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POSITIVE SOLUTIONS FOR SOME COMPETITIVE ELLIPTIC SYSTEMS

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ABSTRACT. Using some potential theory tools and the Schauder fixed point theorem, we prove the existence of positive bounded continuous solutions with a precise global behavior for the semilinear elliptic system $\Delta u = p(x)u^{\alpha}v^{r}$, $\Delta v = q(x)u^{s}v^{\beta}$ in domains D of $\mathbb{R}^n$, $n \geq 3$, with compact boundary (bounded or unbounded) subject to some Dirichlet conditions, where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and the potentials p, q are nonnegative and belong to the Kato class $K(D)$.

1. Introduction

The study of nonlinear elliptic systems has a strong motivation and important research efforts have been made recently for these systems aiming to apply the results of existence and asymptotic behavior of positive solutions in applied fields. Coupled nonlinear Shrödinger systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [1,23]. Stationary elliptic systems arise also in other physical models like non-Newtonian fluids: pseudo-plastic fluids and dilatant fluids [3, 9], non-Newtonian filtration [8] and the turbulent flow of a gas in porous medium [4,7]. They also describe other various nonlinear phenomena such as chemical reactions, pattern formation, population evolution where for example, u and v represent the concentrations of two species in the process. As a consequence, positive solutions of such systems are of interest.

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For some recent results on the qualitative analysis and the applications of positive solutions of nonlinear elliptic systems in both bounded and unbounded domains we refer to [10, 11, 13–20, 26] and the references therein.

In these works various existence results of positive bounded solutions or positive blowing-up ones (called also large solutions) have been established and a precise global behavior is given. We note also that several methods have been used to treat these nonlinear systems such as sub and supersolutions method, variational method and topological methods.

In this paper, we consider a $C^{1,1}$-domain D in $\mathbb{R}^n$ $(n \geq 3)$ with compact boundary. We fix some nonnegative constants a, b, c, d such that $a + c > 0$ and $b + d > 0$ and we assume $c = d = 0$ (consequently $a > 0$ and $b > 0$) if D is bounded. Also we fix two nontrivial nonnegative continuous functions φ and ψ on ∂D and we will deal with the existence of a positive continuous bounded solution (in the sense of distributions) to the system

$$\begin{cases} \Delta u = p(x) u^{\alpha} v^{r}, \quad \text{in } D \\ \Delta v = q(x) u^{s} v^{\beta}, \quad \text{in } D \\ u_{/\partial D} = a\varphi, \quad v_{/\partial D} = b\psi \\ \lim\limits_{x\to\infty} u(x) = c, \quad \lim\limits_{x\to\infty} v(x) = d \\ \text{(whenever } D \text{ is unbounded)}, \end{cases} \tag{1}$$

where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and p, q are two nonnegative functions in the Kato class $K(D)$ introduced and studied in [5] and [22].

Our method is based on some potential theory tools that we apply to give an existence result for equations by an approximation argument, then we use the result for equations to prove, by means of the Schauder fixed point theorem, the existence result for the system (1).

As far as we know, there are no results that contain existence of positive solutions to the elliptic system (1) in the case where $\alpha > 0$ and $\beta > 0$ and the weights $p(x)$ and $q(x)$ are singular functions and when D is an exterior domain.

The study of (1) is motivated by this fact and by the existence results obtained in [14] to the following system

$$\begin{cases} \Delta u = \lambda p(x) g(v), \quad \text{in } D \\ \Delta v = \mu q(x) f(u), \quad \text{in } D \\ u_{/\partial^{\infty} D} = a\varphi \mathbf{1}_{\partial D} + c\mathbf{1}_{\{\infty\}}, \\ v_{/\partial^{\infty} D} = b\psi \mathbf{1}_{\partial D} + d\mathbf{1}_{\{\infty\}}, \end{cases} \tag{2}$$

where λ, μ are nonnegative constants, the functions $f, g\colon [0,\infty) \to [0,\infty)$ are nondecreasing and continuous and $\partial^\infty D = \partial D$ if D is bounded and $\partial^\infty D = \partial D \cup \{\infty\}$ whenever D is unbounded.

More precisely, it was shown in [14] that if the functions p and q are nonnegative and belong to the Kato class, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$ the system (2) has a positive continuous solution (u, v) having the global asymptotic behavior of the unique solution of the associated homogeneous system.

Throughout this paper, we denote by $H_D\varphi$ the unique harmonic function u in D with boundary value φ and satisfying further $\lim\limits_{|x|\to\infty} u(x) = 0$ whenever D is unbounded, where φ is a nonnegative continuous function on ∂D. We denote also by $h = 1 - H_D 1$ and we remark that $h = 0$ if D is bounded and $\lim\limits_{x\to\infty} h(x) = 1$ if D is unbounded.

Taking into account these notations we use some potential theory tools and an approximating sequence in order to prove the following first result concerning the existence of a unique positive continuous solution to the boundary value problem

$$\begin{cases} \Delta u = p(x)u^\gamma, \quad \text{in } D \\ u_{/\partial D} = a\varphi \\ \lim\limits_{x\to\infty} u(x) = c \\ \text{(whenever } D \text{ is unbounded)}, \end{cases} \tag{3}$$

where $\gamma \geq 1$, φ is a nontrivial nonnegative continuous function on ∂D and a, c are two nonnegative constants with $a + c > 0$. More precisely we establish the following.

Theorem 1.1. *Let p be a nonnegative function in the Kato class $K(D)$. Then problem (3) has a unique positive continuous solution satisfying for each $x \in D$*

$$c_0\omega(x) \leq u(x) \leq \omega(x), \tag{4}$$

where $\omega(x) = aH_D\varphi(x) + ch(x)$ and the constant $c_0 \in (0, 1]$.

Next we exploit this result to prove the existence of a positive continuous solution (u, v) to the system (1). More precisely, we denote by $\omega = aH_D\varphi + ch$, $\theta = bH_D\psi + dh$ and we prove the following main result.

Theorem 1.2. *If p, q are two nonnegative functions in the Kato class $K(D)$, then problem (1) has a positive continuous solution (u, v) satisfying for each x in D*

$$\begin{aligned} c_1\omega(x) &\leq u(x) \leq \omega(x), \qquad c_1 \in (0, 1], \\ c_2\theta(x) &\leq v(x) \leq \theta(x), \qquad c_2 \in (0, 1]. \end{aligned} \tag{5}$$

In order to state these results, we give in the sequel some notations and we recall some properties of the Kato class defined by means of the Green function $G(x,y)$ of the Dirichlet Laplacian in D.

Let us denote by $B(D)$ the set of Borel measurable functions in D and by $B^+(D)$ the set of nonnegative ones. We denote also by $C_0(D)$ the set of continuous functions in D having limit zero at $\partial^\infty D$ and by $C_b(D)$ the set of continuous bounded ones. For any $u \in C_b(D)$, we denote by $\|u\|_\infty = \sup_{x\in D} |u(x)|$.

For any nonnegative function f in $B(D)$, we denote by Vf the Green potential of f defined on D by

$$Vf(x) := \int_D G(x,y) f(y)\,\mathrm{d}y$$

and we recall that if $f \in L^1_{\mathrm{loc}}(D)$ and $Vf \in L^1_{\mathrm{loc}}(D)$, then we have in the distributional sense (see [6: p. 52])

$$\Delta(Vf) = -f \qquad \text{in} \quad D. \tag{6}$$

Let $(X_t, t > 0)$ be the Brownian motion in $\mathbb{R}^n$ and P^x be the probability measure on the Brownian continuous paths starting at x. For any nonnegative function $q \in B(D)$, we define the kernel V_q by

$$V_q f(x) = E^x\left(\int_0^{\tau_D} \mathrm{e}^{-\int_0^t q(X_s)\,\mathrm{d}s} f(X_t)\,\mathrm{d}t\right), \tag{7}$$

where E^x is the expectation on P^x and $\tau_D = \inf\{t > 0 :\ X_t \notin D\}$.

If q is a nonnegative function in D such that $Vq < \infty$, the kernel V_q satisfies the following resolvent equation (see [6, 21])

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{8}$$

So for each $u \in B(D)$ such that $V(q|u|) < \infty$, we have

$$(I - V_q(q\,\cdot\,))(I + V(q\,\cdot\,))u = (I + V(q\,\cdot\,))(I - V_q(q\,\cdot\,))u = u \tag{9}$$

and for each $u \in B^+(D)$ we have

$$0 \le V_q(u) \le V(u). \tag{10}$$

Now we recall the definition of the Kato class which contains in particular a wider class of singular functions near the boundary of the domain D.

Definition 1.3. (see [5] and [22]) A Borel measurable function s in D belongs to the Kato class $K(D)$ if

$$\lim_{\alpha\to 0} \sup_{x\in D} \int_{D\cap B(x,\alpha)} \frac{\rho(y)}{\rho(x)} G(x,y)|s(y)|\,\mathrm{d}y = 0$$

and satisfies further

$$\lim_{M\to\infty} \sup_{x\in D} \int\limits_{D\cap\{|y|\geq M\}} \frac{\rho(y)}{\rho(x)} G(x,y)|s(y)|\,\mathrm{d}y = 0 \qquad \text{(whenever } D \text{ is unbounded)},$$

where $\rho(x) = \min(1, \delta(x))$ and $\delta(x)$ denotes the Euclidian distance from x to the boundary ∂D of D.

We remark that in the case where D is bounded and if d denotes its diameter, then

$$\frac{1}{1+d}\delta(x) \leq \rho(x) \leq \delta(x).$$

So in this case, we can replace $\rho(x)$ by $\delta(x)$ in the Definition 1.3.

This Kato class is rich enough as it can be seen in the following example.

Example 1.1. (see [5]) Let $q(x) = \frac{1}{(1+|x|)^{\mu-\lambda}(\delta(x))^{\lambda}}$ for $x \in D$. Then

$$q \in K(D) \iff \lambda < 2 < \mu.$$

Next, we recall some properties of $K(D)$.

Proposition 1.4. (See [5] and [22]) *Let q be a nonnegative function in $K(D)$. Then we have*

(i) $\alpha_q := \sup\limits_{x,y\in D} \int\limits_D \frac{G(x,z)G(z,y)}{G(x,y)} q(z)\,\mathrm{d}z < \infty$.

(ii) *The function $x \mapsto \frac{\delta(x)}{(1+|x|)^{n-1}} q(x)$ is in $L^1(D)$. In particular $q \in L^1_{\mathrm{loc}}(D)$.*

(iii) $Vq \in C_0(D)$.

(iv) *For any nonnegative superharmonic function v in D and all $x \in D$, we have*

$$\int\limits_D G(x,y)v(y)q(y)\,\mathrm{d}y \leq \alpha_q v(x).$$

The following result will play an important role in the proofs of Theorems 1.1 and 1.2.

Proposition 1.5. *Let v be a nonnegative superharmonic function in D and q be a nonnegative function in $K(D)$. Then for each $x \in D$ such that $0 < v(x) < \infty$, we have*

$$\exp(-\alpha_q)v(x) \leq v(x) - V_q(qv)(x) \leq v(x).$$

Proof. Let v be a nonnegative superharmonic function in D. Then by [24: Theorem 2.1] there exists a sequence $(f_k)_k$ of nonnegative measurable functions in D such that the sequence $(v_k)_k$ given in D by

$$v_k(x) := \int_D G_D(x,y) f_k(y)\, \mathrm{d}y$$

increases to v. Let $x \in D$ such that $0 < v(x) < \infty$. Then there exists $k_0 \in \mathbb{N}$ such that $0 < V f_k(x) < \infty$, for $k \geq k_0$.

Now, for a fixed $k \geq k_0$, we consider the function $\chi(r) = V_{rq} f_k(x)$. Since by (7) the function χ is completely monotone on $[0,\infty)$, we deduce from the Hölder inequality and [25: Theorem 12a] that $\log \chi$ is convex on $[0,\infty)$. Therefore,

$$\chi(0) \leq \chi(1) \exp\left(-\frac{\chi'(0)}{\chi(0)}\right),$$

which implies that

$$V f_k(x) \leq V_q f_k(x) \exp\left(\frac{V(qV f_k)(x)}{V f_k(x)}\right).$$

Hence, it follows from Proposition 1.4(iv) that

$$\exp(-\alpha_q) V f_k(x) \leq V_q f_k(x).$$

Consequently, from (8) we obtain

$$\exp(-\alpha_q) V f_k(x) \leq V f_k(x) - V_q(qV f_k)(x) \leq V f_k(x).$$

By letting $k \to \infty$, we deduce the result. □

The following compactness result will be used and it is proved in [22] for bounded domains and in [5] for unbounded ones.

Proposition 1.6. *Let q be a nonnegative function in $K(D)$. Then the family of functions*

$$\mathfrak{F}_q = \left\{ \int_D G(\,\cdot\,,y) p(y)\, \mathrm{d}y : \ |p| \leq q \right\}$$

is equicontinuous in $D \cup \partial^\infty D$ and consequently it is relatively compact in $C_0(D)$.

2. Proof of Theorem 1.1

First we give two Lemmas that will be used for uniqueness.

Lemma 2.1. (see [5]) *Let h be a nonnegative function in $B(D)$ and ϑ be a nonnegative superharmonic function in D. Then for all $z \in B(D)$ such that $V(h|z|) < \infty$ and $z + V(hz) = \vartheta$, we have $0 \leq z \leq \vartheta$.*

Lemma 2.2. *Let u be a nonnegative continuous function in $D \cup \partial^\infty D$. Then*

$$u \text{ is a solution of } (3) \quad \text{if and only if} \quad u = \omega - V(pu^\gamma) \quad \text{in } D.$$

Proof. Let u be a solution of (3). Since u is bounded in $\overline{D} \cup \infty$ and $p \in K(D)$, then $pu^\gamma \in K(D)$. Hence it follows from Proposition 1.4 that $V(pu^\gamma) \in C_0(D)$. This implies that $\Delta(u - \omega + V(pu^\gamma)) = 0$ in the sense of distribution. Consequently, using [12: Corollary 7, p. 294], we deduce that the function $u - \omega + V(pu^\gamma)$ is a classical harmonic function in D with value 0 at $\partial^\infty D$. Hence from the maximum principle, we deduce that $u - \omega + V(pu^\gamma) = 0$ in D. So $u = \omega - V(pu^\gamma)$ in D and this proves necessity.

Now, we prove sufficiency. Let u be a continuous function in $D \cup \partial^\infty D$, then u is bounded and consequently $pu^\gamma \in K(D)$. This implies, by using Proposition 1.4, that $V(pu^\gamma) \in C_0(D)$. Hence $\Delta u = \Delta\omega - \Delta(V(pu^\gamma)) = pu^\gamma$ (in the sense of distributions) and u is a solution of (3). □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. First we show that problem (3) has at most one continuous solution. Let u, v be two continuous solutions of (3). Then, by Lemma 2.2 we have $u = \omega - V(pu^\gamma)$ and $v = \omega - V(pv^\gamma)$ in D. Put $z = v - u$ and $h(x) = \frac{v^\gamma(x) - u^\gamma(x)}{v(x) - u(x)}$ if $u(x) \neq v(x)$ and $h(x) = 0$ whenever $u(x) = v(x)$. Then we have

$$\begin{cases} \Delta\left(z + V(phz)\right) = 0 & \text{in } D \\ z_{/\partial^\infty D} = 0. \end{cases}$$

Now, since z and h are bounded and $p \in K(D)$ we deduce that the function $z + V(phz)$ is a classical harmonic function in D with value 0 at $\partial^\infty D$. Hence from the maximum principle, we deduce that $z + V(phz) = 0$ in D. Using Lemma 2.1, we deduce that $z = 0$ and so $u = v$.

Next, we prove the existence of a positive continuous solution to (3). Let $\omega = aH_D\varphi + ch$, $\widetilde{p} = \gamma p \|\omega^{\gamma-1}\|_\infty$ and put $c_0 = \mathrm{e}^{-\alpha_{\widetilde{p}}}$ where the constant $\alpha_{\widetilde{p}}$ is defined in Proposition 1.4. We define the nonempty closed bounded convex set Λ by

$$\Lambda = \left\{u \in B^+(D) : \; c_0\omega \leq u \leq \omega\right\}.$$

Let T be the operator defined on Λ by

$$Tu := \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^\gamma).$$

We shall prove that T maps Λ to itself. Indeed, for each $u \in \Lambda$ we have

$$\begin{aligned} Tu &= \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^\gamma) \\ &\leq \omega - V_{\widetilde{p}}(pu^\gamma) \\ &\leq \omega. \end{aligned}$$

On the other hand, since the function $\widetilde{p}u - pu^\gamma \geq 0$, we deduce by using Proposition 1.5 that $Tu \geq \omega - V_{\widetilde{p}}(\widetilde{p}\omega) \geq c_0\omega$. Hence $T\Lambda \subset \Lambda$.

Next, we prove that T is nondecreasing on Λ. Let $u_1, u_2 \in \Lambda$ such that $u_1 \leq u_2$. Using the fact that the function $t \mapsto \gamma\|\omega^{\gamma-1}\|_\infty t - t^\gamma$ is nondecreasing on $[0, \|\omega\|_\infty]$ we deduce that

$$\begin{aligned} Tu_2 - Tu_1 &= V_{\widetilde{p}}(\widetilde{p}u_2 - pu_2^\gamma) - V_{\widetilde{p}}(\widetilde{p}u_1 - pu_1^\gamma) \\ &= V_{\widetilde{p}}\left(p[(\gamma\|\omega\|_\infty^{\gamma-1}u_2 - u_2^\gamma) - (\gamma\|\omega\|_\infty^{\gamma-1}u_1 - u_1^\gamma)]\right) \\ &\geq 0. \end{aligned}$$

Now, we consider the sequence $(u_k)_k$ defined by $u_0 = \omega - V_{\widetilde{p}}(\widetilde{p}\omega)$ and $u_{k+1} = Tu_k$.

Clearly $u_0 \in \Lambda$ and $u_1 = Tu_0 \geq u_0$. Thus, using the fact that Λ is invariant under T and the monotonicity of T, we deduce that

$$c_0\omega \leq u_0 \leq u_1 \leq \cdots \leq u_k \leq \omega.$$

Hence, the sequence $(u_k)_k$ converges to a measurable function $u \in \Lambda$. Therefore by applying the monotone convergence theorem, we deduce that u satisfies the following equation

$$u = \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^\gamma) \tag{11}$$

or equivalently

$$u - V_{\widetilde{p}}(\widetilde{p}u) = \omega - V_{\widetilde{p}}(\widetilde{p}\omega) - V_{\widetilde{p}}(pu^\gamma). \tag{12}$$

Applying the operator $(I + V(\widetilde{p}\,\cdot))$ on both sides of (12), we deduce by using (8) and (9) that

$$u = \omega - V(pu^\gamma).$$

Finally, since $p \in K(D)$ and u is bounded, then the function $pu^\gamma \in K(D)$. Hence it follows from Proposition 1.4 that $V(pu^\gamma) \in C_0(D)$. This implies that u is a continuous function and u is a solution of (3). Which completes the proof of Theorem 1.1. □

Next we give the proof of Theorem 1.2.

3. Proof of Theorem 1.2

We give the proof in the comprehensive case where D is unbounded. Let $\omega = aH_D\varphi + ch$ and $\theta = bH_D\psi + dh$. Let $f = \alpha\|\omega^{\alpha-1}\|_\infty\|\theta^r\|_\infty p$, $g = \beta\|\omega^s\|_\infty\|\theta^{\beta-1}\|_\infty q$ and put $c_1 = \mathrm{e}^{-\alpha_f}$ and $c_2 = \mathrm{e}^{-\alpha_g}$, where the nonnegative constants α_f and α_g are defined in Proposition 1.4.

In order to use a fixed point theorem, we consider the nonempty closed convex set Γ defined by

$$\Gamma = \Big\{(u,v)\in\big(C\big(\overline{D}\cup\{\infty\}\big)\big)^2 : \, c_1\omega(x) \le u(x) \le \omega(x) \;\; \& \;\; c_2\theta(x) \le v(x) \le \theta(x)\Big\}$$

Let T be the operator defined on Γ by $T(u,v) := (y,z)$, the unique positive continuous solution of the problem

$$\begin{cases} \Delta y = p(x)y^\alpha v^r, & \text{in } D \\ \Delta z = q(x)u^s z^\beta, & \text{in } D \\ y_{/\partial D} = a\varphi, \quad z_{/\partial D} = b\psi \\ \lim\limits_{x\to\infty} y(x) = c \quad \text{and} & \lim\limits_{x\to\infty} z(x) = d. \end{cases} \tag{13}$$

Then by Theorem 1.1, the solution $(y,z) \in (C(\overline{D}\cup\{\infty\}))^2$ and satisfies the integral equations $y = \omega - V(py^\alpha v^r)$, $z = \theta - V(qu^s z^\beta)$. Moreover we have the following global inequalities $c_1\omega \le y \le \omega$ and $c_2\theta \le z \le \theta$ in D. This implies in particular that $T(\Gamma) \subset \Gamma$.

Next, we aim to prove that the operator T is a compact operator from Γ into itself.

First, we show that $T(\Gamma)$ is equicontinuous on $\overline{D}\cup\{\infty\}$. Let $x, x' \in D$. Then for any $(u,v) \in \Gamma$ with $T(u,v) = (y,z)$ we have

$$|y(x) - y(x')| \le |\omega(x) - \omega(x')| + |V(py^\alpha v^r)(x) - V(py^\alpha v^r)(x')|$$

and

$$|z(x) - z(x')| \le |\theta(x) - \theta(x')| + |V(qu^s z^\beta)(x) - V(qu^s z^\beta)(x')|$$

Now, since the functions y, z, u and v are bounded and $p, q \in K(D)$, then we deduce from Proposition 1.4 that the families of functions $\mathfrak{F}_{p\omega^\alpha\theta^r}$ and $\mathfrak{F}_{q\omega^s\theta^\beta}$ are equicontinuous in $\overline{D}\cup\{\infty\}$. Using this fact and the continuity of ω and θ in $\overline{D}\cup\{\infty\}$, we deduce that $|y(x) - y(x')| \to 0$ and $|z(x) - z(x')| \to 0$ as $\|x - x'\| \to 0$ and for each $\varepsilon > 0$ there exists a compact set K such that for all $x \in D\backslash K$ we have $|y(x) - c| < \varepsilon$ and $|z(x) - d| < \varepsilon$. This implies that $T(\Gamma)$ is equicontinuous on $\overline{D}\cup\{\infty\}$. Since $T(\Gamma)$ is also bounded, then we deduce that $T(\Gamma)$ is relatively compact in $(C(\overline{D}\cup\{\infty\}))^2$.

Next, we shall prove the continuity of T. Let $(u_k, v_k)_k$ be a sequence in Γ that converges to $(u, v) \in \Gamma$ with respect to $\|\cdot\|_\infty$. Put $(y_k, z_k) = T(u_k, v_k)$ and $(y, z) = T(u, v)$. Then we have

$$\begin{aligned} y_k - y &= V(py^\alpha v^r) - V(py_k^\alpha v_k^r) \\ &= V(p(y^\alpha - y_k^\alpha)v^r) + V(py_k^\alpha(v^r - v_k^r)). \end{aligned}$$

Now using the fact that

$$t^\alpha - \eta^\alpha = (t-\eta)\left[\alpha\int_0^1 (\eta + \xi(t-\eta))^{\alpha-1}\,\mathrm{d}\xi\right] \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad \eta \geq 0, \tag{14}$$

we deduce that

$$(y_k - y) + V(p_k(y_k - y)) = V(py_k^\alpha(v^r - v_k^r)), \tag{15}$$

where $p_k(x) = \alpha p(x) v^r(x) \int_0^1 [\xi y_k(x) + (1-\xi)y(x)]^{\alpha-1}\,\mathrm{d}\xi$. Since $p \in K(D)$ and y_k, y, v^r are bounded, then $p_k \in K(D)$. So we can apply $(I - V_{p_k}(p_k\,\cdot))$ to equation (15) to obtain from equation (9) that $y_k - y = V_{p_k}(py_k^\alpha(v^r - v_k^r))$. Hence using the fact that (y_k) is uniformly bounded, the property (10) and the fact that Vp is bounded in $\overline{D}$ we deduce that there exists a positive constant $C > 0$ independent of k such that

$$\|y_k - y\|_\infty \leq C\|Vp\|_\infty \|v^r - v_k^r\|_\infty.$$

On the other hand using the fact that for $r \in (0,1]$ we have $\|v^r - v_k^r\|_\infty \leq \|v - v_k\|_\infty^r$ and for $r > 1$ we have $\|v^r - v_k^r\|_\infty \leq r\|\theta\|_\infty^{r-1}\|v - v_k\|_\infty$ we deduce that $\|v^r - v_k^r\|_\infty \to 0$ as $k \to \infty$. This implies that $\|y_k - y\|_\infty \to 0$ as $k \to \infty$. Similarly we prove that $\|z_k - z\|_\infty \to 0$ as $k \to \infty$.

From the Schauder fixed point theorem there exists $(u, v) \in \Gamma$ such that $T(u, v) = (u, v)$ or equivalently $u = \omega - V(pu^\alpha v^r)$ and $v = \theta - V(qu^s v^\beta)$. The pair (u, v) is clearly a positive continuous solution of (1) in the sense of distributions. This achieves the proof.

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