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SHARING SETS OF Q-DIFFERENCE OF MEROMORPHIC FUNCTIONS

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ABSTRACT. This paper is devoted to proving some uniqueness results for meromorphic functions $f(z)$ share sets with $f(qz)$. We give a partial answer to a question of Gross concerning a zero-order meromorphic function $f(z)$ and its q-difference $f(qz)$.

1. Introduction

In this paper, the term "meromorphic" will always mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory. For a non-constant meromorphic function f and a set S of complex numbers, we define the set $E(S,f) = \bigcup_{a\in S}\{z \mid f(z)-a=0\}$, where a zero of $f-a$ with multiplicity m counts m times in $E(S,f)$. As a special case, when $S=\{a\}$ contains only one element a, if $E(a,f)=E(a,g)$, then we say f and g share the value a CM, see [6].

As usual, by $S(r,f)$ we denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. In particular, we denote by $S_1(r,f)$ any quality satisfying $S_1(r,f)=o(T(r,f))$ for all r on a set of logarithmic density 1.

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In 1976, Gross asked the following question [3: Question 6]:

Question. Can one find (even one set) finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ $(j = 1, 2)$ must be identical?

Since then, many results have been obtained for this and related topics (see [2,7–9]). It is well known that there exists a set S containing seven elements such that if f and g are two non-constant entire functions and $E(S, f) = E(S, g)$, then $f = g$. There are some uniqueness results related to the case when two functions share two sets. We recall the following result given by Gross and Osgood.

Theorem A. ([4]) *Let $S_1 = \{1, -1\}$, $S_2 = \{0\}$. If f and g are non-constant entire functions of finite order such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f = \pm g$ or $fg = 1$.*

Many authors considered the condition that removed the order restriction and we just recall the next two results for meromorphic functions.

Theorem B. ([10]) *Let $S_1 = \{1, \omega, \dots, \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega = \cos(2\pi/n) + \mathrm{i}\sin(2\pi/n)$ and $n \geq 6$ be a positive integer. Suppose that f and g are non-constant meromorphic functions such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f = tg$ or $fg = t$, where $t^n = 1$.*

Theorem C. ([5]) *Let $S_1 = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$, where $n > 2m + 6$, $m \geq 2$ are integers such that n and $n - m$ having no common factors, and let a, b be two non-zero constants such that the algebraic equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. If f and g are non-constant meromorphic functions such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f = g$.*

If g is replaced by q-difference of f in Theorem B and C, similarly as to the above situations, we can consider shared sets problems for $f(z)$ and its q-difference $f(qz)$.

Theorem 1.1. *Let S_1, S_2 be given as in* Theorem B. *Suppose f is a non-constant zero-order meromorphic function such that $E(S_j, f(z)) = E(S_j, f(qz))$ for $j = 1, 2$, and $q \in \mathbb{C} \setminus \{0\}$. If $n \geq 4$, then $f(z) = tf(qz)$, $t^n = 1$ and $|q| = 1$.*

Corollary 1.2. Theorem 1.1 *still holds if f is a non-constant zero-order entire function and $n \geq 3$.*

Example. Let $f(z) = z$, $f(qz) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) z$. Clearly, $f(z)$ and $f(qz)$ share $\left\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right\}$ CM. Moreover, $q = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$ such that $|q| = 1$, and $f(z) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) f(qz)$ satisfying $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)^3 = 1$. This implies that Corollary 1.2 hold.

Corresponding to Corollary 1.2, it is natural to consider the case $n = 2$. The following result is a partial answer as to what may happen if $n = 2$ in Corollary 1.2.

Theorem 1.3. *Let f is a non-constant zero-order entire function and $q \in \mathbb{C} \setminus \{0\}$. If $f(z)$ and $f(qz)$ share the set $\{a(z), -a(z)\}$ CM. where $a(z)$ is a non-vanishing small function of f, then $f(z) = \pm f(qz)$ or $C^2 f(z) = f(q^2 z)$, where C is non-zero constant such that $C^2 \neq 1$ and $|q| = 1$.*

Corollary 1.4. *If a is a non-zero constant in* Theorem 1.3*, then we get $f(z) = \pm f(qz)$.*

Examples.

1. Let $f(z) = z$, $f(-z) = -z$. It is easy to verify that $f(z)$ and $f(-z)$ share $\{1, -1\}$ CM. This implies that Corollary 1.4 may occur.

2. Let $f(z) = z + 1$, $f(-z) = -z + 1$, then we get $f(z)$ and $f(-z)$ share 1 CM, however, the conclusion of Corollary 1.4 does not hold. This example shows that Corollary 1.4 cannot hold when the sharing set contains only one element, which means the assumption of Corollary 1.4 is sharp.

3. Corollary 1.4 is not true, if the order of $f(z)$ is not less than one. This can be seen by considering $f(z) = \mathrm{e}^z$ and $f(-z) = \mathrm{e}^{-z}$. Then $f(z)$ and $f(-z)$ share $\{1, -1\}$ CM, however, we cannot get the conclusion. This means the restriction of order is necessary. Meanwhile, we tried to consider whether Corollary 1.4 is true, if the order of $f(z)$ satisfies $0 < \sigma(f) < 1$. Unfortunately, we have not succeed.

Remarks.

1. From Corollary 1.2 and Corollary 1.4, we obtain that Theorem 1.1 still holds if f is a non-constant zero-order entire function and $n \geq 2$. Furthermore, the assumption $n \geq 2$ is sharp by Example 2.

2. Suppose $f(z)$ and $f(qz)$ share the set $\{a, b\}$ CM in Corollary 1.4, where a, b are two distinct values. Denote $g(z) = f(z) - \frac{a+b}{2}$, we get $g(z)$ and $g(qz)$ share $\left\{\frac{a-b}{2}, \frac{b-a}{2}\right\}$ CM. From Corollary 1.4, easily we know either $f(z) = f(qz)$ or $f(z) + f(qz) = a + b$, where $|q| = 1$.

Let $f(z) = z + 1$, $f(-z) = -z + 1$. Easily, we get that $f(z)$ and $f(-z)$ share $\{3, -1\}$ CM, and $f(z) + f(-z) = 2$. This implies that Remark 2 may occur.

3. As an application of Corollary 1.4, we can consider the existence of solutions of non-linear q-difference equation of type

$$f(z)^2 + f(qz)^2 = a^2,$$

where a is a non-zero constant. In fact, from above equation, we get $f(z)$ and $f(qz)$ share the set $\{\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\}$ CM. Then, we know there does not exist an entire solution of zero-order.

Corollary 1.5. *Under the condition of* Theorem 1.3, *if* $f(z)$ *and* $f(qz)$ *share sets* $\{a(z), -a(z)\}$, $\{0\}$ CM, *then* $f(z) = \pm f(qz)$, *where* $|q| = 1$.

As a q-difference analogue of Theorem C, we get the following result.

Theorem 1.6. *Let* $n \geq 2m + 4$, $m \geq 2$ *with* n *and* $n - m$ *having no common factors, and let* S_j *be given in Theorem C. Suppose* f *is a non-constant zero-order meromorphic function such that* $E(S_j, f(z)) = E(S_j, f(qz))$ *for* $j = 1, 2$, *and* $q \in \mathbb{C} \setminus \{0\}$, *then* $f(z) = f(qz)$ *and* $|q| = 1$.

It is natural to ask what happens if $m = 1$ in Theorem 1.6, we give a partial answer concerning entire functions in the following.

Theorem 1.7. *Let* $S = \{\omega \mid \omega^n + a\omega^{n-1} + b = 0\}$, *where* a *and* b *are two non-zero constants such that the algebraic equation* $\omega^n + a\omega^{n-1} + b = 0$ *has no multiple roots, and* n *is an integer. Suppose* f *is a non-constant zero-order entire function such that* $E(S, f(z)) = E(S, f(qz))$, $q \in \mathbb{C} \setminus \{0\}$. *If* $n \geq 5$, *then* $f(z) = f(qz)$, *and* $|q| = 1$.

2. Some lemmas

Lemma 2.1. ([1: Theorem 2.1]) *Let* f *be a zero-order meromorphic function, and* $q \in \mathbb{C} \setminus \{0\}$. *Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.2. ([11: Theorem 1.1, Theorem 1.3]) *Let* f *be a zero-order meromorphic function, and* $q \in \mathbb{C} \setminus \{0\}$. *Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)) \tag{2.1}$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)) \tag{2.2}$$

on a set of lower logarithmic density 1.

3. Proof of Theorem 1.1

From the assumption of Theorem 1.1, we know $f(z)^n$ and $f(qz)^n$ share 1 and ∞ CM, we obtain that

$$\frac{f(qz)^n - 1}{f(z)^n - 1} = C, \tag{3.1}$$

where C is a non-zero constant. Rewrite (3.1) as

$$f(qz)^n = C\left(f(z)^n - 1 + \frac{1}{C}\right). \tag{3.2}$$

Set

$$G(z) = \frac{f(z)^n}{1 - \frac{1}{C}}.$$

If $C \not\equiv 1$, then we apply the second main theorem to $G(z)$, and get

$$\begin{aligned} nT(r,f) + S(r,f) &= T(r,G) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,G) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f(z)^n - 1 + \frac{1}{C}}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r,f) \\ &\leq 2T(r,f) + T(r,f(qz)) + S(r,f). \end{aligned} \tag{3.3}$$

Combining (3.3) with Lemma 2.2, we get

$$nT(r,f) \leq 3T(r,f) + S_1(r,f),$$

which contradicts that $n \geq 4$. Therefore, $C \equiv 1$, that is, $f(z)^n = f(qz)^n$, so we have $f(z) = tf(qz)$, for a constant t with $t^n = 1$. In the following, we prove $|q| = 1$. Let $F(z) = f(z)^n$ and $F(qz) = f(qz)^n$, then we get $F(z) = F(qz)$. Suppose $|q| < 1$, we get $F(z) = F(q^m z)$. Letting $m \to \infty$, we get $F(z) = F(0)$, a contradiction. Assuming $|q| > 1$, and rewrite $F(z) = F(qz)$ by $F(cz) = F(z)$, where $c = \frac{1}{q}$, we also get a contradiction. Hence, we know $|q| = 1$. This completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

Similarly as Theorem 1.1, we get equation (3.3) as well. From (3.3) and the assumption that f is entire, we get

$$nT(r,f) \leq 2T(r,f) + S(r,f),$$

which contradicts that $n \geq 3$. The assertion now follows as in Theorem 1.1.

5. Proof of Theorem 1.3

Since $f(z)$ is an entire function of zero-order, $f(z)$ and $f(qz)$ share $\{a(z), -a(z)\}$ CM, it follows that

$$(f(qz) - a(z))(f(qz) + a(z)) = C^2(f - a(z))(f + a(z)), \tag{5.1}$$

where C is a non-zero constant.

Case 1. $C^2 \not\equiv 1$. Let $h_1(z) = f(z) - \frac{1}{C}f(qz)$, $h_2(z) = f(z) + \frac{1}{C}f(qz)$, then

$$f(z) = \frac{1}{2}(h_1 + h_2), \quad f(qz) = \frac{C}{2}(h_2 - h_1). \tag{5.2}$$

And from (5.1), we get

$$h_1 h_2 = \left(1 - \frac{1}{C^2}\right)a^2. \tag{5.3}$$

From above equation, we get

$$N\left(r, \frac{1}{h_1}\right) = S(r,f), \qquad N\left(r, \frac{1}{h_2}\right) = S(r,f). \tag{5.4}$$

By the expression of h_1, h_2 and Lemma 2.2, we know

$$T(r,h_i) \leq 2T(r,f) + S_1(r,f),$$

which means $S_1(r,h_i) = o(T(r,f))$ for all r on a set of logarithmic density 1, $i = 1, 2$.

Denote $\alpha = \frac{h_1(qz)}{h_1(z)}$ and $\beta = \frac{h_2(qz)}{h_2(z)}$. From (5.4) and Lemma 2.1, we obtain that

$$\begin{aligned} T(r,\alpha) &= m(r,\alpha) + N\left(r, \frac{1}{h_1}\right) = S_1(r,f), \\ T(r,\beta) &= m(r,\beta) + N\left(r, \frac{1}{h_2}\right) = S_1(r,f). \end{aligned} \tag{5.5}$$

From (5.2), it follows that

$$Ch_2(z) - Ch_1(z) = h_1(qz) + h_2(qz).$$

Dividing above equation with $h_1(z)h_2(z)$, we get

$$(\alpha + C)h_1(z) = (C - \beta)h_2(z). \tag{5.6}$$

By (5.3) and (5.6), we know that

$$(\alpha + C)h_1^2 - (C - \beta)a^2\left(1 - \frac{1}{C^2}\right) = 0. \tag{5.7}$$

Combining (5.5) with (5.7), we get $\alpha = -C$ and $\beta = C$. Otherwise, we know $T(r, h_1) = S_1(r, f)$. By (5.2) and (5.3), it follows that $T(r, f) = S_1(r, f)$, which is a contradiction. From $\alpha = -C$ and $\beta = C$, we get $h_1(qz) = -Ch_1(z)$ and $h_2(qz) = Ch_2(z)$, that is

$$\begin{cases} -C\left(f(z) - \frac{1}{C}f(qz)\right) = f(qz) - \frac{1}{C}f(q^2z) \\ C\left(f(z) + \frac{1}{C}f(qz)\right) = f(qz) + \frac{1}{C}f(q^2z) \end{cases}$$

we see $C^2f(z) = f(q^2z)$.

Case 2. $C^2 \equiv 1$. From (5.1), we get $f(z) = \pm f(qz)$. Using a similar way as Theorem 1.1, we get $|q| = 1$ in Case 1 and Case 2.

6. Proof of Corollary 1.4

Using the same reason of Theorem 1.3, we get equations (5.2) and (5.3) hold as well. From equation (5.3) and the assumption that a is non-zero constant, we get

$$N\left(r, \frac{1}{h_1}\right) = 0, \qquad N\left(r, \frac{1}{h_2}\right) = 0. \tag{6.1}$$

Combining (6.1) with the fact that $h_1(z)$ and $h_2(z)$ are zero-order entire functions, we obtain h_1 and h_2 are non-zero constant. From (5.2), we get $f(z)$ is a constant, which contradicts the assumption. Hence, only Case 2 of Theorem 1.3 holds, we get the conclusion.

7. Proof of Corollary 1.5

It suffices to prove the case $C^2f(z) = f(q^2z)$ in Theorem 1.3 does not hold. Assume that $f(z_0) = 0$, since $f(z)$ and $f(qz)$ share 0 CM, then from (5.2), we get $h_1(z_0) + h_2(z_0) = 0$ and $h_1(qz_0) + h_2(qz_0) = 0$. Therefore,

$$\frac{h_1(qz_0)}{h_1(z_0)}\frac{h_2(z_0)}{h_2(qz_0)} = 1.$$

From the proof of Theorem 1.3, we know $\alpha = \frac{h_1(qz_0)}{h_1(z_0)} = -C$ and $\beta = \frac{h_2(qz_0)}{h_2(z_0)} = C$, which means that

$$\frac{h_1(qz_0)}{h_1(z_0)}\frac{h_2(z_0)}{h_2(qz_0)} = -1.$$

which is a contradiction. Hence 0 must be the Picard exceptional value of $f(z)$ and $f(qz)$. Since $f(z)$ is a zero-order entire function, we know $f(z)$ must be a constant, which contradicts the assumption. So we remove the case $C^2 f(z) = f(q^2 z)$ to get $f(z) = \pm f(qz)$, where $|q| = 1$.

8. Proof of Theorem 1.6

By the condition of Theorem 1.6, we get that

$$\frac{f(qz)^n + af(qz)^{n-m} + b}{f(z)^n + af(z)^{n-m} + b} = C, \tag{8.1}$$

where C is a non-zero constant. Rewrite (8.1) as

$$f(qz)^{n-m}(f(qz)^m + a) = C\left(f(z)^n + af(z)^{n-m} + b - \frac{b}{C}\right). \tag{8.2}$$

If $C \not\equiv 1$, then apply the second main theorem to $f(z)^n + af(z)^{n-m}$ and by the similar argument of Theorem 1.1, we get

$$(n-m-2)T(r, f(z)) \leq (m+1)T(r, f(qz)) + S(r, f). \tag{8.3}$$

From Lemma 2.2 and (8.3), we conclude that

$$(n-m-2)T(r, f(z)) \leq (m+1)T(r, f(z)) + S_1(r, f),$$

that is

$$(n-2m-3)(T(r, f(z)) \leq S_1(r, f), \tag{8.4}$$

which contradicts the assumption that $n \geq 2m+4$. Hence, we get $C \equiv 1$. So we get $f(qz)^n + af(qz)^{n-m} = f(z)^n + af(z)^{n-m}$. Let $h(z) = \frac{f(qz)}{f(z)}$, then

$$f(z)^m(h(z)^n - 1) = -a(h(z)^{n-m} - 1). \tag{8.5}$$

Furthermore, we suppose

$$h(z) - u_j = 0,$$

where $u_j = \mathrm{e}^{\mathrm{i}\frac{2j\pi}{n}}$, $j = 1, 2, \dots n$. Since n and $n-m$ have no common factors, we see that $u_j^{n-m} \neq 1$. From equation (8.5), we know the multiplicity of a u_j-point of h is at least m. Suppose that h is not a constant, then we have

$$2 \geq \sum_{j=1}^{n-1} \Theta(u_j, h) \geq \sum_{j=1}^{n-1}\left(1 - \frac{1}{m}\right) = (n-1)\left(1 - \frac{1}{m}\right),$$

which contradicts to $m \geq 2$ and $n \geq 2m+4$. This shows that h must be a constant. From (8.5), we get $h(z) \equiv 1$. Otherwise, we will deduce that f is a constant. Hence $f(z) = f(qz)$, and by the same way of Theorem 1.1, we get $|q| = 1$. This completes the proof of Theorem 1.6.

9. Proof of Theorem 1.7

Using the same proof of Theorem 1.6 and from the assumption of Theorem 1.7, equation (8.1) holds as well (m=1 here). Suppose $C \not\equiv 1$, since f is entire, (8.4) can be replaced by

$$(n-4)T(r,f) \leq S_1(r,f),$$

which contradicts $n \geq 5$. Hence $C \equiv 1$, and as the same way as Theorem 1.6, we get $f(z)(h(z)^n - 1) = -a(h(z)^{n-1} - 1)$, where $h(z) = \frac{f(qz)}{f(z)}$. If $h(z)$ is a non-constant, then we have

$$f(z) = -a\frac{h^{n-1}-1}{h^n - 1}.$$

Noting that $n \geq 5$, we deduce that 1 is not a Picard value of h^n. Hence, that exists $a_j \in \mathbb{C} \setminus \{1\}$ $(j = 1, 2, \ldots, n-1)$ such that the distinct roots of equation $h^n - 1 = 0$. Thus $f(z)$ must have poles, which contradicts the assumption that f is entire. Therefore, $h(z)$ is a constant. Similarly as Theorem 1.6, we get $f(z) = f(qz)$ and $|q| = 1$.

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