathcal{G}_1(\lambda)V - \mathcal{G}_1(\lambda)W\| &= \left\| \begin{pmatrix} (\lambda - \widehat{S}_{K_1})^{-1}\mathcal{N}_{\sigma_1}\varphi_1 \\ (\lambda - \widehat{S}_{K_2})^{-1}\mathcal{N}_{\sigma_1}\varphi_2 \end{pmatrix} - \begin{pmatrix} (\lambda - \widehat{S}_{K_1})^{-1}\mathcal{N}_{\sigma_1}\psi_1 \\ (\lambda - \widehat{S}_{K_2})^{-1}\mathcal{N}_{\sigma_1}\psi_2 \end{pmatrix} \right\| \\ &\leq \frac{1}{\operatorname{Re}\lambda}\left[1 + \frac{\gamma}{1 - \gamma \mathrm{e}^{-\frac{\operatorname{Re}\lambda}{b}}}\right] \|\omega\|_\infty \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} - \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\| \\ &\leq \Xi(\operatorname{Re}\lambda) \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} - \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|. \end{aligned}$$

Noticing that Ξ is a continuous strictly decreasing function defined on $]0,\infty[$ and

$$\lim_{x \to \infty} \Xi(x) = 0.$$

So, there exists $\lambda_1 \in\,]\max(0, b\log\|K_1\|, b\log\|K_2\|, \infty[$ such that $\Xi(\lambda_1) < 1$. Hence, for $\operatorname{Re}\lambda \geq \lambda_1$, $\mathcal{G}_1(\lambda)$ is a contraction mapping. Using Lemma 3.1 and arguing as in the proof of Theorem 3.1 we show that the operator $\mathcal{G}_2(\lambda)$ is completely continuous on X_p. The Theorem 2.5 concludes the proof. $\square$

Let B_{ij} be defined by Equation (3.12) and let $k_{ij}^{+}(\cdot,\cdot,\cdot)$ (resp. $k_{ij}^{-}(\cdot,\cdot,\cdot)$) denotes the positive part (resp. the negative part) of $k_{ij}(\cdot,\cdot,\cdot)$:

$$k_{ij}(\mu, v, v') = k_{ij}^{+}(\mu, v, v') - k_{ij}^{-}(\mu, v, v') \qquad (\mu, v, v') \in [0,1] \times [a,b] \times [a,b].$$

We define the following non-negative operators:

$$B_{ij}^{\pm} : \psi \to B_{ij}^{\pm}\psi(\mu, v) := \int_a^b k_{ij}^{\pm}(\mu, v, v')\psi(\mu, v')\,\mathrm{d}v'.$$

Clearly,

$$B_{ij} = B_{ij}^{+} - B_{ij}^{-}.$$

Now, let $\mid B_{ij} \mid$ denotes the following non-negative operator:

$$\mid B_{ij} \mid := B_{ij}^{+} + B_{ij}^{-}$$

i.e.

$$|B_{ij}|\psi(\mu, v) = \int_a^b |k_{ij}|(\mu, v, v')\psi(\mu, v')\,\mathrm{d}v', \quad \psi \in X_p.$$

Let us discuss the existence of positive solutions to the boundary value problem. To this purpose we make the hypothesis

(A6) $$K_i[(X_p^1)^+)] \subset (X_p^0)^+$$

where $(X_p^1)^+$ (resp. $(X_p^0)^+$) denotes the positive cone of the space X_p^1 (resp. X_p^0). Let $r > 0$. We define the set B_r^+ by $B_r^+ := B_r \cap X_p^+$.

Theorem 4.2. *Assume that* (A1), (A2), (A3), (A4) *and* (A6) *hold. If* B_{ij} *is a positive operator and* $\mathcal{N}_f(X_p^+) \subset X_p^+$, *then for each* $r > 0$ *there is* $\lambda_r > 0$ *such that for all* $\lambda > \lambda_r$ *the problem* (1.4)–(1.5) *has at least one solution in* B_r^+.

Proof. Obviously, the operators $P_{i,\lambda}$, $\Pi_{i,\lambda}$, $Q_{i,\lambda}$ and $R_{i,\lambda}$ are bounded and positive. Accordingly, a similar reasoning to that in the proof of Theorem 5.3 in [7], gives the desired result. □

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