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GROWTH AND APPROXIMATION OF SOLUTIONS TO A CLASS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbb{R}^N$

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ABSTRACT. In this paper we consider the equation $\nabla^2\varphi + A(r^2)X \cdot \nabla\varphi + C(r^2)\varphi = 0$ for $X \in \mathbb{R}^N$ whose coefficients are entire functions of the variable $r = |X|$. Corresponding to a specified axially symmetric solution φ and set C_n of $(n+1)$ circles, an axially symmetric solution $\Lambda_n^*(x,\eta;C_n)$ and $\Lambda_n(x,\eta;C_n)$ are found that interpolates to $\varphi(x,\eta)$ on the C_n and converges uniformly to $\varphi(x,\eta)$ on certain axially symmetric domains. The main results are the characterization of growth parameters order and type in terms of axially symmetric harmonic polynomial approximation errors and Lagrange polynomial interpolation errors using the method developed in [MARDEN, M.: *Axisymmetric harmonic interpolation polynomials in* $\mathbb{R}^N$, Trans. Amer. Math. Soc. **196** (1974), 385–402] and [MARDEN, M.: *Value distribution of harmonic polynomials in several real variables*, Trans. Amer. math. Soc. **159** (1971), 137–154].

1. Introduction

The work of Morris Marden ([13], [14]) focused on the study of polynomials, entire functions and their geometry in the complex plane. He utilized integral operators based on Laplace type integrals for Legendre polynomials to associate harmonic functions with analytic functions of single complex variable

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and studied the value distribution, interpolation and approximation of harmonic functions in several real variables.

In this paper we apply the methods developed by Morris Marden ([13], [14]) to study the growth and approximation of solutions of the equations of the type

$$T[\varphi] = \nabla^2\varphi + A(r^2)X\cdot\nabla\varphi + C(r^2)\varphi = 0, \qquad X \in \mathbb{R}^N, \tag{1.1}$$

whose coefficients are entire functions. The analysis is based on integral operators developed by S. Bergman [1] which uniquely associate these solutions with harmonic functions.

Regular solutions of the equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial \eta^2} + \frac{(N-2)}{\eta}\frac{\partial H}{\partial \eta} = 0, \qquad N \geq 3 \tag{1.2}$$

are called axisymmetric harmonic functions in $\mathbb{R}^N$ with $x = x_1$, $\eta = (x_2^2 + x_3^2 + \cdots + x_N^2)^{1/2}$ in some neighborhood $\Omega \subset E^N$ of the origin where they are subject to the initial data

$H \in C^2(\Omega)$, satisfies (1.2) for all $(x,\eta) \in \Omega$, $\eta \neq 0$, $\frac{\partial H(x,0)}{\partial \eta} = 0$ on the intersection of Ω with the x axis. Since the x axis is the symmetry axis, the principle coordinates are the cylindrical coordinates (x,η,ϕ) defined by $x_2 = \eta\cos\phi$, $\left(x_3^2 + x_4^2 + \cdots + x_N^2\right)^{1/2} = \eta\sin\phi$, $0 \leq \phi \leq 2\pi$. We shall also work with the spherical coordinates (r,θ,ϕ) defined by $x = r\cos\theta$, $\eta = r\sin\theta$ where the angle "θ" is the colatitude $0 \leq \theta \leq \phi$.

The functions H defined above are necessarily symmetric i.e., satisfy $H(x,\eta) = H(x,-\eta)$. We say that H is regular on $\mathrm{cl}(\Omega)$ if H is regular in some region $\Omega' \supset \mathrm{cl}(\Omega)$.

In axisymmetric regions $\Omega \subset E^N$ about the origin, H may be uniquely expanded as Fourier Jacobi series in terms of the complete system $r^k C_k^{\frac{N}{2}-1}(\cos\theta)$ [3: p. 168], [16]

$$H(r,\theta) = H(x,\eta) = \sum_{k=0}^{\infty} B(N-2,k+1)a_k r^k C_k^{\frac{N}{2}-1}(\cos\theta) \tag{1.3}$$

where the function $C_k^{\frac{N}{2}-1}(u)$ is the so-called Gegenbauer or ultra spherical harmonic polynomial given by [3: p. 81–85]

$$C_k^{\frac{N}{2}-1}(u) = \sum_{j=0}^{[k/2]} (-1)^j \gamma_{kj} u^{k-2j}$$

with coefficients expressed in terms of the gamma function as

$$\gamma_{kj} = 2^{k-2}\frac{\Gamma\left(k-j+\frac{N}{2}-1\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma(j+1)\Gamma(k-2j+1)}.$$

Because of symmetry, H takes constant values on hyper circles $X=(x,\eta)$ in Ω. By a hyper circle (x_k,η_k) we mean the locus of points in $\mathbb{R}^N$ that satisfy simultaneously the two equations $x=x_k$, $\eta=\eta_k$ (cf. [14: p. 151–152]), and by an axisymmetric region $\Omega\subset\mathbb{R}^N$ we mean one such that, if hyper circle $(x_0,\eta_k)\subset\Omega$, then also circle $(x_0,\eta)\subset\Omega$ for $0\le\eta\le\eta_0$. In this case the functions H are harmonic in $\mathbb{R}^N$, i.e., satisfy Laplace's equation

$$\frac{\partial^2 H}{\partial x_1^2}+\frac{\partial^2 H}{\partial x_2^2}+\cdots+\frac{\partial^2 H}{\partial x_N^2}=0.$$

It is well known that every axisymmetric harmonic polynomial in $\mathbb{R}^N$ may be written as a linear combination of the polynomials $P_k(x,\eta)$ defined by the equation

$$P_k(x,\eta)=r^k C_k^{\frac{N}{2}-1}\left(\frac{x}{r}\right),\qquad r^2=x^2+\eta^2.$$

In view of [20: p. 97] we have

$$P_k(x,\eta)=\beta_k\int\limits_L \sigma^k\,\mathrm{d}_{\nu_N}(\xi),\qquad k=0,1,2,\dots,$$

where $\sigma=x+\mathrm{i}\eta\frac{\zeta+\zeta^{-1}}{2}$, $\beta_k=2^{3-N}\Gamma\left(\frac{N-2}{2}\right)^{-2}\left[\frac{(k+N-3)!}{k!}\right]$ with non-negative measure $\mathrm{d}_{\nu_N}(\zeta)=(\zeta-\zeta^{-1})^{N-3}\frac{\mathrm{d}\zeta}{\zeta}$ for ζ on the contour $L\equiv\{\zeta=\mathrm{e}^{\mathrm{i}t}:\ 0\le t\le\pi\}$.

It has to be noted $C_k^{\frac{N}{2}}(1)=P_k(1,0)=\frac{(k+N-3)!}{[k!(N-3)!]}$. For $N=3$, $C_k^{\frac{N}{2}-1}$ clearly reduces to the Legendre polynomial of degree k.

If the origin is contained in Ω, the series (1.3) can be written as

$$H(x,\eta)=\sum_{k=0}^{\infty}B(N-2,k+1)a_kP_k(x,\eta)\tag{1.4}$$

is valid uniformly within N dimensional ball $x^2+\eta^2\le r_0^2$ contained in Ω. For each H with domain Ω, there is a unique associated function f,

$$f(\xi)=\sum_{k=0}^{\infty}a_k\xi^k$$

analytic on the corresponding axi-convex domain $w\subset\mathbb{C}$ [14] whose A_n associate is defined by

$$\begin{aligned}H(x,\eta)=A_n(f)&=\alpha_N\int\limits_L f(\sigma)\,\mathrm{d}\nu_N(\xi),\\ \alpha_N&=\frac{\Gamma(N-2)}{(4\mathrm{i})^{N-2}}\left(\Gamma\frac{N}{2}-1\right).\end{aligned}\tag{1.5}$$

The extension of the representation (1.5) for $H(x,\eta)$ is all of Ω and for $f(\xi)$ in all of w is by harmonic and analytic continuations respectively. By virtue of (1.5) it follows that

$$f(z) = \frac{H(z,0)}{\int\limits_L \mathrm{d}\nu_N(\xi)}, \qquad z = x + \mathrm{i}\eta. \tag{1.6}$$

This implies that $f(z)$ is the analytic continuation of the function given by the right side of (1.6), from the real z to complex z. The extension of the representation (1.5) for $H(x,\eta)$ in all of Ω and for $f(\xi)$ in all of w is by harmonic and analytic continuations respectively.

Now the process extends to solutions $\varphi\colon T[\varphi] = 0$ by the method of ascent; Following S. Bergman [2], a complete set of solutions is given by

$$\begin{aligned} \varphi_k(x,\eta) &= J_k(r^2)B(N-2,k+1)(x^2+\eta^2)^{k/2}C_k^{\frac{N}{2}-1}(\cos\theta) \\ J_k(r^2) = J_k(r^2+\eta^2) &= \int\limits_I E(r^2,\tau)(1-\tau^2)^k \,\mathrm{d}\tau. \end{aligned}$$

for $k = 0,1,2,\ldots$, and $I = [-1,1]$. The functions J_k are considered generalized Bessel functions because the transform is essentially Sonine's integral for Bessel functions of the first kind when the coefficient $A(r^2)$ is identically zero and $C(r^2) = k^2 > 0$. The kernel is the Bergman E-function,

$$\begin{aligned} &E(r,\tau) = \exp\Big(-\frac{1}{2}\int\limits_0^\tau Ar\,\mathrm{d}r\Big)K(r,\tau), \\ &(1-\tau^2)K_{r\tau} - \tau^{-1}(\tau^2+1)K_r + r\tau(K_{rr} + (N-1)r^{-1}Kr + BK) = 0, \\ &B = -\frac{r}{2}A_r - \frac{N}{2}A - \frac{r^2A^2}{4} + C(r^2). \end{aligned} \tag{1.7}$$

The side conditions are that the function $K(r,\tau)$ is regular solution of equation (1.7), that $K_r(r,\tau)/r\tau$ is continuous at $\tau = 0$, $r = 0$, that $K(r,\tau)$ is continuously differentiable including the mixed partial derivatives, and that $K(r,\tau)$ assumes the same values at $\tau = \pm 1$. There are sufficient conditions for the existence of the function $K(r,\tau)$ [2]. One of them is that the equation $T(\varphi) = 0$ has a Green's function on a hyper sphere of sufficiently large radius about the origin to contain the set Ω. In the consequences of these conditions a solution $\varphi \in C^2(\Omega)$ may be expressed as the compactly convergent series

$$\varphi(x,\eta) = \sum_{k=0}^{\infty} a_k\varphi_k(x,\eta), \qquad (x,\eta) \in \Omega \subset E^N. \tag{1.8}$$

Thus the operator B_N associating φ with H follows

$$B_N[H(x,\eta)] = \int_I E(r^2\tau)H(X(1-\tau^2))\,\mathrm{d}\tau, \qquad X=(x,\eta)\in\Omega\subset E^N.$$

Gilbert and Howard ([4], [5]) established the uniqueness of this representation. The operator B_0 associating φ with f is formed under the composition $B_0 = B_N \circ A_N$ as $\varphi = B_N(H) = B_N(A_N[f]) = (B_N\circ A_N)[f] = B_0[f]$. For more details of above integral representation see Gilbert [6].

Following the notations in Morris Marden [13] we shall construct the basic interpolating solutions of (1.1) associated with the harmonic function theory in $\mathbb{R}^N$. Let $C_n = \{(x_k,\eta_k) :\ k = 0,1,2,\dots,n\}$ be a collection of hyper circles. Axially symmetric solutions that interpolate a specified solution "φ" in the sense that

$$\Lambda_n(x_k,\eta_k;C_k) = \varphi(x_k,\eta_k), \qquad k=0,1,2,3,\dots,n. \tag{1.9}$$

These functions can be expanded in the Lagrange interpolation from

$$\Lambda_n(x,\eta,C_n) = \sum_{k=0}^{n} A_k\varphi_k(x,\eta). \tag{1.10}$$

The coefficients A_k are to be determined so that equation (1.9) holds, that is so as to satisfy the system

$$\sum_{k=0}^{n} A_k\varphi_k(x_j,\eta_j) = \varphi(x_j,\eta_j), \qquad j=0,1,\dots,n. \tag{1.11}$$

Eliminating the A_k from (1.10) and system (1.11), we get for Δ the equation

$$\begin{vmatrix} \Lambda_n(x,\eta,C_n) & 1 & \varphi_1(x,\eta) & \dots & \varphi_n(x,\eta) \\ \varphi(x_0,\eta_0) & 1 & \varphi_1(x_0,\eta_0) & \dots & \varphi_n(x_0,\eta_0) \\ \varphi(x_1,\eta_1) & 1 & \varphi_1(x_1,\eta_1) & \dots & \varphi_n(x_1,\eta_1) \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi(x_n,\eta_n) & 1 & \varphi_1(x_n,\eta_n) & \dots & \varphi_n(x_n,\eta_n) \end{vmatrix} = 0. \tag{1.12}$$

Following on the lines of M. Marden [14], we solve (1.12) explicitly for Λ_n to obtain

$$\Lambda_n(x,\eta,C_n) = \frac{\sum_{k=0}^{n}\varphi(x_k,\eta_k)V_k(x,\eta;C_n)}{V(C_n)}, \tag{1.13}$$

provided $V(C_n)\neq 0$. Here $V(C_n) = \det|\varphi_k(x_j,\eta_j)|$, $j,k = 0,1,\dots,n$ and $V_k(x,\eta,C_n) = [V(C_n)]_{(x_k,\eta_k)=(x,\eta)}$ for $k=0,1,\dots,n$.

When the coefficients $A(r^2)$ and $C(r^2)$ are identically zero, the solutions $\varphi\colon T[\varphi] = \nabla^2\varphi = 0$ are harmonic functions and the formulae are essentially those of M. Marden [13]. The restriction $V(C_n)\neq 0$ on the choice of the circles

$C_n = \{(x_k, \eta_k)\}$ has an interesting geometric aspect. For example, let the circles C_n are located on the boundary of the hyper sphere $S(r_0)$ centered at the origin so that $x_j^2 + \eta_j^2 = r_0^2$, $j = 0, 1, 2, \ldots, n$. To study these circles, we normalize the basis functions φ_k as

$$\varphi_k(x, \eta) = \frac{r^k J_k(r^2)}{r_0^k J_k(r_0^2)} B(N-2, k+1)(x^2+\eta^2)^{k/2} C_k^{\frac{N}{2}-1} \frac{x}{r} \tag{1.14}$$

with $x^2 + \eta^2 \le r_0^2$ (r_0-fixed) and $J_k(r_0^2) \neq 0$, $k = 0, 1, 2, \ldots, n$.

Now in the consequence of the equation (1.8) and equations (1.9)–(1.13) with $V(C_n) \neq 0$, we find that

$$\varphi_k(x_j, \eta_j) = B(N-2, k+1) \left(x_j^2 + \eta_j^2\right)^{k/2} C_k^{\frac{N}{2}-1} \frac{x_j}{r_j},$$

$(x_j, \eta_j) \in C_n$, $j = 0, 1, 2, 3, \ldots, n$, and that

$$V(C_n) = r_0^{n(n+1)/2} B(N-2, k+1) \det \left[\left(x_j^2 + \eta_j^2\right)^{k/2} C_k^{\frac{N}{2}-1} \left(\frac{x_j}{r_j}\right) \right],$$

where $j, k = 0, 1, 2, \ldots, n$.

Following the kernel function method developed by R. P. Gilbert and H. C. Howard [5] for the integral representation of Lagrange interpolation, suppose that φ be a solution that is analytic on a closed hyper sphere, say $\mathrm{cl}(\Omega)$, where H is the associated harmonic function determined by the operator $B_N \colon \varphi = B_N[H] = B_N(A_N[f]) = (B_N \circ A_N)[f]$ defined as the integral

$$\varphi(x, \eta) = B_N[H(x, \eta)] = \frac{1}{2\pi \mathrm{i}} \int_L K(r, \zeta) H(x\zeta, \eta\zeta) \, \mathrm{d}\nu_N(\zeta), \tag{1.15}$$

with kernel

$$K(r\zeta) = \sum_{k=0}^{\infty} \frac{r^k J_k(r^2)}{r_0^k J_k(r_0^2)} \zeta^{-1}, \qquad |r| < r_0, \quad \zeta \in L.$$

This transform is valid for φ and $H \in C^2(\Omega) \cap C(\mathrm{cl}(\Omega))$ (see S. Bergman [2]).

Proceeding on the lines of M.Marden [13: pp. 401] the equation (1.13) can be written as

$$\Lambda_n(x, \eta_j, C_n) = \frac{\sum\limits_{k=0}^{n} \int\limits_{T^*} f(\sigma_k) \nu(S_k) M(T^*) \, \mathrm{d}T^*}{\int\limits_{T} \nu(S) M(T) \, \mathrm{d}T} \tag{1.16}$$

where

$$M(T) = \prod_{k=0}^{n} \beta_k \sin^{N-3} tk, \quad M(T_k) = [M(T)]_{t_k = t}, \quad M(T^*) = M(T) \sin^{N-3} t,$$

$\sigma = x + \mathrm{i}\eta\cos t$, $\sigma_k = x_k + \mathrm{i}\eta_k \cos t_k$, $k = 0, 1, 2, \ldots, n$. Here T and T^* are the closed $(n+1)$ and $(n+2)$ dimensional cubes given as

$$\begin{aligned} T &= \{t_0, t_1, \ldots, t_n\}, \quad && 0 \le t_k \le \pi, && k = 0, 1, \ldots, n, \\ T^* &= \{t, t_0, t_1, \ldots, t_n\}, \quad && 0 \le t \le \pi,\ 0 \le t_k \le \pi, \quad && k = 0, 1, 2, \ldots, n \end{aligned}$$

and the notation

$$\begin{aligned} \mathrm{d}T &= \mathrm{d}t_n \ldots \mathrm{d}t_1\, \mathrm{d}t_0, \qquad && \mathrm{d}T^* = \mathrm{d}t_n \ldots \mathrm{d}t_1\, \mathrm{d}t_0\, \mathrm{d}t, \\ S &= \{\sigma_0, \sigma_1, \ldots, \sigma_n\}, && \nu(S) = \nu(\sigma_0, \sigma_1, \ldots, \sigma_n), \\ T_k &= (T)_{t_k = t}, && S_k = (S)_{\sigma_k = \sigma}. \end{aligned}$$

Bearing in mind the equation (1.15) we can write the equation (1.16) as

$$\Lambda_n(x, \eta, C_n) = \frac{\sum\limits_{k=0}^{n} \int\limits_{T^*} K(r, \zeta) f(\sigma_k) \nu(S_k) M(T^*)\, \mathrm{d}T^*}{2\pi\mathrm{i} \int\limits_{T} \nu(S) M(T)\, \mathrm{d}T}. \tag{1.17}$$

Let $l_n(\sigma; S)$ be the associated Lagrange polynomial which interpolates to $f(\sigma)$ at the points $\sigma_k = x_k + i\eta_k \cos t_k$, $k = 0, 1, 2, \ldots, n$. It is known that

$$l_n(\sigma; S) = \sum_{k=0}^{n} \frac{f(\sigma_k) p(\sigma)}{p'(\sigma_k)(\sigma - \sigma_k)} \tag{1.18}$$

where $p(\sigma) = (\sigma - \sigma_0)(\sigma - \sigma_n)$. The equation (1.18) is equivalent to

$$l_n(\sigma; S) = \sum_{k=0}^{\infty} \frac{f(\sigma_k) \nu(S_k)}{\nu(S)}, \tag{1.19}$$

Using (1.19), the equation (1.17) can be written as

$$\Lambda_n(x, \eta; C_n) = \frac{\int\limits_{T^*} K(r, \zeta) l_n(\sigma; S) \nu(S) M(T^*)\, \mathrm{d}T^*}{2\pi\mathrm{i} \int\limits_{T} \nu(S) M(T)\, \mathrm{d}T} \tag{1.20}$$

The equation (1.20) may be reformulated as

$$\Lambda_n(x, \eta; C_n) = \frac{\alpha_N \int\limits_{L} K(r, \zeta) L_n(x\zeta, \eta\zeta; C_n)\, \mathrm{d}\nu_N(S)}{2\pi\mathrm{i} \int\limits_{L} \nu(S)\, \mathrm{d}\nu_N(\zeta)} \tag{1.21}$$

where $L_n(x, \eta; C_n) = \int\limits_{L} l_n(\sigma; S) \nu(S)\, \mathrm{d}\nu_N(S)$ with $\nabla^2 L_n(x, \eta; C_n) = 0$ and $L_n(x_k, \eta_k; C_n) = \varphi(x_k, \eta_k)$ for $k = 0, 1, \ldots, n$.

2. Auxiliary results

In this section we shall prove some auxiliary results which will be used in the sequel.

Lemma 2.1. *Let $\Omega \subset E^N$ be a bounded axi-convex region and $H(x,\eta)$ be an axisymmetric harmonic function which is regular on* $\mathrm{cl}(\Omega)$. *Then there exist axisymmetric harmonic polynomials q_k such that $q_k \to H(x,\eta)$ uniformly on* $\mathrm{cl}(\Omega)$.

P r o o f. It is known that for each $H(x,\eta)$ with domain Ω, there is a unique operator A_N associated function f, analytic on the corresponding axi-convex domain $w \subset \mathbb{C}$ [14]. Since f is analytic on $w \subset \mathbb{C}$, and by Runge's theorem there exist polynomials p_k(of degree k) which converges uniformly to f on w. Let $q_k = A_N(p_k)$. Then q_k is a axi-convex harmonic polynomial (of the same degree as p_k), and for every $(x,\eta) \in \mathrm{cl}(\Omega)$,

$$\begin{aligned}|H(x,\eta) - q_k(x,\eta)| &= |A_N[f - p_n]| \\ &= \left| \alpha_N \int\limits_L [f(\sigma) - p_k(\sigma)]\, \mathrm{d}\nu_N(\sigma) \right| \le \|f - p_k\|,\end{aligned}$$

$\|\cdot\|$ denote the uniform norm over $\mathrm{cl}(\Omega)$. Therefore we obtain

$$\|H(x,\eta) - q_k(x,\eta)\| \le \|f - p_k\| \to 0. \tag{2.1}$$

□

Lemma 2.2. *Let $\Omega \subset E^N$ be a bounded axi-convex region, and ϕ be the conformal mapping of Ω onto the exterior of the unit disk, with $\phi(\infty) = \infty$. Let $H(x,\eta)$ be a solution of the equation* (1.2) *which is regular on* $\mathrm{cl}(\Omega)$. *If δ is the largest number for which $H(x,\eta)$ is regular interior to $\Gamma_\delta = \{z : |\phi(z)| = \delta\}$, for every R where $1 < R < \delta$, there exist axially symmetric harmonic polynomials q_k of degree k such that*

$$\|H(x,\eta) - q_k(x,\eta)\| \le \frac{2m(r,f)}{(R-1)} \left(\frac{5}{4}R\right)^k, \qquad k \ge 0,\ R > \frac{5}{2}, \tag{2.2}$$

where $m(R,f) = \sup\big\{|f(z)| : |z| \le R\big\}$. Further the degree of convergence need not obtained for $R > \delta$.

P r o o f. The proof of this lemma follows on the lines of [2: Theorem 2.2] of Kumar and Arora. □

Now first start the preliminary study of growth with the definition of the maximum moduli for the entire function solutions $\varphi(x,\eta)$ of the equation (1.1) with B_N and A_N associates $H(x,\eta)$ and f respectively as

$$M(r,\varphi) = \sup\big\{|\varphi(x,\eta) : \ x^2+\eta^2 \le r\big\}$$
$$M(r,H) = \sup\big\{|H(x,\eta)| : \ x^2+\eta^2 \le r\big\}$$
$$M(r,f) = \sup\big\{|f(x)| : \ |z| \le r\big\}.$$

Following [7], growth parameters such as order and type of entire function solutions $\varphi(x,\eta)$ of the equation (1.1) are defined as

$$\rho(\varphi(x,\eta)) \equiv \limsup_{r\to\infty} \frac{\log\log M(r,\varphi)}{\log r} \qquad \Theta(\varphi(x,\eta)) \equiv \limsup_{r\to\infty} \frac{\log M(r,\varphi)}{r^{\rho}}.$$

As usual, the orders ρ^*,ρ^{**} and types Θ^*,Θ^{**} of the associated entire harmonic function $H(x,\eta)$ and entire function f are defined as

$$\rho^*(H(x,\eta)) \equiv \limsup_{r\to\infty} \frac{\log\log M(r,H)}{\log r} \qquad \rho^{**}(f) \equiv \limsup_{r\to\infty} \frac{\log\log m(r,f)}{\log r}$$
$$\Theta^*(H(x,\eta)) \equiv \limsup_{r\to\infty} \frac{\log\log M(r,H)}{r^{\rho^*}} \qquad \Theta^{**}(f) \equiv \limsup_{r\to\infty} \frac{\log\log m(r,f)}{r^{\rho^{**}}}.$$

Now we shall prove:

Lemma 2.3. *In axi-convex regions $\Omega \subset E^N$ about the origin, the real entire function solutions $\varphi(x,\eta)$ has finite positive order and type of, and only of, the entire harmonic function B_N associate $H(x,\eta)$ and the entire function A_N associate f has positive orders and types. Then the orders and types are respectively equal.*

Proof. For each φ with domain Ω, there is a unique B_N associated harmonic function H and A_N associated function f analytic on the corresponding axi-convex domain $w \subset \mathbb{C}$ such that $\varphi(x,\eta) = B_N[H(x,\eta)] = B_N[A_N[f]] = (B_N \circ A_N)[f]$. From he normalization and non-negativity of the measure of $(B_N \circ A_N)$ we see that

$$M(r,\varphi) \le M(r,H) \le m(r,f), \qquad r>0. \tag{2.3}$$

Monotonicity of logarithm with the definitions of orders gives

$$\rho(\varphi) \le \rho^*(H) \le \rho^{**}(f). \tag{2.4}$$

Then the order of $\varphi(x,\eta)$ is finite. To prove the reverse inequality let $zr^{-1} = \lambda e^{i\theta}$, $0<\lambda<1$, and

$$f = A_N^{-1}(H(r,\xi)) = \int_{-1}^{1} H(r,\xi) K\Big(\frac{z}{r},\xi\Big)\,d\xi$$

where

$$K\Big(\frac{z}{r},\xi\Big)=\frac{\left(\frac{N}{2}-1\right)\Gamma(N-2)(1-\xi^2)^{N-1}\Big(1-\frac{z^2}{r^2}\Big)}{2^{N-3}\Gamma\left(\frac{N-1}{2}\right)^2\pi\left[1-2\xi\Big(\frac{z}{r}\Big)+\frac{z^2}{r^2}\right]^{N/2}}.$$

For more details of above integral see [3: p. 173]. We get

$$\begin{aligned} &m(r,f)\le K(\lambda)M(r\lambda^{-1},H),\\ &\quad K(\lambda)=\sup\big\{|K(\lambda e^{i\theta},\xi)|:\ 0\le\theta<2\pi,\ -1\le\xi\le+1\big\}. \end{aligned}\tag{2.5}$$

Using the non-negativity of the measure of $(B_N o A_N)^{-1}$, we get

$$f=(B_N\circ A_N)^{-1}[\varphi]$$

or

$$m(r,f)\le K^*M(r,\varphi),\qquad r>0,\quad K^*\equiv K^*(\lambda).\tag{2.6}$$

Similarly, using (2.5) and (2.7) we get

$$\rho^{**}(f)\le\rho^*(H)\qquad\text{and}\qquad\rho^{**}(f)\le\rho(\varphi).\tag{2.7}$$

In view of (2.4) and (2.7) we obtain

$$\rho^{**}(f)=\rho^*(H)=\rho(\varphi).$$

Using the definitions of respective types in (2.3), (2.5) and (2.6), the verification of equality of types can be done. □

3. Main results

In main results the characterizations of order and type of the solutions of equation (1.1) are computed in terms of axisymmetric harmonic polynomial approximation errors and Lagrange interpolation errors in the uniform norm.

First we define the approximation errors as

$$\begin{aligned} E_k(H(x,\eta))&\equiv\inf\big\{\|H-q_k\|:\ q_k\in H_k\big\},\qquad k=0,1,2\dots,\\ \|H-q_k\|&\equiv\sup\big\{|H(x,\eta)-q_k(x,\eta)|:\ x^2+\eta^2=r^2\big\}, \end{aligned}$$

the set H_k of all harmonic polynomials of degree k for every (x,η) in hyper circle X.

Let $h_k=\big\{A_N^{-1}(q_k):\ q_k\in H_k\big\}$. Following Kumar and Arora [12], for the A_N associate f, we define

$$\|f-p_k\|\equiv\sup\{|f(x)-p_k(x)|\},\qquad |x|<r,$$

and

$$\begin{aligned} e(f)&\equiv\inf\big\{\|f-p_k\|:\ p_k\in h_k,\ \ k=0,1,2,\dots\big\},\\ \Delta_n^*&\equiv\inf\big\{\|\varphi(x,\eta)-\Lambda^*(x,\eta;C_n)\|:\ \Lambda^*(x,\eta;C_n)=B_N[H_n]\big\} \end{aligned}$$

and

$$\Delta_n(x,\eta;C_n) \equiv \inf\big\{\|\varphi(x,\eta)-\Lambda_n(x,\eta;C_n)\| : \ x^2+\eta^2=r^2\big\}.$$

The operation Ref provides a transformation from analytic function of a single complex variable to harmonic functions in two dimensions. In a similar manner the A_N integral operator been used to transform results regarding the polynomial approximation of functions analytic in axi-convex region $w \subset \mathbb{C}$ to corresponding approximation theoretic results for generalized axisymmetric potentials H regular on $\mathrm{cl}(\Omega)$. A region Ω will be called axi-convex if $x+\mathrm{i}\eta \in \Omega$ implies $x+\mathrm{i}\beta\eta \in \Omega$ every $\beta \in [-1,1]$. The operator B_N associated the solutions φ of the equation (1.1) to H.

McCoy [17] studied the growth properties of an entire function H on axisymmetric regions $\Omega \subset E^N$ for $N=2$ about the origin, in terms of the errors $E_k(H)$ obtaining global existence criterion for H and coefficient characterizations for order and type. McCoy [18] investigated the accuracy of the Lagrange polynomial approximation and interpolation of the solutions of equation (1.1) using the operators A_N and B_N for $N=3$. He did not study the case for $N>3$ and not characterized the growth parameters in terms of approximation errors $\Delta_n(x,\eta;C_n)$. Kumar and Arora [12] studied the growth parameters (order and type) of generalized axisymmetric potentials (GASP's) H in terms of polynomial approximation errors $E_k(H)$ for $N \geq 3$, but he did not consider the solutions $\varphi(x,\eta)$ of the equation (1.1). In this paper we have tried to fill this gap.

It is significant to mention here that various authors such as Kapoor and Nautiyal [9] and Kumar [11] studied the growth properties of GASP's on axisymmetric region $\Omega \subset E^2$ when GASP is not entire function. Srivastava [19] obtained some results containing lower order and lower type for entire function GASP on $\Omega \subset E^2$. Tersenov [21] proved the existence of a solution of the Cauchy problem for a system of equations of ultra parabolic type. He [22] also studied the first boundary value problem as well as the Cauchy problem for a certain class of ultra parabolic equations. A priori estimates of the solutions had obtained in the special Holder spaces.

Marichev [15] studied the boundary value problems for equation of mixed type with two lines of degeneracy. Isamukhamedov and Oramov [8] discussed these problems for an equation of mixed type of the second kind with non-smooth degeneration lines. The study of Khe Kan Cher [10] devoted to the singular Tricomi problem for an equation of mixed type with two lines degeneracy. In [10] he also proved the uniqueness of solutions of the Tricomi problem for equations with two lines of degeneracy. In this paper our results and approach are different from all these authors.

Now we prove the following theorem.

THEOREM 3.1. *Let $\varphi \in \mathbb{C}^2(\Omega) \cap \mathbb{C}(\mathrm{cl}(\Omega))$ be a solution of the equation* (1.1) *on the domain $\Omega \subset E^N$ which is bounded axi-convex region and φ be regular on* $\mathrm{cl}(\Omega)$. *Then a necessary and sufficient condition for φ to have an analytic continuation as an entire function is that*

$$\lim_{n\to\infty} \Delta_n^{*1/n}(\varphi) = 0. \tag{3.1}$$

Proof. Let $\varphi(x,\eta)$ be regular on $\mathrm{cl}(\Omega)$ and let

$$\lim_{n\to\infty} \Delta_n^{*1/n}(\varphi) = 0. \tag{3.2}$$

Then we have

$$\Delta^*(x,\eta,C_n) = \frac{1}{2\pi \mathrm{i}} \int\limits_L K(r,\zeta) q_n(x\zeta,\eta\zeta;C_n)\,\mathrm{d}\nu_N(\zeta)$$

where

$$q_n(x,\eta;C_n) = \alpha_N \int\limits_L p_n(\sigma)\,\mathrm{d}\nu_N(\zeta)$$

with $\Lambda^2 q_n(x,\eta;C_n) = 0$ and $q_n(x_k,\eta_k;C_n) = \varphi(x_k,\eta_k)$ for $k = 0,1,2,\ldots,n$.

Since

$$\begin{aligned}|H(x,\eta) - q_n(x,\eta;C_n)| = |A_N(f-p_n)| &= \left|\alpha_N \int\limits_L [f(\sigma) - p_n(\sigma)]\,\mathrm{d}\nu_N(\zeta)\right| \\ &\le \|f - p_n\|.\end{aligned}$$

Hence

$$\begin{aligned}|\varphi(x,\eta) - \Lambda_n^*(x,\eta;C_n)| &= \left|\frac{1}{2\pi \mathrm{i}} \int\limits_L K(r,\zeta)[H(x\zeta,\eta\zeta) - q_n(x\zeta,\eta\zeta;C_n)]\,\mathrm{d}\nu_N(\zeta)\right| \\ &\le \frac{\alpha_N}{2\pi \mathrm{i}} \int\limits_L K(r,\zeta)(\|f-p_n\|)\,\mathrm{d}\nu N(\zeta) \le K\|f-p_n\|\end{aligned}$$

or

$$\lim_{n\to\infty} \Delta_n^{*1/n}(\varphi) \le \lim_{n\to\infty} K^{1/n}(e_n(f))^{1/n} = 0.$$

By our assumption (3.2) the $(B_N \circ A_N)$ associate of $\varphi(x,\eta)$ satisfies the Bernstein limit

$$\lim_{n\to\infty} e_n^{1/n}(f) = 0$$

and is necessarily an entire function $f = f(z)$, z in axi-convex domain $w \subset \mathbb{C}$. Using the fact "In axisymmetric regions $\Omega \subset E^N$ about the origin, for each $\varphi(x,\eta)$ regular in the hyper sphere S_N, there is a unique $(B_n \circ A_N)$ associated function f analytic in the disk D_R and conversely", the function $\varphi(x,\eta)$ is an entire function.

In order to prove the converse part let us assume that $\varphi(x,\eta)$ be regular on $\mathrm{cl}(\Omega)$ and have analytic continuation as an entire function. Then by Lemma 2.2, with the result

$$|\varphi(x,\eta)-\Lambda_n^*(x,\eta;C_n)| = \left|\frac{1}{2\pi \mathrm{i}}\int\limits_L K(r,\zeta)[H(x\zeta,\eta\zeta)-q_n(x\zeta,\eta\zeta;C_n)]\,\mathrm{d}\nu_N(\zeta)\right|$$

we obtain

$$\Delta_n^*(\varphi) \le \frac{K^* 2m(R,f)}{(R-1)}\left(\frac{5}{4}R\right)^n, \qquad n\ge 0,\quad R>5/2, \tag{3.3}$$

where $K^*=\frac{1}{2\pi\mathrm{i}}\int\limits_L K(r,\zeta)\,\mathrm{d}\nu_N(\zeta)$, is a constant independent of n.

Hence

$$\lim_{n\to\infty}(\Delta_n^*(\varphi))^{1/n}\to 0 \qquad \text{as} \quad R\to\infty.$$

Hence the proof is completed. □

Theorem 3.2. *Let $\varphi\in\mathbb{C}^2(\Omega)\cap\mathbb{C}(\mathrm{cl}(\Omega))$ be a solution of the equation (1.1) on the bounded axi-convex region $\Omega\subset E^N$ and φ be regular on $\mathrm{cl}(\Omega)$. Then*

$$\rho(\varphi)=\limsup_{n\to\infty}\frac{n\log n}{-\log\Delta_n^*(\varphi)} \tag{3.4}$$

is non-negative and finite if, and only if, the function φ has analytic continuation as an entire function of finite order $\rho(\varphi)$.

Proof. Let

$$\limsup_{n\to\infty}\frac{n\log n}{-\log\Delta_n^*(\varphi)}=\lambda^*.$$

First we consider the case $0<\lambda^*<\infty$. then for arbitrary $\varepsilon>0$, we have

$$\log\Delta_n^*(\varphi)\ge\frac{n\log n}{(\lambda^*+\varepsilon)}. \tag{3.5}$$

Using (3.3) with (3.5), we get

$$\log m(R,f)\ge\frac{n\log n}{\lambda^*+\varepsilon}+\log(R-1)-\log 2-\log K^*+n\log\left(\frac{4R}{5}\right).$$

The maximum value of right hand side is attained at $\frac{R}{\mathrm{e}}=(n)^{1/(\lambda^*+\varepsilon)}$ and we have

$$=\frac{R}{\mathrm{e}}\log\frac{R}{\mathrm{e}}+\log\frac{R-1}{2}K^*+\left(\frac{R}{\mathrm{e}}\right)^{(\lambda^*+\varepsilon)}\log\frac{4R}{5}$$

$$=\left(\frac{R}{\mathrm{e}}\right)^{(\lambda^*+\varepsilon)}\log\frac{4R^2}{5\mathrm{e}}+\log\frac{R-1}{2}K^*$$

or

$$\log\log m(R,f)\ge(\lambda^*+\varepsilon)\log\frac{R}{\mathrm{e}}+\log\log\frac{4R^2}{5\mathrm{e}}+0(1).$$

It gives that

$$\rho(f) = \limsup_{R\to\infty} \frac{\log\log m(R,f)}{\log R} \geq \lambda^*. \tag{3.6}$$

This inequality holds for $\lambda^* = 0$ and of $\lambda^* = \infty$ then $\rho(f) = \infty$. In order to prove reverse inequality in (3.6) using

$$f = (B_N \circ A_N)^{-1}[\varphi] \qquad \text{or} \qquad \|f - p_n\| \leq K_1 \|\varphi(x,\eta) - \Lambda_n^*(x,\eta;C_n)\|$$

or

$$e_n(f) \leq K_1 \Delta_n^*(\varphi) \tag{3.7}$$

Using the monotonicity of the logarithm in above inequality, we obtain

$$\limsup_{n\to\infty} \frac{n\log n}{-\log \Delta_n^*(\varphi)} \geq \limsup_{n\to\infty} \frac{n\log n}{-\log e_n(f)} = \rho(f). \tag{3.8}$$

Combining the inequalities (3.6) and (3.8) the result is immediate. Hence the proof is completed. □

Theorem 3.3. *Let $\varphi \in \mathbb{C}^2(\Omega) \cap \mathbb{C}(\mathrm{cl}(\Omega))$ be a solution of the equation* (1.1) *on the bounded axi-convex region $\Omega \subset E^N$ and φ be regular on $\mathrm{cl}(\Omega)$. Then φ has an analytic continuation as an entire function of order $\rho(\varphi)$ and finite type $\Theta(\varphi)$ if, and only if*

$$\frac{\Theta(f)}{M} = \limsup_{n\to\infty} \frac{n}{\mathrm{e}\rho(f)} [\Delta_n^*(\varphi)]^{\rho(f)/n}$$

where $M = \left(\frac{4}{5}\right)^{\rho(f)}$.

P r o o f. The proof follows in a similar manner as Theorem 3.2 with Lemma 2.2 and equation (3.7). So we omit the detailed proof here. □

4. Growth of $\varphi(x,\eta)$ in terms of Lagrange polynomial approximation errors

We can write the function φ as

$$\varphi(x,\eta) = \frac{\int\limits_{T^*} K(r,\zeta) f(\sigma)\nu(S)M(T^*)\,\mathrm{d}T^*}{2\pi\mathrm{i}\int\limits_{T} \nu(S)M(T)\,\mathrm{d}T}.$$

So, the Lagrange polynomial approximation error is defined as

$$\Delta_n(x,\eta;C_n) \equiv \inf\left\{\left\|\left[\left\{\int\limits_{T^*} K(r,\zeta)[f(\sigma)-l_n(\sigma;S)]\nu(S)M(T^*)\,\mathrm{d}T^*\right\}\right]\right.\right. \\ \left.\left. \times\left[2\pi\mathrm{i}\int\limits_{T}\nu(S)M(T)\,\mathrm{d}T\right]\right\|\right\}.$$

Bearing in mind that the restriction of φ to the sphere $S(r_0)$ is simply the associated harmonic function H and that the maximum principle holds, using M.Marden [14] to bound the error

$$|\Delta_n(x,\eta;C_n)| \le \frac{\int\limits_{T^*} |K(r,\zeta)||f(\sigma)-l_n(\sigma,S)||\nu(S)||M(T^*)||\mathrm{d}(T^*)|}{2\pi\left|\int\limits_{T}\nu(S)M(T)\,\mathrm{d}T\right|}.$$

Since

$$\Xi(C_n) = \frac{\int\limits_{T^*}\nu(S)|M(T^*)\,\mathrm{d}T^*|}{\left|2\pi\int\limits_{T}\nu(S)M(T)\,\mathrm{d}T\right|}$$

is bounded i.e., $\le M^*$(some constant), it follows that

$$|\Delta_n(x,\eta;C_n)| \le \inf\{\|f-l_n(\sigma,S)\|\} \qquad \text{for} \quad n\ge N_0 \ \text{ and } \ \sigma\in\overline{D}(r_0)\times T^*. \tag{4.1}$$

In view of a result of Winiarski [23: Lemma 3.3] that: "Let $E\subset\mathbb{C}^N$ be a compact set such that $V(\eta^{(n)})\ne 0$ for $n=1,2,\dots$, If $f\colon E\to B$ is defined and bounded on E, then

$$e_n(f,E) \le \|f-L_n\|_E \le (1+\nu_*)e_n(f,E) \qquad \text{for} \quad n=1,2,\dots,$$

where L_n is the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of E", and (4.1)

We can characterize the Theorems 3.1, 3.2 and 3.3 in terms of Lagrange interpolation and approximation errors $\Delta_n(x,\eta;C_n)$ in place of $\Delta_n^*(\varphi)$.

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