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EXISTENCE AND STABILITY OF ANTI-PERIODIC SOLUTIONS FOR IMPULSIVE FUZZY COHEN-GROSSBERGE NEURAL NETWORKS ON TIME SCALES

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ABSTRACT. By applying the method of coincidence degree and constructing suitable Lyapunov functional, some sufficient conditions are established for the existence and global exponential stability of anti-periodic solutions for a kind of impulsive fuzzy Cohen-Grossberg neural networks on time scales. Moreover an example is given to illustrate our results.

1. Introduction

In recent years, Cohen and Grossberg neural networks [1] have been extensively studied and applied in many different fields such as associative memory, signal processing and some optimization problems. In such applications, it is of prime importance to ensure that the designed neural networks are stable [2]. In practice, due to the finite speeds of the switching and transmission of signals, time delays do exist in a working network and thus should be incorporated into the model equation. In recent years, the dynamical behaviors on the existence and global stability of equilibrium, periodic solutions of Cohen-Grossberg neural

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networks with constant delays or time-varying delays or distributed delays have been studied (see for example Refs. [3–15] and the references therein).

In this paper, we would like to integrate fuzzy operations into Cohen-Grossberg neural networks. Speaking of fuzzy operations, Yang and Yang [18] first introduced fuzzy cellular neural networks (FCNNs) combining those operations with cellular neural networks. So far researchers have found that FCNNs are useful in image processing, and some results have been reported on stability and periodicity of FCNNs [18–24].

Arising from problems in applied sciences, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [25–29]). It is worth continuing the investigation of the existence and stability of anti-periodic solutions of impulsive fuzzy Cohen-Grossberg neural networks. To the best of our knowledge, there are few published papers considering the anti-periodic solutions of impulsive fuzzy Cohen-Grossberg neural networks.

Motivated by the above discussions, we consider the following impulsive fuzzy Cohen-Grossberg neural networks on time scales.

$$\begin{cases} x_i^{\Delta}(t) = -a_i(t,x_i(t))\Big[b_i(t,x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\ \qquad - \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\ \qquad - \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) - I_i(t)\Big], \qquad t \neq t_k \\ \Delta_i(x_i(t_k)) = J_{ik}(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-), \qquad k \in \mathbb{N}, \quad i = 1,2,\ldots,n. \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is an $\frac{\omega}{2}$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. n corresponds to the number of units in the neural networks. For $i = 1,2,\ldots,n$, $x_i(t)$ corresponds to the state of the ith neuron. $f_j(\cdot)$, $g_j(\cdot)$ are signal transmission functions. $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the jth unit to the ith unit and satisfies $0 \leq \tau_{ij}(t) \leq \tau$ (τ is a constant). $a_i(x_i(t)) > 0$ represents an amplification function at time t. $b_i(x_i(t))$ is an appropriately behaved function at time t; $c_{ij}(t)$ represents the elements of the feedback template. $I_i(t)$ is external input to the ith unit. $\alpha_{ij}(t), \beta_{ij}(t)$ are elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively; $x_i(t_k^+), x_i(t_k^-)$ represent the right and left limit of $x_i(t_k)$ in the sense of time scales, t_k is a sequence of real numbers such that $0 < t_1 < t_2 < \ldots$, $\lim_{k\to+\infty} t_k = +\infty$. There exists a positive integer p such that $t_{k+p} = t_k + \frac{\omega}{2}$, $J_{i(k+p)}(x_i(t_{k+p}) = -J_{ik}(-(x_i(t_k))), k \in \mathbb{N}$. Without loss of generality, we also assume that $\left[0, \frac{\omega}{2}\right)_{\mathrm{T}} \cap \{t_k : \ k \in \mathbb{N}\} = \{t_1, t_2, \dots, t_q\}$. Let $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{T}^+ = \mathbb{R}^+ \cap \mathbb{T}$.

Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in C(\mathbb{T}, \mathbb{R}^n)$, $\|x\| = \sum\limits_{i=1}^{n} \max\limits_{t \in [0,\omega]_{\mathbb{T}}} |x_i(t)|$. The initial conditions associated with system (1.1) are of the form

$$x_i(t) = \varphi_i(t), \qquad t \in [-\tau, 0]_{\mathbb{T}}, \quad \tau = \max_{1 \le i,j \le n} \sup_{t \in \mathrm{T}} \{\tau_{ij}(t)\} \tag{1.2}$$

where $\varphi_i(t)$, $i = 1, 2, \dots, n$, are continuous functions on $[-\tau, 0]_{\mathrm{T}}$.

For the sake of convenience, we introduce some notations

$$\overline{I}_i = \max_{t \in [0,\omega]_{\mathrm{T}}} |I_i(t)|, \qquad \overline{c}_{ij} = \max_{t \in [0,\omega]_{\mathrm{T}}} |c_{ij}(t)|, \qquad \overline{\alpha}_{ij} = \max_{t \in [0,\omega]_{\mathrm{T}}} |\alpha_{ij}(t)|.$$

$$\overline{\beta}_{ij} = \max_{t \in [0,\omega]_{\mathrm{T}}} |\beta_{ij}(t)|, \quad \|f\|_2 = \left(\int_0^{\omega} |f(t)|^2 \Delta t \right)^{1/2}, \quad \|g\|_2 = \left(\int_0^{\omega} |g(t)|^2 \Delta t \right)^{1/2}.$$

where f is an ω-periodic function.

Throughout this paper, we make the following assumptions:

(A1) $c_{ij}(t) \ge 0$, $\alpha_{ij}(t) \ge 0$, $\beta_{ij}(t) \ge 0$, $c_{ij}(t + \frac{\omega}{2}) = c_{ij}(t)$, $\alpha_{ij}(t + \frac{\omega}{2}) = \alpha_{ij}(t)$, $\beta_{ij}(t + \frac{\omega}{2}) = \beta_{ij}(t)$, $\tau_{ij}(t + \frac{\omega}{2}) = \tau_{ij}(t)$, $I_i(t + \frac{\omega}{2}) = -I_i(t)$, $i, j = 1, 2, \dots, n$.

(A2) $a_i \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R}^+)$, $a_i(t + \frac{\omega}{2}, -u) = a_i(t, u)$, and there exist positive constants $\underline{a}_i, \overline{a}_i$ such that $0 < \underline{a}_i \le a_i(\cdot) \le \overline{a}_i$, $i = 1, 2, \dots, n$.

(A3) $b_i \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R}^+)$ are delta differentiable, $b_i(t, 0) = 0$, $b_i(t + \frac{\omega}{2}, -u) = -b_i(t, u)$ and there exist positive constants ϱ_i, δ_i such that $0 < \varrho_i \le \frac{\partial b_i(t,u)}{\partial u} \le \delta_i$, for all $u \in \mathbb{R}$, $i = 1, 2, \dots, n$.

(A4) $f_j, g_j \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $f_j(t + \frac{\omega}{2}, -u) = f_j(t, u)$, $g_j(t + \frac{\omega}{2}, -u) = g_j(t, u)$, and there exist M_f, M_g, κ_j, ν_j $(j = 1, 2, \dots, n)$ such that

$$\begin{aligned} |f_j(t, u)| &\le M_f, \qquad |f_j(t, u) - f_j(t, v)| \le \kappa_j |u - v|, \\ |g_j(t, u)| &\le M_g, \qquad |g_j(t, u) - g_j(t, v)| \le \nu_j |u - v|. \end{aligned}$$

(A5) $J_{ik} \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants ρ_{ik} such that

$$|J_{ik}(u) - J_{ik}(v)| \le \rho_{ik} |u - v|, \qquad \text{for all} \quad u, v \in \mathbb{R}, \quad k \in \mathbb{N}, \quad i = 1, 2, \dots, n.$$

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, by using coincidence degree, we establish sufficient conditions for the existence of the anti-periodic solutions of system (1.1). In Section 4, by constructing Lyapunov functional, we shall derive sufficient conditions for the global exponential stability of anti-periodic solutions of system (1.1). An example is given to demonstrate the effectiveness of our results in Section 5. Conclusions are drawn in Section 6.

2. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho\colon \mathbb{T} \to \mathbb{T}$ and the graininess $\mu\colon \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} :\ s < t\}, \qquad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f\colon \mathbb{T} \to R$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for given $\varepsilon > 0$, there exists a neighborhood U of t such that $|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - y(s)]| < \varepsilon|\sigma(t) - s|$ for all $s \in U$. If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t. Let y be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s = Y(t) - Y(a)$.

Definition 2.1. ([30]) If $a \in \mathbb{T}$, $\sup \mathbb{T} = \mathbb{R}$, and f is rd-continuous on $[0, \infty)$, then we define the improper integral by

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Definition 2.2. ([32]) For each $t \in \mathbb{T}$, let N be a neighborhood of t, then, for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$. Define $D^+V^\Delta(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of t such that

$$\frac{V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))}{\mu(t,s)} < D^+V^\Delta(t, x(t)) + \varepsilon.$$

for each $s \in N_\varepsilon$, $s > t$, where $\mu(t,s) = \sigma(t) - s$. If t is rd and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}.$$

Definition 2.3. ([15]) We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the least positive p is called the period of the time scale. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with periodic p. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is $\frac{\omega}{2}$ anti-periodic if there exists a natural number n such that $\frac{\omega}{2} = np$, $f(t + \frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$ and $\frac{\omega}{2}$ is the least number such that $f(t + \frac{\omega}{2}) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is $\frac{\omega}{2}$ anti-periodic if $\frac{\omega}{2}$ is is the least positive number such that $f(t + \frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$.

A function $r\colon \mathbb{T} \to R$ is called regressive if $1 + \mu(t)r(t) \neq 0$, for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}, \qquad s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_{h(z)} = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0 \\ z, & h = 0 \end{cases}$$

Let $p, q\colon \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq; \qquad p \ominus q := p \oplus (\ominus q); \qquad \ominus p := \frac{p}{1 + \mu p}.$$

Lemma 2.1. ([17]) *Let p, q be regressive functions on $\mathbb{T}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(iv) $e_p^\Delta(\cdot, s) = pe_p(\cdot, s)$.

Lemma 2.2. ([16]) *Assume that $f, g\colon \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

Lemma 2.3. ([17]) *Let $t_1, t_2 \in [0, \omega]_\mathbb{T}$, if $x\colon \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$x(t) \le x(t_1) + \int_0^\omega |x^\Delta(s)|\Delta s, \qquad x(t) \ge x(t_2) - \int_0^\omega |x^\Delta(s)|\Delta s.$$

Lemma 2.4. ([14]) *Let $a, b \in \mathbb{T}$, for rd-continuous functions $f, g\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$, we have*

$$\int_a^b |f(t)||g(t)|\Delta t \leq \bigg(\int_a^b |f(t)|^2\Delta t\bigg)^{1/2}\bigg(\int_a^b |g(t)|^2\Delta t\bigg)^{1/2}.$$

Lemma 2.5. ([31]) *Let f be continuous on $[a.b]_{\mathbb{T}}$ and delta differentiable on $[a,b)_{\mathbb{T}}$, then there exist $\xi, \varsigma \in [a,b)_{\mathbb{T}}$ such that $f^{\Delta}(\xi)(b-a) \leq f(b) - f(a) \leq f^{\Delta}(\varsigma)(b-a)$.*

Lemma 2.6. ([18]) *Suppose x and y are two states of system* (1.1)*, then we have*

$$\bigg|\bigwedge_{j=1}^{n} \alpha_{ij}(t)g_j(x) - \bigwedge_{j=1}^{n} \alpha_{ij}(t)g_j(y)\bigg| \leq \sum_{j=1}^{n} |\alpha_{ij}(t)||g_j(x) - g_j(y)|,$$

and

$$\bigg|\bigvee_{j=1}^{n} \beta_{ij}(t)g_j(x) - \bigvee_{j=1}^{n} \beta_{ij}(t)g_j(y)\bigg| \leq \sum_{j=1}^{n} |\beta_{ij}(t)||g_j(x) - g_j(y)|.$$

Definition 2.4. The anti-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ of system (1.1) with initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$ is said to be globally exponentially stable if there exists a positive constant $M = M(\eta) \geq 1$ and $\varepsilon > 0$ such that, for every $\eta \in \mathbb{T}$,

$$\|x_i(t) - x_i^*\| \leq Me_{\ominus\varepsilon}(t,\eta)\|\varphi - x^*\|.$$

where $\|\varphi - x^*\| = \sup\limits_{\eta\in(-\tau,0]_{\mathbb{T}}} \max\limits_{1\leq i\leq n} |\varphi_i(\eta) - x_i^*(\eta)|$.

3. Existence of anti-periodic solution

In this section, based on Mawhin's continuation theorem, we shall study the existence of at least one anti-periodic solution of (1.1). To do so, the following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 3.1. ([33]) *Let $\mathbb{X}, \mathbb{Y}$ be two Banach space, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L\colon \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $\operatorname{Dom} L \cap \overline{\Omega} \neq \phi$ and $N\colon \overline{\Omega} \to \mathbb{Y}$ is L-compact. Furthermore suppose that*

$$Lx - Nx \neq \lambda(-Lx - N(-x)), \qquad \text{for each} \quad \lambda \in (0,1], \quad x \in \partial\Omega \cap \operatorname{Dom} L.$$

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$, where $\overline{\Omega}$ is the closure to Ω, $\partial\Omega$ is the boundary of Ω.

Theorem 3.1. *Under condition* (A1)–(A5)*, and suppose further* $E=(e_{ij})_{n\times n} > 0$*, where*

$$e_{ij} = \underline{a}_i \varrho_i \omega (1-\overline{a}_i \delta_i \omega) - (1+\underline{a}_i \varrho_i \omega)\sum_{k=1}^{2q} \rho_{ik}$$
$$- (1+\underline{a}_i \varrho_i \omega)\sum_{j=1}^{n} (\overline{c}_{ij} M_f + \overline{\alpha}_{ij} M_g + \overline{\beta}_{ij} M_g)\overline{a}_i \omega.$$

Then system (1.1) *has at least one* $\frac{\omega}{2}$ *anti-periodic solution* $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$.

P r o o f. Let

$$C^m\big([0,\omega; t_1,\dots,t_q,t_{q+1},\dots,t_{2q}]_{\mathbb{T}}, R^n\big)$$
$$= \big\{x\colon [0,\omega]_{\mathbb{T}} \to \mathbb{R}^n \mid x^{(m)}(t) \text{ is piecewise continuous map with first-class discontinuous points in } [0,\omega]_{\mathbb{T}} \cap \{t_k:\ k\in\mathbb{N}\} \text{ and at each discontinuous point it is continuous on the left}\big\},$$
$$m = 0,1.$$

Take

$$\mathbb{X} = \Big\{x \in C^m\big([0,\omega;t_1,\dots,t_q,t_{q+1},\dots,t_{2q}]_{\mathbb{T}}, R^n\big):\ x\big(t+\tfrac{\omega}{2}\big) = -x(t),\ t\in[0,\tfrac{\omega}{2}]_{\mathbb{T}}\Big\},$$

and $\mathbb{Y} = \mathbb{X}\times\mathbb{R}^{n\times q}$. with the norm defined by $\|x\|_{\mathbb{X}} = \sum\limits_{i=1}^{n}|x_i|$, where $|x_i| = \max\limits_{t\in[0,\omega]_T}|x_i(t)|$, $\|y\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} + \|z\|$, where $x\in\mathbb{X}$, $z\in R^{n\times q}$, $\|\cdot\|$ is any norm of $R^{n\times q}$, then $\mathbb{X}$, $\mathbb{Y}$ are Banach space.

Set

$$L\colon \operatorname{Dom} L\cap\mathbb{X}\to\mathbb{Y},\quad x\mapsto (x^{\Delta}, \Delta x(t_1),\dots,\Delta x(t_q)),\qquad \text{and}\qquad N\colon\mathbb{X}\to\mathbb{Y}.$$

where

$$\operatorname{Dom} L = \Big\{x\in C^1[0,\omega;t_1,\dots,t_{2q}]_{\mathbb{T}}:\ x\big(t+\tfrac{\omega}{2}\big) = -x(t),\ t\in\big[0,\tfrac{\omega}{2}\big]_{\mathbb{T}}\Big\}$$

$$Nx = \left(\begin{pmatrix} A_1(x)\\ \vdots\\ A_n(x)\end{pmatrix}, \begin{pmatrix}\Delta x_1(t_1)\\ \vdots\\ \Delta x_n(t_1)\end{pmatrix}, \begin{pmatrix}\Delta x_1(t_2)\\ \vdots\\ \Delta x_n(t_2)\end{pmatrix}, \dots, \begin{pmatrix}\Delta x_1(t_q)\\ \vdots\\ \Delta x_n(t_q)\end{pmatrix}\right)$$

$$
\begin{aligned}
A_i(t) = -a_i(t, x_i(t))\Big[b_i(t, x_i(t)) &- \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\
&- \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\
&- \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) - I_i(t)\Big],
\end{aligned}
$$

for $i = 1, 2, \ldots, n$. It is easy to see that $\operatorname{Ker} L = \{0\}$,

$$
\operatorname{Im} L = \Big\{ z = (f, c_1, \ldots, c_q) \in \mathbb{Y} : \int_0^{\omega} f(s)\Delta s = 0 \Big\} \equiv \mathbb{Y}.
$$

Thus $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 0$. So, $\operatorname{Im} L$ is closed in $\mathbb{Y}$, L is a Fredholm mapping of index zero. Define the project operators P and Q as

$$
Px = \int_0^{\omega} x(t)\Delta t = 0, \qquad x \in \mathbb{X},
$$

$$
Qz = Q(f, C_1, \ldots, C_q) = \Big(\frac{1}{\omega} \int_0^{\omega} f(s)\Delta s, 0, \ldots, 0, 0 \Big), \qquad z \in \mathbb{Y}
$$

Obviously, P and Q are continuous projectors and satisfy

$$
\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).
$$

Denoting by $L_p^{-1} = L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}$ and generalized inverse $K_p = L_P^{-1}$

$$
(K_p z)(t) = \int_0^{t} f(s)\Delta s + \sum_{t > t_k} C_k - \frac{1}{2} \int_0^{\frac{\omega}{2}} f(s)\Delta s - \frac{1}{2} \sum_{k=1}^{q} C_k.
$$

in which $C_{q+i} = -C_i$, $1 \leq i \leq q$. Similar to [19], it is not difficult to show that $QN(\overline{\Omega})$, $K_P(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

Now, we only need to search for an appropriate open bounded subset Ω for the application of Lemma 3.1. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$
\begin{cases}
x_i^{\Delta}(t) = \dfrac{1}{1+\lambda} G_i(t, x) - \dfrac{\lambda}{1+\lambda} G_i(t, -x), & t \in \mathbb{T}^+, \ t \neq t_k, \ k \in \mathbb{N} \\
\Delta_i(x_i(t_k)) = \dfrac{1}{1+\lambda} J_{ik}(x_i(t_k)) - \dfrac{\lambda}{1+\lambda} J_{ik}(-x_i(t_k)), & i = 1, 2, \ldots, n.
\end{cases}
\tag{3.1}
$$

where

$$
\begin{aligned}
G_i(t,x) = -a_i(t,x_i(t))\Big[b_i(t,x_i(t)) &- \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\
&- \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) \\
&- \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t)))x_j(t) - I_i(t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
G_i(t,-x) = -a_i(t,-x_i(t))\Big[b_i(t,-x_i(t)) &- \sum_{j=1}^{n} c_{ij}(t) f_j(t, -x_j(t-\tau_{ij}(t)))(-x_j(t)) \\
&- \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, -x_j(t-\tau_{ij}(t)))(-x_j(t)) \\
&- \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, -x_j(t-\tau_{ij}(t)))(-x_j(t)) - I_i(t)\Big]
\end{aligned}
$$

Suppose that $x = (x_1, x_2, \ldots, x_n)^T$ is a solution of system (3.1) for a certain $\lambda \in (0,1)$, set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$, together with (A2)–(A4), Lemma 2.4, we get

$$
\begin{aligned}
\int_0^{\omega} |x_i^{\Delta}(t)|\Delta t = &\sum_{k=1}^{2q+1} \int_{t_{k-1}^+}^{t_k} |x_i^{\Delta}(t)|\Delta t + \sum_{k=1}^{2q} |J_{ik}(x_i(t_k))| \\
\le &\int_0^{\omega} \left| \frac{1}{1+\lambda} G_i(t,x) - \frac{\lambda}{1+\lambda} G_i(t,-x) \right| \Delta t \\
&+ \sum_{k=1}^{2q} \left| \frac{1}{1+\lambda} J_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} J_{ik}(-x_i(t_k)) \right| \\
\le &\,\overline{a}_i \Big[\delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{j=1}^{n} \overline{c}_{ij} M_f \sqrt{\omega} \|x_j\|_2 + \bigwedge_{j=1}^{n} \overline{\alpha}_{ij} M_g \sqrt{\omega} \|x_j\|_2 \\
&+ \bigvee_{j=1}^{n} \overline{\beta}_{ij} M_g \sqrt{\omega} \|x_j\|_2 + \omega \overline{I}_i \Big] + \sum_{k=1}^{2q} \rho_{ik} |x_i|_0 + \sum_{k=1}^{2q} |I_{ik}(0)| \quad (3.2)
\end{aligned}
$$

Integrating (3.1) from 0 to ω, and applying (A2)–(A4), we obtain

$$
\begin{aligned}
&\bigg|\int_0^\omega \bigg[\frac{1}{1+\lambda}a_i(t,x_i(t))b_i(t,x_i(t)) - \frac{\lambda}{1+\lambda}a_i(t,-x_i(t))b_i(t,-x_i(t))\bigg]\Delta t\bigg| \\
&= \bigg|\int_0^\omega a_i(t,x_i(t))b_i(t,x_i(t))\Delta t\bigg| \\
&= \bigg|\frac{1}{1+\lambda}\int_0^\omega a_i(t,x_i(t))\sum_{j=1}^n c_{ij}(t)f_j(t,x_j(t-\tau_{ij}(t)))x_j(t)\Delta t \\
&\quad - \frac{\lambda}{1+\lambda}\int_0^\omega a_i(t,-x_i(t))\sum_{j=1}^n c_{ij}(t)f_j(t,-x_j(t-\tau_{ij}(t)))(-x_j(t))\Delta t \\
&\quad + \frac{1}{1+\lambda}\int_0^\omega a_i(t,x_i(t))\bigwedge_{j=1}^n \alpha_{ij}(t)g_j(t,x_j(t-\tau_{ij}(t)))x_j(t)\Delta t \\
&\quad - \frac{\lambda}{1+\lambda}\int_0^\omega a_i(t,-x_i(t))\bigwedge_{j=1}^n \alpha_{ij}(t)g_j(t,-x_j(t-\tau_{ij}(t)))(-x_j(t))\Delta t \\
&\quad + \frac{1}{1+\lambda}\int_0^\omega a_i(t,x_i(t))\bigvee_{j=1}^n \beta_{ij}(t)g_j(t,x_j(t-\tau_{ij}(t)))x_j(t)\Delta t \\
&\quad - \frac{\lambda}{1+\lambda}\int_0^\omega a_i(t,-x_i(t))\bigvee_{j=1}^n \beta_{ij}(t)g_j(t,-x_j(t-\tau_{ij}(t)))(-x_j(t))\Delta t \\
&\quad + \frac{1}{1+\lambda}\int_0^\omega a_i(t,x_i(t))I_i(t)\Delta t - \frac{\lambda}{1+\lambda}\int_0^\omega a_i(t,-x_i(t))I_i(t)\Delta t \\
&\quad + \frac{1}{1+\lambda}\sum_{k=1}^{2q} J_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda}\sum_{k=1}^{2q} J_{ik}(-x_i(t_k))\bigg| \\
&\le \overline{a}_i\bigg[\sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \omega\overline{I}_i\bigg] \\
&\quad + \sum_{k=1}^{2q}\rho_{ik}|x_i| + \sum_{k=1}^{2q}|J_{ik}(0)|
\end{aligned}
$$

Applying Lemma 2.4 and (A3), we have

$$\begin{aligned}
&\Big|\int_0^\omega a_i(t,x_i(t))x_i(t)\Delta t\Big| \\
&\leq \frac{\overline{a}_i}{\varrho_i}\Big[\sum_{j=1}^{n}\overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2+\bigwedge_{j=1}^{n}\overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2+\omega\overline{I}_i+\bigvee_{j=1}^{n}\overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2\Big] \\
&\quad+\frac{1}{\varrho_i}\sum_{k=1}^{2q}\rho_{ik}|x_i|+\frac{1}{\varrho_i}\sum_{k=1}^{2q}|J_{ik}(0)| \qquad (3.3)
\end{aligned}$$

For any $t_1^i, t_2^i \in [0,\omega]_{\mathbb{T}}$, $i=1,2,\ldots,n$, applying Lemma 2.3, we have

$$\begin{aligned}
&\int_0^\omega a_i(t,x_i(t))x_i(t)\Delta t \\
&\leq \int_0^\omega a_i(t,x_i(t))x_i(t_1^i)\Delta t+\int_0^\omega a_i(t,x_i(t))\bigg(\int_0^\omega |x_i^{\Delta}(s)|\Delta s\bigg)\Delta t \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\omega a_i(t,x_i(t))x_i(t)\Delta t \\
&\geq \int_0^\omega a_i(t,x_i(t))x_i(t_2^i)\Delta t-\int_0^\omega a_i(t,x_i(t))\bigg(\int_0^\omega |x_i^{\Delta}(s)|\Delta s\bigg)\Delta t \qquad (3.5)
\end{aligned}$$

Dividing by $\int_0^\omega a_i(x_i(t))\Delta t$ on both sides of (3.4) and (3.5), respectively, we obtain

$$x_i(t_1^i) \geq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t}\int_0^\omega a_i(x_i(t))x_i(t)\Delta t-\int_0^\omega |x_i^{\Delta}(s)|\Delta s, \quad \text{for} \quad i=1,2,\ldots,n. \qquad (3.6)$$

and

$$x_i(t_2^i) \leq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t}\int_0^\omega a_i(x_i(t))x_i(t)\Delta t+\int_0^\omega |x_i^{\Delta}(s)|\Delta s, \quad \text{for} \quad i=1,2,\ldots,n. \qquad (3.7)$$

Let $\overline{t}_i, \underline{t}_i \in [0,\omega]_{\mathbb{T}}$ such that $x_i(\overline{t}_i) = \max_{t\in[0,\omega]_{\mathbb{T}}} x_i(t)$, $x_i(\underline{t}_i) = \min_{t\in[0,\omega]_{\mathbb{T}}} x_i(t)$, from (3.6) and (3.7), we have

$$\begin{aligned}
x_i(\underline{t_i}) \geq & -\frac{1}{\underline{a_i}\omega}\bigg|\int_0^\omega a_i(x_i(t))x_i(t)\Delta t\bigg| - \int_0^\omega |x_i^\Delta(s)|\Delta s \\
\geq & -\frac{\overline{a}_i}{\underline{a}_i\varrho_i\omega}\bigg[\sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 \\
& + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \overline{I}_i\omega\bigg] + \frac{1}{\underline{a}_i\varrho_i\omega}\sum_{k=1}^{2q}\rho_{ik}|x_i| + \frac{1}{\underline{a}_i\varrho_i\omega}\sum_{k=1}^{2q}|J_{ik}(0)| \\
& - \overline{a}_i\bigg[\delta_i\sqrt{\omega}\|x_i\|_2 + \sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 \\
& + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \overline{I}_i\omega\bigg] + \sum_{k=1}^{2q}\rho_{ik}|x_i| + \sum_{k=1}^{2q}|J_{ik}(0)|
\end{aligned}$$

and

$$\begin{aligned}
x_i(\overline{t_i}) \leq & \frac{1}{\underline{a_i}\omega}\bigg|\int_0^\omega a_i(x_i(t))x_i(t)\Delta t\bigg| + \int_0^\omega |x_i^\Delta(s)|\Delta s \\
\leq & \frac{\overline{a}_i}{\underline{a}_i\varrho_i\omega}\bigg[\sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 \\
& + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \overline{I}_i\omega\bigg] + \frac{1}{\underline{a}_i\varrho_i\omega}\sum_{k=1}^{2q}\rho_{ik}|x_i| + \frac{1}{\underline{a}_i\varrho_i\omega}\sum_{k=1}^{2q}|J_{ik}(0)| \\
& + \overline{a}_i\bigg[\delta_i\sqrt{\omega}\|x_i\|_2 + \sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 \\
& + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \overline{I}_i\omega\bigg] + \sum_{k=1}^{2q}\rho_{ik}|x_i| + \sum_{k=1}^{2q}|J_{ik}(0)|
\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}
|x_i| \leq & \frac{1}{\underline{a_i}\omega}\bigg|\int_0^\omega a_i(x_i(t))x_i(t)\Delta t\bigg| + \int_0^\omega |x_i^\Delta(s)|\Delta s \\
\leq & \frac{\overline{a}_i}{\underline{a}_i\varrho_i\omega}\bigg[\sum_{j=1}^n \overline{c}_{ij}M_f\sqrt{\omega}\|x_j\|_2 + \bigwedge_{j=1}^n \overline{\alpha}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \bigvee_{j=1}^n \overline{\beta}_{ij}M_g\sqrt{\omega}\|x_j\|_2 + \overline{I}_i\omega\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\underline{a}_i \varrho_i \omega} \sum_{k=1}^{2q} \rho_{ik} |x_i| + \frac{1}{\underline{a}_i \varrho_i \omega} \sum_{k=1}^{2q} |J_{ik}(0)| \\
&+ \overline{a}_i \Big[\delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{j=1}^{n} \overline{c}_{ij} M_f \sqrt{\omega} \|x_j\|_2 + \bigwedge_{j=1}^{n} \overline{\alpha}_{ij} M_g \sqrt{\omega} \|x_j\|_2 \\
&+ \bigvee_{j=1}^{n} \overline{\beta}_{ij} M_g \sqrt{\omega} \|x_j\|_2 + \overline{I}_i \omega \Big] + \sum_{k=1}^{2q} \rho_{ik} |x_i| + \sum_{k=1}^{2q} |J_{ik}(0)|
\end{aligned}
\tag{3.8}
$$

In addition, we have

$$
\|x_i\|_2 = \left(\int_0^{\omega} |x_i(s)|^2 \Delta s \right)^{1/2} \leq \sqrt{\omega} \max_{t \in [0,\omega]_{\mathbb{T}}} |x_i(t)| = \sqrt{\omega} |x_i|_0 \tag{3.9}
$$

It follows from (3.8) and (3.9) that

$$
\begin{aligned}
\underline{a}_i \varrho_i \omega |x_i|_0 \leq{}& \overline{a}_i \Big[\sum_{j=1}^{n} \overline{c}_{ij} M_f \omega |x_j|_0 + \bigwedge_{j=1}^{n} \overline{\alpha}_{ij} M_g \omega |x_j|_0 + \bigvee_{j=1}^{n} \overline{\beta}_{ij} M_g \omega |x_j|_0 + \overline{I}_i \omega \Big] \\
&+ \sum_{k=1}^{2q} \rho_{ik} |x_i| + \sum_{k=1}^{2q} |J_{ik}(0)| \\
&+ \underline{a}_i \varrho_i \omega \Big[\overline{a}_i \delta_i \omega |x_i|_0 + \overline{a}_i \sum_{j=1}^{n} \overline{c}_{ij} M_f \omega |x_j|_0 + \bigwedge_{j=1}^{n} \overline{a}_i \overline{\alpha}_{ij} M_g \omega |x_j|_0 \\
&+ \bigvee_{j=1}^{n} \overline{a}_i \overline{\beta}_{ij} M_g \omega |x_j|_0 + \overline{a}_i \overline{I}_i \omega + \sum_{k=1}^{2q} \rho_{ik} |x_i| + \sum_{k=1}^{2q} |J_{ik}(0)| \Big] \\
\leq{}& \overline{a}_i \Big[\sum_{j=1}^{n} \overline{c}_{ij} M_f \omega |x_j|_0 + \sum_{j=1}^{n} \overline{\alpha}_{ij} M_g \omega |x_j|_0 + \sum_{j=1}^{n} \overline{\beta}_{ij} M_g \omega |x_j|_0 + \overline{I}_i \omega \Big] \\
&+ \sum_{k=1}^{2q} \rho_{ik} |x_i| + \sum_{k=1}^{2q} |J_{ik}(0)| \\
&+ \underline{a}_i \varrho_i \omega \Big[\overline{a}_i \delta_i \omega |x_i|_0 + \overline{a}_i \sum_{j=1}^{n} \overline{c}_{ij} M_f \omega |x_j|_0 + \sum_{j=1}^{n} \overline{a}_i \overline{\alpha}_{ij} M_g \omega |x_j|_0 \\
&+ \sum_{j=1}^{n} \overline{a}_i \overline{\beta}_{ij} M_g \omega |x_j|_0 + \overline{a}_i \overline{I}_i \omega + \sum_{k=1}^{2q} \rho_{ik} |x_i| + \sum_{k=1}^{2q} |J_{ik}(0)| \Big]
\end{aligned}
$$

That is

$$\left[\underline{a}_i\varrho_i\omega(1-\overline{a}_i\delta_i\omega)-(1+\underline{a}_i\varrho_i\omega)\sum_{k=1}^{2q}\rho_{ik}\right]|x_i|_0-(1+\underline{a}_i\varrho_i\omega)$$
$$\times\sum_{j=1}^{n}(\overline{c}_{ij}M_f+\overline{\alpha}_{ij}M_g+\overline{\beta}_{ij}M_g)\overline{a}_i\omega|x_j|_0\le\Upsilon_i \tag{3.10}$$

where

$$\Upsilon_i=(1+\underline{a}_i\varrho_i\omega)\bigg(\sum_{k=1}^{2q}|J_{ik}(0)|+\overline{a}_i\overline{I}_i\omega\bigg),\qquad i=1,2,\ldots,n.$$

Denote $|x|=(|x_1|,|x_2|,\ldots,|x_n|)^T$ and $\Upsilon=(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_n)^T$, then (3.10) can be written in the matrix form

$$E|x|\le\Upsilon,$$

From the conditions of Theorem 3.1, $E=(e_{ij})_{n\times n}>0$, so

$$|x|\le E^{-1}\Upsilon:=(D_1,D_2,\ldots,D_n)^T. \tag{3.11}$$

Let $D=\sum_{i=1}^{n}D_i+1$, D is independent of λ. Then take $\Omega=\{x\in\mathbb{X}:\ \|x\|_{\mathbb{X}}<D\}$. Obviously Ω satisfies all the requirement in Lemma 3.1 and the condition (a) is satisfied. Then system (1.1) has at least $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. □

4. Global exponential stability of anti-periodic solutions

Suppose that $x^*(t)=(x_1^*(t),x_2^*(t),\ldots,x_n^*(t))^T$ is an $\frac{\omega}{2}$-anti-periodic solution of system (1.1) with the initial conditions $x_i(s)=\varphi_i^*(s), s\in(-\infty,0]_{\mathbb{T}}$, $i=1,2,\ldots,n$. We will construct some suitable Lyapunov functions to prove the global exponential stability of this anti-periodic solution.

Theorem 4.1. *Assume that all conditions of* Theorem 3.1 *are satisfied, suppose further that*

(A6) *The impulsive operators $J_{ik}(x_i(t_k))=-\gamma_{ik}(x_i(t_k))$, $x_i(t_k^-)=x_i(t_k)$, $0<\gamma_{ik}<2$, $i=1,2,\ldots,n$, $k\in\mathbb{N}$.*

(A7) *There exist positive constants l_j such that*

$$|a_i(t,u)-a_i(t,v)|\le l_j|u-v|,\qquad u,v\in\mathbb{R},\quad i,j=1,2,\ldots,n.$$

(A8) *There exist* $\xi_i > 0$ $(i = 1, 2, \dots, n)$ *such that*

$$\Theta_i = (l_i \overline{I}_i - \underline{a}_i \varrho_i)\xi_i + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} M_f + \overline{\alpha}_{ij} M_g + \overline{\beta}_{ij} M_g)\xi_i + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} D_0 \kappa_j + \overline{\alpha}_{ij} D_0 \nu_j + \overline{\beta}_{ij} D_0 \nu_j)\xi_j \leq 0$$

where $D_0 = \max_{1 \leq i \leq n} D_i$.

Then the $\frac{\omega}{2}$*-anti-periodic solution of* (1.1) *is globally exponentially stable.*

P r o o f. According to Theorem 3.1, we know that system (1.1) has an $\frac{\omega}{2}$-anti-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ with the initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T$ and $|x_i|_0 \leq D_0$.

Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be an arbitrary solution of system (1.1) with initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$. Set $y(t) = x(t) - x^*(t)$, then from (1.1), we have

$$\begin{cases} y_i^{\Delta}(t) = & -\left[a_i(t, x_i(t)) b_i(t, x_i(t)) - a_i(t, x_i^*(t)) b_i(t, x_i^*(t))\right] \\ & + \Bigg[a_i(t, x_i(t)) \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j(t - \tau_{ij}(t))) \\ & - a_i(t, x_i^*(t)) \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j^*(t - \tau_{ij}(t)))\Bigg] \\ & + \Bigg[a_i(t, x_i(t)) \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j(t - \tau_{ij}(t))) \\ & - a_i(t, x_i^*(t)) \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j^*(t - \tau_{ij}(t)))\Bigg] \\ & + \Bigg[a_i(t, x_i(t)) \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t - \tau_{ij}(t))) \\ & - a_i(t, x_i^*(t)) \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j^*(t - \tau_{ij}(t)))\Bigg] \\ & + \left[a_i(t, x_i(t)) - a_i(t, x_i^*(t))\right] x_i(t), \qquad t \neq t_k \\ \Delta y_i(t_k) = & -\gamma_{ik} y_i(t_k), \qquad i = 1, 2, \dots, n \end{cases} \tag{4.1}$$

Also $|y_i(t_k + 0)| = |1 - \gamma_{ik}||y_i(t_k)| \leq |y_i(t_k)|$, $i = 1, 2, \dots, n$, $k \in \mathbb{N}$.

The initial conditions of (4.1) is $\phi_i(s) = \varphi_i(s) - x_i^*(s), s \in [-\tau, 0]_{\mathbb{T}}$, $i = 1, 2, \dots, n$.

If (A8) holds, it can always find a small enough constant $\varepsilon > 0$, satisfying for all $t \in \mathbb{T}$, $1 - \mu(t)\varepsilon > 0$, such that

$$
\begin{aligned}
&(\varepsilon + l_i \overline{I}_i - \underline{a}_i \varrho_i)\xi_i + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} M_f + \overline{\alpha}_{ij} M_g + \overline{\beta}_{ij} M_g)\xi_i \\
&\quad + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} D_0 \kappa_j + \overline{\alpha}_{ij} D_0 \nu_j + \overline{\beta}_{ij} D_0 \nu_j) e_\varepsilon(t, t - \tau_{ij}(t))\xi_j \le 0
\end{aligned}
\tag{4.2}
$$

Consider a Lyapunov function

$$
V(t) = (V_1(t), V_2(t), \ldots, V_n(t))^T
$$

where $V_i(t) = e_\varepsilon(t, \eta)|y_i(t)|$, $\eta \in [-\tau, 0]_{\mathbb{T}}$, $i = 1, 2, \ldots, n$.

For $t \in \mathbb{T}^+$, $t \neq t_k, k \in \mathbb{N}$, calculating the upper right derivative of $V_i(t)$ along the solution of system (4.1), we have

$$
\begin{aligned}
&D^+ |V_i(t)|^\Delta \\
&= \varepsilon |y_i(t)| e_\varepsilon(t, \eta) + e_\varepsilon(\sigma(t), \eta) \operatorname{sign}\Big\{ -[a_i(t, x_i(t)) b_i(t, x_i(t)) \\
&\quad - a_i(t, x_i^*(t)) b_i(t, x_i^*(t))] + \Big[a_i(t, x_i(t)) \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j(t - \tau_{ij}(t))) \\
&\quad - a_i(t, x_i^*(t)) \sum_{j=1}^{n} c_{ij}(t) f_j(t, x_j^*(t - \tau_{ij}(t))) \Big] \\
&\quad + \Big[a_i(t, x_i(t)) \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j(t - \tau_{ij}(t))) \\
&\quad - a_i(t, x_i^*(t)) \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j(t, x_j^*(t - \tau_{ij}(t))) \Big] \\
&\quad + \Big[a_i(t, x_i(t)) \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t - \tau_{ij}(t))) \\
&\quad - a_i(t, x_i^*(t)) \bigvee_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j^*(t - \tau_{ij}(t))) \Big] \\
&\quad + [a_i(t, x_i(t)) - a_i(t, x_i^*(t))]\, x_i(t) \Big\} \\
&\le (1 + \mu(t)\varepsilon) \Big\{ (\varepsilon + l_i \overline{I}_i - \underline{a}_i \varrho_i) V_i(t) + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} M_f + \overline{\alpha}_{ij} M_g + \overline{\beta}_{ij} M_g) V_i(t) \\
&\quad + \overline{a}_i \sum_{j=1}^{n} (\overline{c}_{ij} D_0 \kappa_j + \overline{\alpha}_{ij} D_0 \nu_j + \overline{\beta}_{ij} D_0 \nu_j) e_\varepsilon(t, t - \tau_{ij}(t)) V_j(t) \Big\}
\end{aligned}
\tag{4.3}
$$

Define the curve $F = \{w(h): \ w_i = \xi_i h, \ h > 0, \ i = 1, 2, \dots, n\}$ and set $\Omega(w) = \{u: \ 0 \le u \le w, \ w \in F\}$, $S_i(w) = \{u \in \Omega(w): \ u_i = w_i, \ 0 \le u \le w\}$. Clearly, if $h = \bar{h}$, then $\Omega(w(h)) \subset \Omega(w(\bar{h}))$.

We shall prove that the zero solution of (4.1) is globally exponentially stable.

Let $\overline{\xi} = \max\limits_{1 \le i \le n}\{\xi_i\}$, $\underline{\xi} = \min\limits_{1 \le i \le n}\{\xi_i\}$, $h_0 = (1-\eta)|\phi_i|_0/\underline{\xi}$, where $\eta \le 0$ is a constant, $|\phi_i|_0 = \max\limits_{\eta \in [-\tau, 0]_{\mathbb{T}}} |\phi_i(\eta)|$. Then $\{|V|: \ |V| = e_\varepsilon(t, \eta)|\phi_i(\eta)|, \ -\tau \le t \le \eta \le 0\} \subset \Omega(w(h_0))$, that is $|V_i(\eta)| = e_\varepsilon(t, \eta)|\phi_i(\eta)| < \xi_i h_0$, $-\tau \le \eta \le 0$, $i = 1, 2, \dots, n$.

We claim that $|V_i(t)| < \xi_i h_0$, for $t \in \mathbb{T}^+$, $i = 1, 2, \dots, n$. If it is not true, then there exist some $i \in \{1, 2, \dots, n\}$ and $t^* \in \mathbb{T}^+$ such that $|V_i(t^*)| = \xi_i h_0$, $D^+V_i(t^*)^\Delta \ge 0$ and $|V_i(t)| \le \xi_i h_0$ for $t \in [-\tau, t^*]_{\mathbb{T}}$, $i = 1, 2, \dots, n$. However, from (4.2) and (4.3), we have

$$
\begin{aligned}
&D^+|V_i(t^*)|^\Delta \\
&\le (1+\mu(t)\varepsilon)\Big\{(\varepsilon + l_i\overline{I}_i - \underline{a}_i\varrho_i)\xi_i + \overline{a}_i\sum_{j=1}^n(\overline{c}_{ij}M_f + \overline{\alpha}_{ij}M_g + \overline{\beta}_{ij}M_g)\xi_i \\
&\quad + \overline{a}_i\sum_{j=1}^n(\overline{c}_{ij}D_0\kappa_j + \overline{\alpha}_{ij}D_0\nu_j + \overline{\beta}_{ij}D_0\nu_j)e_\varepsilon(t, t-\tau_{ij}(t))\xi_j\Big\}h_0 < 0, \\
&\quad t \in \mathbb{T}^+, \quad t \ne t_k, \quad k \in \mathbb{N}
\end{aligned}
$$

This is a contradiction, so $|V_i(t)| < \xi_i h_0$, for $t \in \mathbb{T}^+$, $t \ne t_k$, $k \in \mathbb{N}$, $i = 1, 2, \dots, n$. Also

$$V_i(t_k+0) = e_\varepsilon(t_k+0, \eta)|y_i(t_k+0)| \le e_\varepsilon(t_k, \eta)|y_i(t_k)| = V_i(t_k), \qquad k \in \mathbb{N}.$$

Then

$$|y_i(t)| < e_{\theta\varepsilon}(t, \eta)\xi_i h_0 = e_{\theta\varepsilon}(t, \eta)\xi_i(1-\eta)|\phi_i|_0/\underline{\xi}, \qquad t \in \mathbb{T}^+, \quad i = 1, 2, \dots, n.$$

Which means that

$$\|x - x^*(t)\| = \|y\| \le Me_{\theta\varepsilon}(t, \eta)\|\varphi - x^*\|, \qquad t \in \mathbb{T}^+.$$

where $M = M(\eta) = \frac{\overline{\xi}(1-\eta)}{\underline{\xi}} > 1$. From Definition 2.4, the $\frac{\omega}{2}$-anti-periodic solution $x^*(t)$ of system (1.1) is globally exponentially stable. This completes the proof. $\square$

5. An example

Example 1. Consider the following impulsive Cohen-Grossberg fuzzy neural networks with time-varying delays

$$\begin{cases} x_i^{\Delta}(t) = & -a_i(t,x_i(t))\Big[b_i(t,x_i(t)) - \sum_{j=1}^{2} c_{ij}(t) f_j(t,x_j(t-\tau_{ij}(t)))x_j(t) \\ & + \bigwedge_{j=1}^{2} \alpha_{ij}(t) g_j(x_j(t,t-\tau_{ij}(t)))x_j(t) \\ & - \bigvee_{j=1}^{2} \beta_{ij}(t) g_j(t,x_j(t-\tau_{ij}(t)))x_j(t) - I_i(t)\Big], \qquad t \in \mathbb{T}^+ \\ \Delta x_i(t_k)) = & -0.003 x_i(t_k), \qquad t = t_k, \quad i=1,2, \quad k=1,2 \end{cases} \tag{5.1}$$

where $a_1(t,u) = 3.0 + 0.2\sin|u|$, $a_2(t,u) = 2.9 + 0.1\sin|u|$, $b_1(t,u) = b_2(t,u) = u$, $f_j(t,u) = \frac{1}{20}|\sin u|$, $g_j(t,u) = \frac{1}{20}|\cos u|$ $(j=1,2)$.

$$(c_{ij})_{2\times 2} = \begin{pmatrix} 0.07|\sin(8\pi t)| & 0.09|\cos(8\pi t)| \\ 0.08|\cos(8\pi t)| & 0.06|\sin(8\pi t)| \end{pmatrix},$$

$$(\alpha_{ij})_{2\times 2} = \begin{pmatrix} 0.12|\cos(8\pi t)| & 0.11|\sin(8\pi t)| \\ 0.09|\sin(8\pi t)| & 0.08|\cos(8\pi t)| \end{pmatrix},$$

$$(\beta_{ij})_{2\times 2} = \begin{pmatrix} 0.04|\sin(8\pi t)| & 0.05|\cos(8\pi t)| \\ 0.07|\cos(8\pi t)| & 0.06|\sin(8\pi t)| \end{pmatrix}, \qquad (I_i)_{2\times 1} = \begin{pmatrix} 0.5\sin(8\pi t) \\ 0.7\cos(8\pi t) \end{pmatrix}$$

then, we have $\overline{a}_1 = 3.2$, $\underline{a}_1 = 2.8$, $\overline{a}_2 = 3$, $\underline{a}_2 = 2.8$, $M_f = M_g = \kappa_j = \nu_j = \frac{1}{10}$ $(j=1,2)$, $\overline{c}_{11} = 0.07$, $\overline{c}_{22} = 0.06$, $\overline{c}_{12} = 0.09$, $\overline{c}_{21} = 0.08$, $\overline{\alpha}_{11} = 0.12$, $\overline{\alpha}_{22} = 0.08$, $\overline{\alpha}_{12} = 0.11$, $\overline{\alpha}_{21} = 0.09$, $\overline{\beta}_{11} = 0.04$, $\overline{\beta}_{22} = 0.06$, $\overline{\beta}_{12} = 0.05$, $\overline{\beta}_{21} = 0.07$, $\overline{I}_1 = 0.5$, $\overline{I}_2 = 0.7$, $\rho_{ik} = 0.003 (i,k=1,2)$, $\varrho_i = \delta_i = 1$ $(i=1,2)$.

Computing by MATLAB, we have

$$E = \begin{pmatrix} 0.1142 & 0.1128 \\ 0.1495 & 0.1520 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 0.6902 \\ 0.9027 \end{pmatrix}, \quad \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} 1.2365 \\ 0.1952 \end{pmatrix}$$

So $D_0 = 1.2365$, take $l_1 = 0.2$, $l_2 = 0.1$, $\xi_i = 1$ $(i=1,2)$. Then $\Theta_1 = -0.7239 < 0$, $\Theta_2 = -0.8774 < 0$.

Now, we can see that conditions (A1)–(A8) hold. By Theorem 3.1 and Theorem 4.1, system (1.1) has a $\frac{1}{8}$-anti-periodic solution which is globally exponentially stable.

6. Conclusion

In this paper, we have studied the existence, globally exponential stability of the anti-periodic solution for impulsive fuzzy Cohen-Grossberg neural networks with time delays. Some sufficient conditions set up here are easily verified and these conditions are correlated with parameters of the system (1.1). The obtained criteria can be applied to design globally exponential stable of anti-periodic fuzzy Cohen-Grossberg neural networks.

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