

# CONVERGENCE OF SERIES IN THREE PARAMETRIC MITTAG-LEFFLER FUNCTIONS

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**ABSTRACT.** In this paper we consider a family of 3-index generalizations of the classical Mittag-Leffler functions. We study the convergence of series in such functions in the complex plane. First we find the domains of convergence of such series and then study their behaviour on the boundaries of these domains. More precisely, Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems are proved as analogues of the classical theorems for the power series.

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## 1. Introduction

The special functions, defined in the whole complex plane  $\mathbb{C}$  by the power series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (1.1)$$

with  $\alpha, \beta \in \mathbb{C}$ ,  $\operatorname{Re}(\alpha) > 0$ , are known as Mittag-Leffler (M-L) functions ([4: Section 18.1]). The first was introduced by Mittag-Leffler (1902-1905) who investigated some of its properties, while the other first appeared in a paper of Wiman (1905). The main results in the classical theory of these functions are

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presented in the handbook by Erdelyi et al. ([4: Section 18.1]), while modern results are given in the books by Dzherbashyan [2, 3]: asymptotic formulae in different parts of the complex plane, distribution of the zeros, kernel functions of inverse Borel type integral transforms, various relations and representations. Detailed accounts of the properties of these functions can be found in the contemporary monographs of Kilbas et al. [8] and Podlubny [22], see also [6, 10–12]. Recently the interest to these functions has grown up due to their applications in some evolution problems [1] and their various generalizations appearing in the solutions of fractional order differential and integral equations.

Prabhakar [23] generalizes (1.1) by introducing the function  $E_{\alpha,\beta}^\gamma$  of the form

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0, \quad (1.2)$$

where  $(\gamma)_k$  is the Pochhammer symbol ([4: Section 2.1.1])

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1).$$

For  $\gamma = 1$  this function coincides with  $E_{\alpha,\beta}$ , while for  $\gamma = \beta = 1$  with  $E_\alpha$ , i.e.:

$$E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z), \quad E_{\alpha,1}^1(z) = E_\alpha(z).$$

As a matter of fact, Prabhakar introduced this function for positive  $\gamma$ , and in this case it is an entire function of  $z$  of order  $\rho = 1/\operatorname{Re}(\alpha)$ , as mentioned in [8] and [14] and type  $\sigma = 1$ .

Prabhakar studied some properties of the generalized three parametric Mittag-Leffler function (1.2) and of an integral operator containing it as a kernel-function, and applied the obtained results to prove the existence and uniqueness of the solution of a corresponding integral equation. Further, some properties of  $E_{\alpha,\beta}^\gamma(z)$ , including classical and fractional order differentiations and integrations, are proved by Kilbas, Saigo and Saxena [7]. An integral operator with such a kernel-function is also studied in the space  $L(a, b)$ . The functions (1.2) and series in them have been used recently to express solutions of the generalized Langevin equation, by Sandev, Tomovski and Dubbeldam [28]. Another type of 3-parameter Mittag-Leffler function ( $q$ -analogue of the M-L function) is also considered, see for example in Rajkovic, Marinkovic and Stankovic [24].

In the previous papers ([19, 20]), the author considered series in systems of Mittag-Leffler functions and, resp. in [15]–[18], series in the multi-index ( $2m$ -indices) analogues (in the sense of [6, 10–12]) of the M-L functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus ([12]). Their convergence in the complex plane  $\mathbb{C}$  is studied and Cauchy-Hadamard, Abel and Tauberian type theorems are proved. Various properties of such classes of functions are studied by many authors, among them see e.g. [8, 10, 11, 22, 24], etc.

In this paper is studied the convergence of series in Prabhakar's three parametric Mittag-Leffler functions.

## 2. Auxiliary statements

Consider now the generalized Mittag-Leffler functions (1.2) for indices of the kind  $\beta = n$ ;  $n = 0, 1, 2, \dots$ , namely:

$$E_{\alpha,n}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!}, \quad \alpha, \gamma \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0, \quad n \in \mathbb{N}_0. \quad (2.1)$$

In this section we give some results related to the above functions that are obtained in [21], namely an asymptotic formula for “large” values of indices of the functions (2.1). Furthermore, we apply these results to prove the main theorems.

**Note 2.1.** Given a number  $\gamma$ , suppose that some coefficients in (2.1) equal to zero, that is, there exists a number  $p \in \mathbb{N}_0$ , such that the representation (2.1) can be written as follows:

$$E_{\alpha,n}^{\gamma}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}. \quad (2.2)$$

More precisely, as it is proved in [21], if  $\gamma$  is different from zero, then  $p = 0$  for each positive integer  $n$  and  $p = 1$  for  $n = 0$ .

Furthermore, an asymptotic formula for “large” values of the indices  $n$  is proved, namely:

**THEOREM 2.1.** *Let  $z, \alpha, \gamma \in \mathbb{C}$ ,  $n \in \mathbb{N}_0$ ,  $\gamma \neq 0$ ,  $\operatorname{Re}(\alpha) > 0$ . Then there exist entire functions  $\theta_{\alpha,n}^{\gamma}$  such that the generalized Mittag-Leffler functions (2.1) have the following asymptotic formulae*

$$E_{\alpha,n}^{\gamma}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p + n)} z^p (1 + \theta_{\alpha,n}^{\gamma}(z)) \quad \text{and} \quad \theta_{\alpha,n}^{\gamma}(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.3)$$

with the corresponding  $p$ , depending on  $\gamma$ . Moreover, on the compact subsets of the complex plane  $\mathbb{C}$ , the convergence is uniform and

$$\theta_{\alpha,n}^{\gamma}(z) = O\left(\frac{1}{n^{\operatorname{Re}(\alpha)}}\right) \quad (n \in \mathbb{N}). \quad (2.4)$$

**Note 2.2.** If  $\gamma = 0$ , the functions (2.1) take the simplest form

1.  $E_{\alpha,n}^0(z) = \frac{1}{\Gamma(n)}$  for  $n \in \mathbb{N}$ ,
2.  $E_{\alpha,n}^0(z) = 0$  for  $n = 0$ .

**Note 2.3.** According to the asymptotic formula (2.3), it follows there exists a positive integer  $M$  such that the functions  $E_{\alpha,n}^\gamma$  have no zeros for  $n > M$ , possibly except at the origin.

### 3. Series in Mittag-Leffler functions

We introduce the following auxiliary functions, related to the generalized Mittag-Leffler functions (1.2), namely:

$$\begin{aligned}\tilde{E}_{\alpha,0}^0(z) &= 0, & \tilde{E}_{\alpha,n}^0(z) &= \Gamma(n)z^n E_{\alpha,n}^0(z), & n \in \mathbb{N}, \\ \tilde{E}_{\alpha,n}^\gamma(z) &= \frac{\Gamma(\alpha p + n)}{(\gamma)_p} z^{n-p} E_{\alpha,n}^\gamma(z), & \gamma \neq 0, & n \in \mathbb{N}_0,\end{aligned}$$

and consider series in these functions of the form:

$$\sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha,n}^\gamma(z), \quad (3.1)$$

with complex coefficients  $a_n$  ( $n = 0, 1, 2, \dots$ ).

Our main objective is to study the convergence of the series (3.1) in the complex plane and prove theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems for the power series. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions (as for example, in Kiryakova and Al-Saqabi [9] and Sandev, Tomovski and Dubbeldam [28]).

The same type convergence theorems have been obtained for series in other special functions, e.g. for series in Laguerre and Hermite polynomials, by Rusev [25]–[27], and resp. by the author – for series in Bessel functions and their Wright’s 2-, 3-, and 4-indices generalizations, in [15]–[18], and in Mittag-Leffler functions, in [19, 20].

### 4. Cauchy-Hadamard and Abel type theorems

In the beginning we give a theorem of Cauchy-Hadamard type for the series (3.1).

**THEOREM 4.1** (Cauchy-Hadamard type). *The domain of convergence of the series (3.1) with complex coefficients  $a_n$  is the disk  $|z| < R$  with a radius of convergence  $R = 1/\Lambda$ , where*

$$\Lambda = \limsup_{n \rightarrow \infty} (|a_n|)^{1/n}. \quad (4.1)$$

*The cases  $\Lambda = 0$  and  $\Lambda = \infty$  can be included in the general case, provided  $1/\Lambda$  means  $\infty$ , respectively 0.*

**P r o o f.** We give only the idea of the proof. The case  $\gamma = 0$  follows immediately, because the series (3.1) reduces to the power series  $\sum_{n=1}^{\infty} a_n z^n$ . For the other cases, using the asymptotic formula (2.3), we evaluate the absolute value of the general term of the series (3.1). Further the proof goes separately in the three cases:  $\Lambda = 0$ ,  $0 < \Lambda < \infty$ ,  $\Lambda = \infty$ . We show the absolute convergence of the series (3.1) in the circular domain  $\{z : z \in \mathbb{C}, |z| < R\}$ . In the second case we prove that the series is divergent for  $|z| > R$  and in the third case — divergent for all the complex  $z \neq 0$ .  $\square$

Let  $z_0 \in \mathbb{C}$ ,  $0 < R < \infty$ ,  $|z_0| = R$  and  $g_\varphi$  be an arbitrary angular domain with size  $2\varphi < \pi$  and with a vertex at the point  $z = z_0$ , which is symmetric with respect to the straight line defined by the points 0 and  $z_0$ . The following theorem is valid.

**THEOREM 4.2** (Abel type). *Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence of complex numbers, and let  $K = \{z : z \in \mathbb{C}, |z| < R\}$ , with  $R := 1/\Lambda$ ,  $0 < \Lambda < \infty$  as defined in (4.1). If  $f(z; \alpha, \gamma)$  is the sum of the series (3.1) on the domain  $K$ , and this series converges at the point  $z_0$  of the boundary of  $K$ , then:*

$$\lim_{z \rightarrow z_0} f(z; \alpha, \gamma) = \sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha, n}^{\gamma}(z_0), \quad (4.2)$$

provided  $|z| < R$  and  $z \in g_\varphi$ .

**P r o o f.** The case  $\gamma = 0$  follows immediately, because the series (3.1) reduces to a power series. When  $\gamma \neq 0$ , we consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha, n}^{\gamma}(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (\tilde{E}_{\alpha, n}^{\gamma}(z_0) - \tilde{E}_{\alpha, n}^{\gamma}(z)) \quad (4.3)$$

and represent it in the form

$$\Delta(z) = \sum_{n=0}^k a_n (\tilde{E}_{\alpha, n}^{\gamma}(z_0) - \tilde{E}_{\alpha, n}^{\gamma}(z)) + \sum_{n=k+1}^{\infty} a_n (\tilde{E}_{\alpha, n}^{\gamma}(z_0) - \tilde{E}_{\alpha, n}^{\gamma}(z)).$$

Let  $p > 0$ . By using the notations

$$\begin{aligned} \beta_m &= \sum_{n=k+1}^m a_n \tilde{E}_{\alpha, n}^{\gamma}(z_0), \quad m > k, \quad \beta_k = 0, \\ \gamma_n(z) &= 1 - \tilde{E}_{\alpha, n}^{\gamma}(z) / \tilde{E}_{\alpha, n}^{\gamma}(z_0), \end{aligned}$$

and the Abel transformation (see in ([13: Vol. 1, Ch. 1, p. 32, 3.4:7]), we obtain consecutively:

$$\begin{aligned} \sum_{n=k+1}^{k+p} a_n(\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) &= \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z) \\ &= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z) - \gamma_n(z)), \end{aligned}$$

i.e.

$$\begin{aligned} \sum_{n=k+1}^{k+p} a_n(\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) \\ = (1 - \tilde{E}_{\alpha,k+p}^{\gamma}(z)/\tilde{E}_{\alpha,k+p}^{\gamma}(z_0)) \sum_{n=k+1}^{k+p} a_n \tilde{E}_{\alpha,n}^{\gamma}(z_0) \\ - \sum_{n=k+1}^{k+p-1} \left( \sum_{s=k+1}^n a_s \tilde{E}_{\alpha,s}^{\gamma}(z_0) \right) \left( \frac{\tilde{E}_{\alpha,n}^{\gamma}(z)}{\tilde{E}_{\alpha,n}^{\gamma}(z_0)} - \frac{\tilde{E}_{n+1}^{\gamma}(z)}{\tilde{E}_{n+1}^{\gamma}(z_0)} \right). \end{aligned}$$

According to Note 2.3, there exists a positive integer  $M$  such that  $\tilde{E}_{\alpha,n}^{\gamma}(z_0) \neq 0$  when  $n > M$ . Let  $k > M$ . Then, for all the natural  $n > k$ :

$$\begin{aligned} \tilde{E}_{\alpha,n}^{\gamma}(z)/\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n+1}^{\gamma}(z)/\tilde{E}_{\alpha,n+1}^{\gamma}(z_0) \\ = (z/z_0)^n \frac{(1 + \theta_{\alpha,n}^{\gamma}(z))(1 + \theta_{\alpha,n+1}^{\gamma}(z_0)) - (z/z_0)(1 + \theta_{\alpha,n+1}^{\gamma}(z))(1 + \theta_{\alpha,n}^{\gamma}(z_0))}{(1 + \theta_{\alpha,n}^{\gamma}(z_0))(1 + \theta_{\alpha,n+1}^{\gamma}(z_0))}. \end{aligned} \quad (4.4)$$

For the right hand side of (4.4) we apply the Schwartz lemma. Then we get that there exists a constant  $C$  such that:

$$|\tilde{E}_{\alpha,n}^{\gamma}(z)/\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n+1}^{\gamma}(z)/\tilde{E}_{\alpha,n+1}^{\gamma}(z_0)| \leq C|z - z_0||z/z_0|^n.$$

Analogously, there exists a constant  $B$ :

$$|1 - \tilde{E}_{\alpha,k+p}^{\gamma}(z)/\tilde{E}_{\alpha,k+p}^{\gamma}(z_0)| \leq B|z - z_0| \leq 2B|z_0|.$$

Let  $\varepsilon$  be an arbitrary positive number and choose  $N(\varepsilon)$  so large that for  $k > N(\varepsilon)$  the inequality

$$\left| \sum_{s=k+1}^n a_s \tilde{E}_{\alpha,s}^{\gamma}(z_0) \right| < \min(\varepsilon \cos \varphi / (12B|z_0|), \varepsilon \cos \varphi / (6C|z_0|))$$

holds for all the positive integers  $n > k$ . Therefore, for  $k > \max(M, N(\varepsilon))$ :

$$\left| \sum_{s=k+1}^{\infty} a_s \tilde{E}_{\alpha,s}^{\gamma}(z_0) \right| \leq \min(\varepsilon \cos \varphi / (12B|z_0|), \varepsilon \cos \varphi / (6C|z_0|)),$$

and

$$\begin{aligned} \left| \sum_{n=k+1}^{\infty} a_n (\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) \right| &\leq (\varepsilon \cos \varphi / 6) (1 + \sum_{n=k+1}^{\infty} |z_0|^{-1} |z - z_0| |z/z_0|^n) \\ &\leq (\varepsilon \cos \varphi / 6) (1 + |z - z_0| / (|z_0| - |z|)). \end{aligned}$$

But near the vertex of the angular domain  $g_{\varphi}$  in the part  $d_{\varphi}$  closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle, we have  $|z - z_0| / (|z_0| - |z|) < 2 / \cos \varphi$ , i.e.  $|z - z_0| \cos \varphi < 2(|z_0| - |z|)$ . That is why the inequality

$$\left| \sum_{n=k+1}^{\infty} a_n (\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) \right| < (\varepsilon \cos \varphi) / 6 + \varepsilon / 3 \leq \varepsilon / 2 \quad (4.5)$$

holds for  $z \in d_{\varphi}$  and  $k > \max(M, N(\varepsilon))$ . Fix some  $k > \max(M, N(\varepsilon))$  and after that choose  $\delta(\varepsilon)$  such that if  $|z - z_0| < \delta(\varepsilon)$  then the inequality

$$\left| \sum_{n=0}^k a_n (\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) \right| < \varepsilon / 2 \quad (4.6)$$

holds inside  $d_{\varphi}$ . We get

$$|\Delta(z)| = \left| \sum_{n=0}^{\infty} a_n (\tilde{E}_{\alpha,n}^{\gamma}(z_0) - \tilde{E}_{\alpha,n}^{\gamma}(z)) \right|$$

for the module of the difference (4.3). From (4.5) and (4.6) it follows that the equality (4.2) is satisfied.  $\square$

## 5. $(E, z_0)$ -summation

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \dots \quad (5.1)$$

To define its Abel summability ([5: p. 20, 1.3(2)]), we consider also the power series  $\sum_{n=0}^{\infty} a_n z^n$ .

**DEFINITION 5.1.** The series (5.1) is called A-summable, if the series  $\sum_{n=0}^{\infty} a_n z^n$  converges in the disk  $D = \{z : z \in \mathbb{C}, |z| < 1\}$  and moreover there exists  $\lim_{z \rightarrow 1-0} \sum_{n=0}^{\infty} a_n z^n = S$ . The complex number  $S$  is called A-sum of the series (5.1) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S(A).$$

**Note 5.1.** The A-summation is regular. It means that if the series (5.1) converges, then it is A-summable, and its A-sum is equal to its usual sum.

**Note 5.2.** The A-summability of the series (5.1) does not imply in general its convergence. But, with additional conditions on the growth of the general term of the series (5.1), the convergence can be ensured.

Note that each of the functions  $\tilde{E}_{\alpha,n}^{\gamma}(z)$  ( $n \in \mathbb{N}$ ), being an entire function, not identically zero, has no more than finite number of zeros in the closed and bounded set  $|z| \leq R$  ([13: Vol. 1, Ch. 3, §6, 6.1, p. 305]). Moreover, because of Note 2.3, no more than finite number of these functions have some zeros, possibly except of zero.

Let  $z_0 \in \mathbb{C}$ ,  $|z_0| = R$ ,  $0 < R < \infty$  and  $\tilde{E}_{\alpha,n}^{\gamma}(z_0) \neq 0$ . For the sake of brevity, denote

$$E_{\alpha,n,\gamma}^*(z; z_0) = \frac{\tilde{E}_{\alpha,n}^{\gamma}(z)}{\tilde{E}_{\alpha,n}^{\gamma}(z_0)}$$

and  $E_{\alpha,0,0}^*(z; z_0) = 0$ , just for completeness.

**DEFINITION 5.2.** The series (5.1) is said to be  $(E, z_0)$ -summable if the series

$$\sum_{n=0}^{\infty} a_n E_{\alpha,n,\gamma}^*(z; z_0), \quad (5.2)$$

converges in the disk  $|z| < R$  and, moreover, there exists the limit

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_{\alpha,n,\gamma}^*(z; z_0), \quad (5.3)$$

provided  $z$  remains on the segment  $[0, z_0]$ .

**Note 5.3.** Every  $(E, z_0)$ -summation is regular, and this property is just a particular case of Theorem 4.2.

## 6. Tauberian type theorems

A Tauberian theorem is a statement that relates the Abel summability and the standard convergence of a numerical series by means of some assumptions imposed on the general term of the series under consideration. A classical result in this direction is given by [5: Theorem 85].

In this paper we extend the validity of such type of assertion to series in three parametric Mittag-Leffler functions. Tauber type theorems are given also for summations by means of Laguerre polynomials [25], classical Mittag-Leffler and Bessel type functions by the author [15, 16], [19]–[20].



**THEOREM 6.1** (Tauber type). *If the series (5.1) is  $(E, z_0)$ -summable, and*

$$\lim_{n \rightarrow \infty} na_n = 0, \quad (6.1)$$

*then it is convergent.*

**Proof.** The case  $\gamma = 0$  follows immediately, because the series (5.2) reduces to a power series. Let now,  $\gamma \neq 0$  and  $z$  belong to the segment  $[0, z_0]$ . Taking into account the asymptotic formula (2.3) for the Mittag-Leffler functions, we obtain:

$$a_n E_{\alpha, n, \gamma}^*(z; z_0) = a_n \left( \frac{z}{z_0} \right)^n \frac{1 + \theta_{\alpha, n}^\gamma(z)}{1 + \theta_{\alpha, n}^\gamma(z_0)} = a_n \left( \frac{z}{z_0} \right)^n \left( 1 + \tilde{\theta}_n(z; z_0) \right),$$

where

$$\tilde{\theta}_n(z; z_0) = \frac{\theta_{\alpha, n}^\gamma(z) - \theta_{\alpha, n}^\gamma(z_0)}{1 + \theta_{\alpha, n}^\gamma(z_0)}.$$

Then, due to (2.4):

$$|\tilde{\theta}_n(z; z_0)| = O\left(1/n^{\operatorname{Re}(\alpha)}\right). \quad (6.2)$$

Let us write (5.2) in the form

$$\sum_{n=0}^{\infty} a_n E_{\alpha, n, \gamma}^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left( \frac{z}{z_0} \right)^n \left( 1 + \tilde{\theta}_n(z; z_0) \right). \quad (6.3)$$

Denoting

$$w_n(z) = a_n \left( \frac{z}{z_0} \right)^n \tilde{\theta}_n(z; z_0)$$

we consider the series  $\sum_{n=0}^{\infty} w_n(z)$ . Since  $|w_n(z)| \leq |a_n| |\tilde{\theta}_n(z; z_0)|$  and according to condition (6.1) and the relationship (6.2), there exists a constant  $C$ , such that  $|w_n(z)| \leq C/n^{1+\operatorname{Re}(\alpha)}$ . Since  $\sum_{n=1}^{\infty} 1/n^{1+\operatorname{Re}(\alpha)}$  converges, the series  $\sum_{n=0}^{\infty} w_n(z)$  is also convergent, even absolutely and uniformly on the segment  $[0, z_0]$ . Therefore (since  $\lim_{z \rightarrow z_0} w_n(z) = 0$ )

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z \rightarrow z_0} w_n(z) = 0.$$

Obviously, the assumption that the series (5.1) is  $(E, z_0)$ -summable implies the existence of the limit (5.3). Then, having in mind that (6.3) can be written in the form

$$\sum_{n=0}^{\infty} a_n E_{\alpha, n, \gamma}^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left( \frac{z}{z_0} \right)^n + \sum_{n=0}^{\infty} a_n \left( \frac{z}{z_0} \right)^n \tilde{\theta}_n(z; z_0),$$

we conclude that there exists the limit

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n \left( \frac{z}{z_0} \right)^n \quad (6.4)$$

and, moreover,

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_{\alpha, n, \gamma}^*(z; z_0) = \lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n \left( \frac{z}{z_0} \right)^n.$$

From the existence of the limit (6.4) it follows that the series (5.1) is A-summable. Then according to [5: Theorem 85], the series (5.1) converges.  $\square$

At first sight it seems that the condition  $a_n = o(1/n)$  is essential. Nevertheless, Littlewood succeeded to weaken it and obtained the strengthened version of the Tauber theorem ([5: Theorem 90]).

A Littlewood generalization of the  $o(n)$  version of a Tauber type theorem (Theorem 6.1) is also given in this part. We proved similar theorem for series in Bessel type functions, see e.g. [17].

**THEOREM 6.2** (Littlewood type). *If the series (5.1) is  $(E, z_0)$ -summable, and*

$$a_n = O(1/n)$$

*then the series (5.1) converges.*

**Proof.** Using [5: Theorem 90], at the place of [5: Theorem 85], the proof of the  $(E, z_0)$  summability follows the line of that of Theorem 6.1 and the ideas of the proofs of the cases of  $(E, z_0)$ - and  $(E_\alpha, z_0)$ -summabilities, given in [19], respectively [18].  $\square$

## 7. Conclusion

In conclusion, note that the case  $\gamma = 1$  gives analogous results related to the classical Mittag-Leffler functions (1.1). Additionally, if the parameters  $\alpha$  and  $\beta$  are positive, then we obtain the previous results, published in the papers [19] and [20].

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# CONVERGENCE OF SERIES IN THREE PARAMETRIC MITTAG-LEFFLER FUNCTIONS

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