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OSCILLATION OF CERTAIN THIRD-ORDER QUASILINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, new oscillation criteria for the third-order quasilinear neutral differential equation

$$\big(a(t)(z''(t))^{\gamma}\big)' + q(t)x^{\gamma}(\tau(t)) = 0, \qquad t \ge t_0,$$

are established, where $z(t) = x(t) + p(t)x(\delta(t))$, and γ is a ratio of odd positive integers. Those results extend the oscillation criteria due to Sun [SUN, Y. G.: *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping*, J. Math. Anal. Appl. **291** (2004) 341–351] to the equation, and complement the existing results in literature. Two examples are provided to illustrate the relevance of our main theorems.

1. Introduction

Consider the third-order quasilinear neutral differential equation

$$(a(t)(z''(t))^{\gamma})' + q(t)x^{\gamma}(\tau(t)) = 0, \qquad t \ge t_0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\delta(t))$, γ is a ratio of odd positive integers. Throughout this paper, we assume that

(H1) $a \in C^1([t_0,\infty),(0,\infty))$ with $a'(t) \ge 0$, and $\int\limits_{t_0}^{\infty} a^{-1/\gamma}(s)\,\mathrm{d}s = \infty$;

(H2) $\delta, \tau \in C([t_0,\infty),\mathbb{R})$ with $\delta(t) \le t$, $\tau(t) \le t$, and $\lim\limits_{t\to\infty} \delta(t) = \lim\limits_{t\to\infty} \tau(t) = \infty$;

(H3) $p, q \in C([t_0,\infty),\mathbb{R})$ with $-\mu \le p(t) \le p < 1$ for $\mu \in (0,1)$, and $q(t) \ge 0$.

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We recall that a function $x \in C^1([T_x,\infty),\mathbb{R})$, $T_x \ge t_0$, is said to be a solution of Eq. (1.1) if $x(t)$ has the property $a(z'')^\gamma \in C^1([T_x,\infty),\mathbb{R})$ and satisfies Eq. (1.1) for all $t \ge T_x$. In the sequel, we consider only those solutions $x(t)$ of Eq. (1.1) which satisfy $\sup\{|x(t)| : \ t \ge T\} > 0$ for all $t \ge T_x$. We assume that Eq. (1.1) always possesses such a solution. As is customary, a solution of Eq. (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise, it is said to be nonoscillatory.

Neutral differential equations have been found numerous applications in electric networks. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [14].

The problem of the oscillation of solutions of delay differential equations has been widely studied by many authors and by many techniques. In the past 30 years, the oscillation theory for second order neutral differential equations and third-order retarded differential equations have been well developed; see, for example, the monographs [5,11] and papers [1–3,6,8–10,12,15,16,20], and the references cites therein. Compared to second order neutral differential equations, it seems that no much work has been done concerning with the oscillation and asymptotic behavior of third order neutral differential equations [4,13]. Very recently, Baculíková and Džurina [4] studied the oscillation behavior of Eq. (1.1) and extended Nehari's theorems [17] to Eq. (1.1).

Motivated by the recent works [4,19], in this paper, we establish some new Kamenev-type oscillation criteria which extend the main results [19] to Eq. (1.1), and give sufficient conditions which insure that any solution of Eq. (1.1) oscillates or converges to zero. The results obtained here complement the existing results in [4]. Two examples are also provided to show the relevance of our main results.

2. Oscillation criteria for $0 \le p(t) \le p < 1$

In this section, we will establish new oscillation criteria for Eq. (1.1) for $0 \le p(t) \le p < 1$. For this, we need the following lemmas.

Lemma 2.1. (see [4: Lemma 1]) *Let $x(t)$ be a positive solution of Eq. (1.1). Then there are only the following two cases for $z(t)$:*

(i) $z(t) > 0, \ \ z'(t) > 0, \ \ z''(t) > 0;$

(ii) $z(t) > 0, \ \ z'(t) < 0, \ \ z''(t) > 0,$

for $t \ge t_1$, where t_1 is sufficiently large.

Lemma 2.2. (see [4: Lemma 2]) *Let $x(t)$ be a positive solution of Eq.* (1.1) *and $z(t)$ satisfy* Lemma 2.1(ii). *If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty} q(s)\mathrm{d}s\right]^{1/\gamma}\mathrm{d}u\,\mathrm{d}v=\infty, \tag{2.1}$$

then $\lim\limits_{t\to\infty} x(t)=\lim\limits_{t\to\infty} z(t)=0$.

Lemma 2.3. (see [4: Lemma 3]) *Let $u\in C^2([t_0,\infty),\mathbb{R})$. Assume that $u(t)>0$, $u'(t)\ge 0$, and $u''(t)\le 0$ on $[t_0,\infty)$. Then for each $k_1\in(0,1)$ there exist a $T_1\ge t_0$ such that*

$$\frac{u(\tau(t))}{u(t)}\ge k_1\frac{\tau(t)}{t},\qquad t\ge T_1.$$

Lemma 2.4. *Let $u\in C^3([t_0,\infty),\mathbb{R})$. Assume that $u(t)>0$, $u'(t)>0$, $u''(t)>0$, $u'''(t)\le 0$ on $[t_0,\infty)$, then for each $k_2\in(0,1)$ there exists a $T_2\ge t_0$ such that*

$$\frac{u(t)}{u'(t)}\ge\frac{1}{2}k_2 t,\qquad t\ge T_2.$$

The proof of this lemma proceeds along the lines of that of [4: Lemma 4] and hence is omitted.

Following the ideas of Philos [18] and Sun [19], we now establish new Kamenev-type oscillation [19] for Eq. (1.1). Firstly, we introduce a function class $\Re$. The function $H\in C(E,\mathbb{R})$, where $E=\big\{(t,s,l):\ t_0\le l\le s\le t\le\infty\big\}$, is said to belong to the function class $\Re$, defined by $H\in\Re$, and if H satisfies the following two conditions:

(i) $H(t,t,l)=0$, $H(t,l,l)=0$, $H(t,s,l)\ne 0$ for $l<s<t$;

(ii) $H(t,s,l)$ has the partial derivative $\partial H/\partial s$ on E such that $\partial H/\partial s$ is locally integrable with respect to s in E, and, for some $h\in C(E,\mathbb{R})$,

$$\frac{\partial H(t,s,l)}{\partial s}=h(t,s,l)H(t,s,l). \tag{2.2}$$

In the sequel, we will use the following notation. For each $k_1,k_2\in(0,1)$, define, for $t\ge t_0$,

$$\widetilde{Q}_1(t)=q(t)\big[1-p(\tau(t))\big]^{\gamma}\left(\frac{\tau^2(t)}{t}\right)^{\gamma},$$

$$Q_1(t)=\left(\frac{k_1k_2}{2}\right)^{\gamma}\widetilde{Q}_1(t),\qquad Q_{11}(t)=q(t)\big[1-p(\tau(t))\big]\frac{\tau^2(t)}{t},$$

and

$$A(t)=\int_{t_0}^{t}\frac{\mathrm{d}s}{a^{1/\gamma}(s)},\qquad A_0(t)=\int_{t_0}^{t}\frac{\mathrm{d}s}{a(s)}.$$

For $\alpha,\beta>\gamma$, define H_0 by

$$H_0=\sum_{i=0}^{\gamma+1}C_{\gamma+1}^{\gamma-i+1}(-1)^{\gamma-i+1}\alpha^{\gamma-i+1}\beta^{i}\frac{\Gamma(\alpha+i-\gamma)\Gamma(\beta-i+1)}{\Gamma(\alpha+\beta-\gamma+1)},\qquad \gamma\in\mathbb{Z}^{+},$$

where

$$\Gamma(s)=\int_{0}^{+\infty}x^{s-1}\mathrm{e}^{-x}\mathrm{d}x,\quad s>0,\qquad\text{and}\qquad k_0=\frac{1}{(\gamma+1)^{(\gamma+1)}}.$$

Theorem 2.1. *Let (2.1) hold. Assume that there exist functions $H\in\Re$ and $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ with $\rho'(t)\ge 0$ such that, for any $k_1,k_2\in(0,1)$,*

$$\limsup_{t\to\infty}\int_{l}^{t}H(t,s,l)\rho(s)\Big[Q_1(s)-k_0a(s)\Big(h(t,s,l)+\frac{\rho'(s)}{\rho(s)}\Big)^{\gamma+1}\Big]\,\mathrm{d}s>0.\quad(2.3)$$

Then every solution $x(t)$ of Eq. (1.1) is oscillatory or satisfies $\lim\limits_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1) on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)>0$, $x(\delta(t))>0$, $x(\tau(t))>0$ for $t\ge t_1\ge t_2$. Clearly, $z(t)>0$, $t\ge t_1$. By Lemma 2.1, we can see that $z(t)$ satisfies Lemma 2.1(i) or (ii).

Firstly, we assume that $z(t)$ satisfies Lemma 2.1(ii). Note that (2.1) holds, it follows from Lemma 2.2 that $\lim\limits_{t\to\infty}x(t)=0$.

Secondly, we assume that $z(t)$ satisfies Lemma 2.1(i). Note that

$$x(t)=z(t)-p(t)x(\delta(t))\ge z(t)-p(t)z(\delta(t))\ge z(t)[1-p(t)],$$

which follows from Eq. (1.1) that

$$(a(t)(z''(t))^{\gamma})'=-q(t)x^{\gamma}(\tau(t))\le -q(t)z^{\gamma}(\tau(t))\big[1-p(\tau(t))\big]^{\gamma},\qquad(2.4)$$

and

$$\big(a(t)(z''(t))^{\gamma}\big)'\le 0.$$

The last inequality together with $a'(t)\ge 0$ and $z''(t)>0$ gives $z'''(t)\le 0$. So there exists a $t_2\ge t_1$ such that $z(t)$ satisfies

$$z(\tau(t))>0,\quad z'(t)>0,\quad z''(t)>0,\quad z'''(t)\le 0,\quad t\ge t_2.$$

Let

$$\omega(t)=\rho(t)a(t)\Big(\frac{z''(t)}{z'(t)}\Big)^{\gamma},\qquad t\ge t_2.$$

Then $\omega(t) > 0$. By (2.4),

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)q(t)\big[1 - p(\tau(t))\big]^{\gamma}\left(\frac{z(\tau(t))}{z'(t)}\right)^{\gamma} - \gamma\frac{\omega^{(\gamma+1)/\gamma}(t)}{\big(\rho(t)a(t)\big)^{1/\gamma}}. \quad (2.5)$$

Using Lemma 2.3 with $u(t) = z'(t)$, we get

$$\frac{1}{z'(t)} \ge k_1\frac{\tau(t)}{t}\frac{1}{z'(\tau(t))}, \qquad t \ge t_3 \ge t_2.$$

By Lemma 2.4,

$$\frac{z(\tau(t))}{z'(t)} \ge k_1\frac{\tau(t)}{t}\frac{z(\tau(t))}{z'(\tau(t))} \ge \frac{k_1k_2}{2}\frac{\tau^2(t)}{t}, \qquad t \ge t_3. \quad (2.6)$$

Combining (2.5) and (2.6) gives

$$\omega'(t) \le -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \gamma\frac{\omega^{(\gamma+1)/\gamma}(t)}{\big(\rho(t)a(t)\big)^{1/\gamma}},$$

i.e.,

$$\rho(t)Q_1(t) \le -\omega'(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \gamma\frac{\omega^{(\gamma+1)/\gamma}(t)}{\big(\rho(t)a(t)\big)^{1/\gamma}}. \quad (2.7)$$

Multiplying (2.7) by $H(t,s,l)$, integrating from l ($l \ge t_3$) to t, and by (2.2), we have, for all $t \ge l$,

$$\begin{aligned}
&\int_l^t H(t,s,l)\rho(s)Q_1(s)\mathrm{d}s \\
\le\ & -\int_l^t H(t,s,l)\omega'(s)\mathrm{d}s + \int_l^t H(t,s,l)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - \gamma\frac{\omega^{(\gamma+1)/\gamma}(s)}{\big(\rho(s)a(s)\big)^{1/\gamma}}\right]\mathrm{d}s \quad (2.8)\\
=\ & \int_l^t H(t,s,l)\left[\left(h(t,s,l) + \frac{\rho'(s)}{\rho(s)}\right)\omega(s) - \gamma\frac{\omega^{(\gamma+1)/\gamma}(s)}{\big(\rho(s)a(s)\big)^{1/\gamma}}\right]\mathrm{d}s.
\end{aligned}$$

Set

$$F(\nu) = \left(h(t,s,l) + \frac{\rho'(s)}{\rho(s)}\right)\nu - \gamma\frac{\nu^{(\gamma+1)/\gamma}}{\big(\rho(s)a(s)\big)^{1/\gamma}}.$$

A simple calculation implies when

$$\nu = k_0\rho(s)a(s)\left(h(t,s,l) + \frac{\rho'(s)}{\rho(s)}\right)^{\gamma},$$

$F(\nu)$ has the maximum

$$k_0\rho(s)a(s)\bigg(h(t,s,l)+\frac{\rho'(s)}{\rho(s)}\bigg)^{\gamma+1},$$

that is,

$$F(\nu)\le F_{\max}=k_0\rho(s)a(s)\bigg(h(t,s,l)+\frac{\rho'(s)}{\rho(s)}\bigg)^{\gamma+1}. \tag{2.9}$$

In view of (2.8) and (2.9),

$$\int_l^t H(t,s,l)\rho(s)\bigg[Q_1(s)-k_0a(s)\bigg(h(t,s,l)+\frac{\rho'(s)}{\rho(s)}\bigg)^{\gamma+1}\bigg]\,\mathrm{d}s\le 0.$$

Taking the super limit in the above inequality, we get

$$\limsup_{t\to\infty}\int_l^t H(t,s,l)\rho(s)\bigg[Q_1(s)-k_0a(s)\bigg(h(t,s,l)+\frac{\rho'(s)}{\rho(s)}\bigg)^{\gamma+1}\bigg]\,\mathrm{d}s\le 0$$

which contradicts (2.3). Hence, we complete the proof. □

Efficient oscillation theorems can be derived from Theorem 2.1 with the appropriate choice the kernel functions $H(t,s,l)$. Next, we will give two forms of H and establish the following results.

Let

$$H(t,s,l)=(t-s)^\sigma(s-l)^\nu,\qquad \sigma,\nu>\gamma,\quad (t,s,l)\in E.$$

Then

$$h(t,s,l)=\frac{\nu t-(\sigma+\nu)s+\sigma l}{(t-s)(s-l)}.$$

By Theorem 2.1, we obtain

Theorem 2.2. *Let* (2.1) *hold. Assume that, for each $l\ge t_0$, there exist $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ with $\rho'(t)\ge 0$ and two constants σ, $\nu>\gamma$ such that, for any $k_1,k_2\in(0,1)$,*

$$\begin{aligned}\limsup_{t\to\infty}\int_l^t &(t-s)^\sigma(s-l)^\nu\rho(s)\\ &\times\bigg[Q_1(s)-k_0a(s)\bigg(\frac{\rho'(s)}{\rho(s)}+\frac{\nu t-(\sigma+\nu)s+\sigma l}{(t-s)(s-l)}\bigg)^{\gamma+1}\bigg]\,\mathrm{d}s>0.\end{aligned}\tag{2.10}$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.*

Put

$$H(t,s,l) = \big[A(t) - A(s)\big]^{\alpha}\big[A(s) - A(l)\big]^{\beta}, \qquad \alpha, \beta > \gamma, \quad (t,s,l) \in E.$$

Then

$$h(t,s,l) = \frac{\beta A(t) - (\alpha+\beta)A(s) + \alpha A(l)}{a^{1/\gamma}(s)\big[A(t) - A(s)\big]\big[A(s) - A(l)\big]}.$$

By Theorem 2.1, we get:

Theorem 2.3. *Let* (2.1) *hold. Assume that for each* $l \ge t_0$*, there exist* $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ *with* $\rho'(t) \ge 0$ *and two constants* $\alpha, \beta > \gamma$ *such that, for any* $k_1, k_2 \in (0,1)$*,*

$$\limsup_{t\to\infty} \int_l^t \big[A(t) - A(s)\big]^{\alpha}\big[A(s) - A(l)\big]^{\beta}\rho(s)\Bigg[Q_1(s) - k_0 a(s) \times \left(\frac{\rho'(s)}{\rho(s)} + \frac{\beta A(t) - (\alpha+\beta)A(s) + \alpha A(l)}{a^{1/\gamma}(s)\big[A(t) - A(s)\big]\big[A(s) - A(l)\big]}\right)^{\gamma+1}\Bigg]\mathrm{d}s > 0. \tag{2.11}$$

Then every solution $x(t)$ *of Eq.* (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$*.*

Next, let γ be an odd positive integer and $\rho(t) \equiv 1$ in Theorem 2.3, we can get the following interesting results for Eq. (1.1).

Corollary 2.1. *Let* (2.1) *hold, and let* γ *be an odd positive integer. Assume that for each* $l \ge t_0$*, there exist two constants* $\alpha, \beta > \gamma$ *such that, for any* $k_1, k_2 \in (0,1)$*,*

$$\limsup_{t\to\infty} \frac{1}{\big[A(t) - A(l)\big]^{\alpha+\beta-\gamma}} \int_l^t \big[A(t) - A(s)\big]^{\alpha} \times \big[A(s) - A(l)\big]^{\beta} Q_1(s)\mathrm{d}s > k_0 H_0. \tag{2.12}$$

Then every solution $x(t)$ *of Eq.* (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$*.*

Proof. In the view of Theorem 2.3 with $\rho(t) = 1$, it is sufficient to show that (2.12) implies (2.11) holds. Recall that

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Put $y = \frac{u}{v}$, we get

$$\begin{aligned}\int_0^v (v-u)^{\alpha+i-\gamma-1} u^{\beta-i} \mathrm{d}u &= \int_0^1 v^{\alpha+\beta-\gamma} (1-y)^{\alpha+i-\gamma-1} y^{\beta-i} \mathrm{d}y \\ &= v^{\alpha+\beta-\gamma} \frac{\Gamma(\alpha+i-\gamma)\Gamma(\beta-i+1)}{\Gamma(\alpha+\beta-\gamma+1)}.\end{aligned} \tag{2.13}$$

Let $u = A(s) - A(l)$, $v = A(t) - A(l)$. Then, by (2.3),

$$\begin{aligned}&\int_l^t a(s)(A(t)-A(s))^{\alpha}(A(s)-A(l))^{\beta}\left(\frac{\beta A(t)-(\alpha+\beta)A(s)+\alpha A(l)}{a^{1/\gamma}(s)\big[A(t)-A(s)\big]\big[A(s)-A(l)\big]}\right)^{\gamma+1} \mathrm{d}s \\ &= \int_0^v (v-u)^{\alpha-\gamma-1} u^{\beta-\gamma-1} (\beta(v-u)-\alpha u)^{\gamma+1} \mathrm{d}u \\ &= \sum_{i=0}^{\gamma+1} C_{\gamma+1}^{\gamma-i+1} (-1)^{\gamma-i+1} \alpha^{\gamma-i+1} \beta^i \int_0^v u^{\beta-i}(v-u)^{\alpha+i-\gamma-1} \mathrm{d}u = \big[A(t)-A(l)\big]^{\alpha+\beta-\gamma} H_0.\end{aligned} \tag{2.14}$$

Hence, by (2.12) and (2.14), we get (2.11) holds. This completes the proof. □

Corollary 2.2. *Let* (2.1) *hold, and let* γ *be an odd positive integer. Assume that for each* $l \ge t_0$*, there exist two constants* $\alpha, \beta > \gamma$ *such that*

$$\limsup_{t\to\infty} \frac{1}{\big[A(t)-A(l)\big]^{\alpha+\beta-\gamma}} \int_l^t \big[A(t)-A(s)\big]^{\alpha} \big[A(s)-A(l)\big]^{\beta} \widetilde{Q}_1(s)\,\mathrm{d}s > 2^{\gamma} k_0 H_0. \tag{2.15}$$

Then every solution $x(t)$ *of Eq.* (1.1) *is oscillatory or satisfies* $\lim\limits_{t\to\infty} x(t) = 0$.

P r o o f. We shall show (2.15) implies (2.12). Note that

$$Q_1(t) = \left(\frac{k}{2}\right)^{\gamma} \widetilde{Q}_1(t), \tag{2.16}$$

where $k = k_1 k_2$. On the other hand, (2.15) implies, for the $k \in (0,1)$,

$$\limsup_{t\to\infty} \frac{1}{\big[A(t)-A(l)\big]^{\alpha+\beta-\gamma}} \int_l^t \big[A(t)-A(s)\big]^{\alpha} \big[A(s)-A(l)\big]^{\beta} \widetilde{Q}_1(s)\,\mathrm{d}s > \frac{1}{k^{\gamma}} 2^{\gamma} k_0 H_0. \tag{2.17}$$

Combining (2.16) and (2.17), we get that (2.12) holds. Hence, by Corollary 2.1, we complete the proof. □

Let $\gamma = 1$. Then Eq. (1.1) reduces to the following equation

$$(a(t)z''(t))' + q(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{2.18}$$

where $z(t) = x(t)+p(t)x(\delta(t))$, and functions $a(t)$, $\delta(t)$, $\tau(t)$, $p(t)$ and $q(t)$ satisfy (H1)–(H3). Then (2.1) becomes

$$\int_{t_0}^{\infty}\int_{v}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v = \infty. \tag{2.19}$$

Then, by Corollary 2.2, we have

Corollary 2.3. *Let* (2.19) *hold. Assume that there exists a constant* $\alpha > 1$, *such that for each* $l \geq t_0$. *If one of the following holds,*

1. $\limsup\limits_{t\to\infty} \frac{1}{A_0^{\alpha+1}(t)} \int_l^t \big[A_0(t) - A_0(s)\big]^2 \big[A_0(s) - A_0(l)\big]^\alpha Q_{11}(s)\,\mathrm{d}s > \frac{2\alpha}{(\alpha-1)(\alpha+1)}$;
2. $\limsup\limits_{t\to\infty} \frac{1}{A_0^{\alpha+1}(t)} \int_l^t \big[A_0(t) - A_0(s)\big]^\alpha \big[A_0(s) - A_0(l)\big]^2 Q_{11}(s)\,\mathrm{d}s > \frac{2\alpha}{(\alpha-1)(\alpha+1)}$,

then every solution $x(t)$ *of Eq.* (2.18) *is oscillatory or satisfies* $\lim\limits_{t\to\infty} x(t) = 0$.

Example 2.1. Consider the third-order differential equation

$$\left[t^{3/2}\left[\left(x(t) + \frac{t}{t+1}x\left(\frac{t}{2}\right)\right)''\right]^3\right]' + \frac{\kappa}{t^{5/2}}x^3\left(\frac{t}{2}\right) = 0, \qquad t > 1, \tag{2.20}$$

where

$$a(t) = t^{3/2}, \quad p(t) = \frac{t}{t+1}, \quad \tau(t) = \delta(t) = \frac{t}{2}, \quad q(t) = \frac{\kappa}{t^{5/2}}, \quad \gamma = 3, \quad \kappa > 0.$$

It is easy to get

$$\begin{aligned}\int_1^{\infty}\int_v^{\infty} \left[\frac{1}{a(u)}\int_u^{\infty} q(s)\,\mathrm{d}s\right]^{1/\gamma} \mathrm{d}u\,\mathrm{d}v &= \int_1^{\infty}\int_v^{\infty} \left[\frac{1}{u^{3/2}}\int_u^{\infty} \frac{\kappa}{s^{5/2}}\,\mathrm{d}s\right]^{1/3} \mathrm{d}u\,\mathrm{d}v \\ &= \left(\frac{2\kappa}{3}\right)^{1/3} \int_1^{\infty}\int_v^{\infty} \frac{1}{u}\,\mathrm{d}u\,\mathrm{d}v = \infty.\end{aligned}$$

Hence, (2.1) holds. Note that

$$\widetilde{Q}_1(t) = \frac{\kappa}{8}t^{1/2}(t+2)^3, \qquad A(t) = 2(t^{1/2} - 1).$$

Let $\alpha = 4$, $\beta = 5$. It follows that $2^\gamma k_0 H_0 = \frac{75}{64}$, and, for $t > l > 1$, the left side of (2.15) takes

$$\limsup_{t\to\infty} \frac{1}{\left[2(t^{1/2} - l^{1/2})\right]^6} \int_l^t \left[2(t^{1/2} - s^{1/2})\right]^4 \left[2(s^{1/2} - l^{1/2})\right]^5 \frac{\kappa}{8} s^{1/2}(s+2)^3 \,\mathrm{d}s = \frac{\kappa}{15}.$$

Hence, by Corollary 2.2, we know every nonoscillatory solution of Eq. (2.20) converges to zero provided that $\kappa > \frac{1125}{64}$.

3. Oscillation criteria for $-\mu \le p(t) \le 0$

In this section, we shall establish some oscillation criteria for Eq. (1.1) under the case when $-\mu \le p(t) \le 0$ for $\mu \in (0,1)$. Here, the following lemmas are need.

Lemma 3.1. (see [4: Lemma 7]) *Let $x(t)$ be a positive solution of Eq.* (1.1). *Then there are only the following four cases for $z(t)$:*

(j) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$;

(jj) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$;

(jjj) $z(t) < 0$, $z'(t) < 0$, $z''(t) > 0$;

(jv) $z(t) < 0$, $z'(t) < 0$, $z''(t) < 0$,

for $t \ge t_1$, where t_1 is sufficiently large.

Lemma 3.2. (see [4: Lemma 8]) *Let $x(t)$ be a positive solution of Eq.* (1.1) *and $z(t)$ satisfy* Lemma 3.1(jj). *If* (2.1) *holds, then* $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} z(t) = 0$.

For simplicity, for each k_1, $k_2 \in (0,1)$, define, for $t \ge t_0$,

$$\widetilde{Q}_2(t) = q(t)\left(\frac{\tau^2(t)}{t}\right)^\gamma, \qquad Q_2(t) = \left(\frac{k_1 k_2}{2}\right)^\gamma \widetilde{Q}_2(t), \qquad Q_{22}(t) = q(t)\frac{\tau^2(t)}{t}.$$

Theorem 3.1. *Assume that* (2.1) *hold. Assume that there exist function $H \in \Re$ and $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ with $\rho'(t) \ge 0$ such that, for any $k_1, k_2 \in (0,1)$,*

$$\limsup_{t\to\infty} \int_l^t H(t,s,l)\rho(s)\left[Q_2(s) - k_0 a(s)\left(h(t,s,l) + \frac{\rho'(s)}{\rho(s)}\right)^{\gamma+1}\right]\mathrm{d}s > 0. \quad (3.1)$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies* $\lim\limits_{t\to\infty} x(t) = 0$.

P r o o f. Assume that $x(t)$ is a nonoscillatory positive solution of Eq. (1.1). We claim $x(t)$ is bounded. To prove this, we may assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence t_m such that $\lim\limits_{m\to\infty} t_m = \infty$; moreover $\lim\limits_{m\to\infty} x(t_m) = \infty$, and

$$x_{t_m} = \max\bigl\{x(s):\ t_0 \le s \le t_m\bigr\}, \qquad m \ge 2.$$

Since $\delta(t) \to \infty$ as $t \to \infty$, we can choose m sufficiently large such that $\delta(m) > t_2$. As $\delta(t) \le t$, we have

$$x(\delta(t_m)) \le \max\bigl\{x(s):\ t_0 \le s \le \delta(t_m)\bigr\} \le \max\bigl\{x(s):\ t_0 \le s \le t_m\bigr\} = x(t_m).$$

Therefore, for all large m,

$$z(t_m) = x(t_m) + p(t_m)x(\delta(t_m)) \ge (1-\mu)x(t_m).$$

Thus, $z(t_m) \to \infty$ as $m \to \infty$. So $z(t)$ is positive and unbounded. It follows from Lemma 3.1 that case (j) has to hold.

Since $0 < z(t) < x(t)$, Eq. (1.1) can be written as follows:

$$(a(t)(z''(t))^{\gamma})' + q(t)z^{\gamma}(\tau(t)) < 0, \qquad \text{and} \qquad (a(t)(z''(t))^{\gamma})' < 0.$$

Note that $a'(t) \ge 0$ and $z''(t) > 0$, we have $z'''(t) \le 0$. So there exists a $t_2 \ge t_1$ such that $z(t)$ satisfies

$$z(\tau(t)) > 0, \qquad z'(t) > 0, \qquad z''(t) > 0, \qquad z'''(t) \le 0, \qquad t \ge t_2.$$

Defined $\omega(t)$ as in proof of Theorem 2.1, and similar to the proof of Theorem 2.1, we get

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)Q_2(t) - \gamma\frac{\omega^{(\gamma+1)/\gamma}(t)}{\bigl(\rho(t)a(t)\bigr)^{1/\gamma}}. \tag{3.2}$$

Multiplying (3.2) by $H(t,s,l)$, integrating from l to t, and by (2.2), we have

$$\limsup_{t\to\infty} \int_l^t H(t,s,l)\rho(s)\left[Q_2(s) - k_0 a(s)\left(h(t,s,l) + \frac{\rho'(s)}{\rho(s)}\right)^{\gamma+1}\right]\mathrm{d}s \le 0.$$

which contradicts (3.1). So we can conclude that both $x(t)$ and $z(t)$ are bounded. Lemma 3.1 now implies that for $z(t)$ either Lemma 3.1(jj) or Lemma 3.1(jjj) holds.

If Lemma 3.1(jj) holds, then Lemma 3.2 ensures that $\lim\limits_{t\to\infty} x(t) = 0$.

If Lemma 3.1(jjj) holds, then there exists a finite $\lim\limits_{t\to\infty} z(t) = -c < 0$. We know $x(t) > 0$ is bounded, so

$$\limsup_{t\to\infty} x(t) = x_0, \qquad 0 \le x_0 < \infty.$$

We claim that $x_0 = 0$. If not, then there exists a sequence t_k such that $\lim_{k\to\infty} t_k = \infty$, and $\lim_{k\to\infty} x(t_k) = x_0$. It is easy to see that for $\varepsilon = \frac{x_0(1-\mu)}{2\mu} > 0$, we have $x(\delta(t_k)) < x_0 + \varepsilon$. Moreover

$$0 > -c = \lim_{k\to\infty} z(t_k) = \lim_{k\to\infty} \big[x(t_k) - \mu(x_0 + \varepsilon)\big] = \frac{x_0}{2}(1-\mu) > 0,$$

this is a contradiction. Thus $x_0 = 0$ and $\lim_{t\to\infty} x(t) = 0$. (jjj). Hence, we complete the proof. $\square$

Choosing the function H as same as that of Theorems 2.2 and 2.3 and in the view of Theorem 3.1, we get:

Theorem 3.2. *Let* (2.1) *hold. Assume that, for each $l \geq t_0$, there exist $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ with $\rho'(t) \geq 0$ and two constants σ, $\nu > \gamma$ such that, for any $k_1, k_2 \in (0,1)$,*

$$\limsup_{t\to\infty} \int_l^t (t-s)^\sigma (s-l)^\nu \rho(s) \bigg[Q_2(s) - k_0 a(s) \times \left(\frac{\rho'(s)}{\rho(s)} + \frac{\nu t - (\sigma+\nu)s + \sigma l}{(t-s)(s-l)}\right)^{\gamma+1}\bigg] \mathrm{d}s > 0. \tag{3.3}$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Theorem 3.3. *Let* (2.1) *hold. Assume that for each $l \geq t_0$, there exist $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ with $\rho'(t) \geq 0$ and two constants $\alpha, \beta > \gamma$ such that, for any $k_1, k_2 \in (0,1)$,*

$$\limsup_{t\to\infty} \int_l^t \big[A(t) - A(s)\big]^\alpha \big[A(s) - A(l)\big]^\beta \rho(s) \bigg[Q_2(s) - k_0 a(s) \times \left(\frac{\rho'(s)}{\rho(s)} + \frac{\beta A(t) - (\alpha+\beta)A(s) + \alpha A(l)}{a^{1/\gamma}(s)\big[A(t)-A(s)\big]\big[A(s)-A(l)\big]}\right)^{\gamma+1}\bigg] \mathrm{d}s > 0. \tag{3.4}$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Corollary 3.1. *Let* (2.1) *hold, and let γ be an odd positive integer. Assume that for each $l \geq t_0$, there exist two constants $\alpha, \beta > \gamma$ such that, for any $k_1, k_2 \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{1}{\big[A(t)-A(l)\big]^{\alpha+\beta-\gamma}} \int_l^t \big[A(t)-A(s)\big]^{\alpha}\big[A(s)-A(l)\big]^{\beta} Q_2(s)\,\mathrm{d}s > k_0 H_0. \tag{3.5}$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Corollary 3.2. *Let* (2.1) *hold, and let γ be an odd positive integer. Assume that for each $l \geq t_0$, there exist two constants $\alpha, \beta > \gamma$ such that*

$$\limsup_{t\to\infty} \frac{1}{\big[A(t)-A(l)\big]^{\alpha+\beta-\gamma}} \int_l^t \big[A(t)-A(s)\big]^{\alpha}\big[A(s)-A(l)\big]^{\beta} \widetilde{Q}_2(s)\mathrm{d}s > 2^{\gamma} k_0 H_0. \tag{3.6}$$

Then every solution $x(t)$ of Eq. (1.1) *is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Let $\gamma = 1$. Similar to Corollary 2.3, we have:

Corollary 3.3. *Let* (2.19) *hold. Assume that there exists a constant $\alpha > 1$, such that for each $l \geq t_0$, if one of the following holds,*

1. $\limsup_{t\to\infty} \frac{1}{A_0^{\alpha+1}(t)} \int_l^t \big[A_0(t)-A_0(s)\big]^2 \big[A_0(s)-A_0(l)\big]^{\alpha} Q_{22}(s)\mathrm{d}s > \frac{2\alpha}{(\alpha-1)(\alpha+1)}$;
2. $\limsup_{t\to\infty} \frac{1}{A_0^{\alpha+1}(t)} \int_l^t \big[A_0(t)-A_0(s)\big]^{\alpha} \big[A_0(s)-A_0(l)\big]^{2} Q_{22}(s)\mathrm{d}s > \frac{2\alpha}{(\alpha-1)(\alpha+1)}$,

then every solution $x(t)$ of Eq. (2.18) *is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Example 3.1. Consider the third-order differential equation

$$\left[t^2\left[\left(x(t)-\frac{1}{3}x\left(\frac{t}{4}\right)\right)''\right]^3\right]' + \frac{\kappa}{t^5}x^3\left(\frac{t}{4}\right) = 0, \qquad t > 1, \tag{3.7}$$

where

$$a(t) = t^2, \quad p(t) = -\frac{1}{3}, \quad \tau(t) = \delta(t) = \frac{t}{4}, \quad q(t) = \frac{\kappa}{t^5}, \quad \gamma = 3, \quad \kappa > 0.$$

It is easy to get

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty}q(s)\,\mathrm{d}s\right]^{1/\gamma}\mathrm{d}u\mathrm{d}v=\int_{1}^{\infty}\int_{v}^{\infty}\left[\frac{1}{u^2}\int_{u}^{\infty}\frac{\kappa}{s^5}\,\mathrm{d}s\right]^{1/\gamma}\mathrm{d}u\,\mathrm{d}v$$
$$=\left(\frac{\kappa}{2}\right)^{1/3}\int_{1}^{\infty}\frac{1}{v}\,\mathrm{d}v=\infty,$$

i.e, (2.1) holds. Note that

$$\widetilde{Q}_2(t)=\frac{\kappa}{4096t^2},\qquad A(t)=3(t^{1/3}-1).$$

Let $\alpha=4$, $\beta=5$. It follows that $2^{\gamma}k_0H_0=\frac{75}{64}$, and, the left side of (3.6) takes

$$\limsup_{t\to\infty}\frac{1}{\left[3\left(t^{1/3}-l^{1/3}\right)\right]^6}\int_{l}^{t}\left[3\left(t^{1/3}-s^{1/3}\right)\right]^4\left[3\left(s^{1/3}-l^{1/3}\right)\right]^5\frac{\kappa}{4096s^2}\,\mathrm{d}s$$
$$=\frac{27\kappa}{40960}.$$

Hence, by Corollary 3.2, we know that every nonoscillatory solution of Eq. (3.7) converges to zero provided that $\kappa>\frac{16000}{9}$.

In particular, let $\kappa=\frac{11}{1024}$, one such solution is $x(t)=t^{-1/2}$.

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