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MAJORIZATION PROPERTIES FOR A NEW SUBCLASS OF θ-SPIRAL FUNCTIONS OF ORDER γ

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ABSTRACT. In the present paper, we introduce a new subclass $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$ of θ-spiral functions of order γ defined by a class of generalized multiplier transformations. Majorization properties for functions belonging to the class $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$ are considered. Moreover, we point out some new or known consequences of our main result.

1. Introduction and preliminaries

Let f and g be two analytic functions in the open unit disk

$$\Delta = \{z \in C:\ |z| < 1\}. \tag{1.1}$$

We say that f is majorized by g in Δ (see [1]) and write

$$f(z) \ll g(z) \qquad (z \in \Delta), \tag{1.2}$$

if there exists a function φ, analytic in Δ such that

$$|\varphi(z)| \le 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \qquad (z \in \Delta). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

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Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in N = \{1, 2, \dots\}), \tag{1.4}$$

that are analytic and multivalent in the open unit disk Δ. For simplicity, we write $A = A_1$.

For functions $f_j \in A_p$ given by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2;\;\; p \in N), \tag{1.5}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z).$$

Let $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$ ($q, s \in N \cup \{0\}$, $q \le s+1$) be complex numbers such that $\beta_j \ne 0, -1, -2, \dots,$ for $j \in \{1, 2, \dots, s\}$. We now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by (see, for example, [2])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, \dots, (\alpha_q)_k}{(\beta_1)_k, \dots, (\beta_s)_k} \cdot \frac{z^k}{k!} \qquad (z \in \Delta), \tag{1.6}$$

where $(x)_k$ denotes the Pochhammer symbol defined by

$$(x)_k = x(x+1)(x+2)\dots(x+k-1) \quad \text{for } \; k \in N \qquad \text{and} \qquad (x)_0 = 1.$$

Let

$$\begin{aligned} h_{p,q,s}(\alpha_1, \beta_1; z) &= z^p_q F_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= z^p + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k, \dots, (\alpha_q)_k}{(\beta_1)_k, \dots, (\beta_s)_k (1)_k} z^{k+p}, \end{aligned} \tag{1.7}$$

and using the Hadamard product, we define the following operator $I^{m,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)$: $A_p \to A_p$ by

$$\begin{aligned} I^{0,l}_{p,q,s,\lambda}(\alpha_1, \beta_1) f(z) &= f(z) * h_{p,q,s}(\alpha_1, \beta_1; z); \\ I^{1,l}_{p,q,s,\lambda}(\alpha_1, \beta_1) f(z) &= (1-\lambda)(f(z) * h_{p,q,s}(\alpha_1, \beta_1; z)) \\ &\quad + \frac{\lambda}{(p+l) z^{l-1}} (z^l f(z) * h_{p,q,s}(\alpha_1, \beta_1; z))'; \end{aligned}$$

and

$$I^{m,l}_{p,q,s,\lambda}(\alpha_1, \beta_1) f(z) = I^{1,l}_{p,q,s,\lambda}(I^{m-1,l}_{p,q,s,\lambda}(\alpha_1, \beta_1) f(z)). \tag{1.8}$$

If $f \in A_p$, then from (1.4) and (1.8), we can easily see that

$$I^{m,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m \times \frac{(\alpha_1)_{k-p}, \dots, (\alpha_q)_{k-p}}{(\beta_1)_{k-p}, \dots, (\beta_s)_{k-p}(1)_{k-p}} a_k z^k, \tag{1.9}$$

where $m \in N_0 = N \cup \{0\}$, λ, $l \geq 0$ and $p \in N$.

We note here some special cases.

(i) $I^{m,0}_{1,q,s,\lambda}(\alpha_1, \beta_1)f(z) = D^m_\lambda(\alpha_1, \beta_1)f(z)$ (see Selvaraj and Karthikeyan [3]);

(ii) $I^{0,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = H_{p,q,s}(\alpha_1, \beta_1)f(z)$ (see Dziok and Srivastava [4,5]);

(iii) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$ and $\beta_j = 1$ $(j = 1, \dots, s)$, we get the operator $I_p(m, \lambda, l)$ (see Catas [6]);

(iv) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$ and $\lambda = 1$, we obtain the operator $I_p(m, l)$ (see Kumar et al. [7]);

(v) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $l = 0$ and $\lambda = 1$, we have the operator D^m_p (see Kamali and Orhan [8] and Aouf and Mostafa [9]);

(vi) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$ and $p = \lambda = 1$, we get the operator I^m_l (see Cho and Srivastava [10] and Cho and Kim [11]);

(vii) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $p = 1$ and $l = 0$, we obtain the operator D^m_λ (see Al-Oboudi [12]);

(viii) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $p = \lambda = 1$ and $l = 0$, we obtain the operator D^m (see Salagean [13]).

Also, by specializing $m, \lambda, l, p, q, s, \alpha_i$ $(i = 1, \dots, q)$ and β_j $(j = 1, \dots, s)$, we obtain various new operators, e.g.,

(i) $I^{m,l}_{p,2,1,\lambda}(n+p, 1; 1)f(z) = z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m \frac{(p+n)_{k-p}}{(1)_{k-p}} a_k z^k$
$(n > -p;\ p, n \in N)$;

(ii) $I^{m,l}_{p,2,1,\lambda}(a, 1; c)f(z) = z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k$
$(a \in R;\ c \in R \setminus \overline{z_0} = \{0, -1, \dots\})$;

(iii) $I^{m,l}_{p,2,1,\lambda}(p+1, 1; n+p)f(z) = z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m \frac{(p+1)_{k-p}}{(n+p)_{k-p}} a_k z^k$
$(n \in Z;\ p \in N;\ n > -p)$;

(iv) $I^{m,l}_{p,2,1,\lambda}(p+1,1;p+1-\delta)f(z)=z^p+\sum\limits_{k=p+1}^{\infty}\left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m\frac{(p+1)_{k-p}}{(p+1-\delta)_{k-p}}a_kz^k$ $(p\in N;\ 0\le\delta<1)$;

(v) $I^{m,l}_{p,2,1,\lambda}(p+\delta,c;a)f(z)=z^p+\sum\limits_{k=p+1}^{\infty}\left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m\frac{(p+\delta)_{k-p}(c)_{k-p}}{(a)_{k-p}(1)_{k-p}}a_kz^k$ $(a,c\in R\setminus\overline{z_0};\ \delta>-p;\ p\in N)$;

(vi) $I^{m,l}_{p,2,1,\lambda}(p+\delta,1;p+\delta+1)f(z)=z^p+\sum\limits_{k=p+1}^{\infty}\left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m\frac{(p+\delta)_{k-p}}{(p+\delta+1)_{k-p}}a_kz^k$ $(\delta>-p;\ p\in N)$.

It can be easily verified from the definition (1.9) that

$$\begin{aligned}\lambda z\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\right)'&=(p+l)I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\\&\quad-[p(1-\lambda)+l]I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\qquad(\lambda>0)\end{aligned}\tag{1.10}$$

and

$$\begin{aligned}z\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\right)'&=\alpha_1I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)f(z)\\&\quad-(\alpha_1-p)I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z).\end{aligned}\tag{1.11}$$

Using the operator $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)$, we now define the class $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$ as follows.

Definition 1.1. A function $f(z)\in A_p$ is said to be in the class $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$, if and only if

$$\operatorname{Re}\left\{e^{i\theta}\frac{z\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\right)'}{I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}\right\}>\gamma\cos\theta$$

$$\left(q\le s+1;\ q,s,m\in N_0;\ p\in N;\ \lambda,l\ge0;\ 0\le\gamma<1;\ -\frac{\pi}{2}<\theta<\frac{\pi}{2};\ z\in\Delta\right).\tag{1.12}$$

Clearly, when $p=1$, $q=2$, $s=1$, $\alpha_1=\beta_1$ and $\alpha_2=1$, we have the following relationships:

(i) $S^{0,l}_{1,2,1,\lambda}(\gamma,\theta)=S^*_\theta(\gamma)$;

(ii) $S^{0,l}_{1,2,1,\lambda}(\gamma,0)=S^*(\gamma)$;

(iii) $S^{0,l}_{1,2,1,\lambda}(0,\theta)=S^*_\theta$.

The classes $S^*_\theta(\gamma)$ and $S^*(\gamma)$ are said to be the classes of θ-spiral-like and starlike functions of order γ in Δ, which were studied by Libera [14] and Robertson

[15], while S^*_θ denotes the class of θ-spiral-like functions in Δ considered by Spacek [16].

A majorization problem for the class $S^* = S^*(0)$ has been investigated by MacGregor [1]. Also, majorization problems for starlike functions of complex order have recently been investigated by Altintas et al. [17], Goyal et al. [18, 19], and Goswami et al. [20, 21]. Motivated by these works, in this paper, we investigate a majorization problem for the class $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$.

2. Majorization problem for the class $S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$

We begin by proving the following result.

Theorem 2.1. *Let the function $f \in A_p$ and suppose that $g \in S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$. If $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)$ is majorized by $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)$ in Δ, then*

(i)

$$\left|I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\right| \le \left|I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)\right| \qquad (|z| \le r_1), \tag{2.1}$$

(ii)

$$\left|I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)f(z)\right| \le \left|I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)\right| \qquad (|z| \le r_2), \tag{2.2}$$

where

$$\begin{aligned} r_1 &= r_1(p,\lambda,l,\gamma,\theta) \\ &= \frac{\eta_1 - \sqrt{\eta_1^2 - 4|p+l+(1-p)\lambda||2\lambda(1-\gamma)\cos\theta - [p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|}}{2|2\lambda(1-\gamma)\cos\theta - [p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|} \end{aligned} \tag{2.3}$$

with

$$\eta_1 = 2\lambda + |p+l+(1-p)\lambda| + |2\lambda(1-\gamma)\cos\theta - [p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|$$

and

$$|p+l+(1-p)\lambda| \ge |2\lambda(1-\gamma)\cos\theta - [p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|$$

and

$$\begin{aligned} r_2 &= r_2(p,\alpha_1,\gamma,\theta) \\ &= \frac{\eta_2 - \sqrt{\eta_2^2 - 4|1+\alpha_1-p||2(1-\gamma)\cos\theta - (1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|}}{2|2(1-\gamma)\cos\theta - (1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|} \end{aligned} \tag{2.4}$$

with

$$\eta_2 = 2 + |1+\alpha_1-p| + |2(1-\gamma)\cos\theta - (1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|$$

and

$$|1+\alpha_1-p| \geq |2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|$$

$$(p \in N; \quad \lambda, l \geq 0; \quad \alpha_1 \in C; \quad 0 \leq \gamma < 1; \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}).$$

P r o o f. We first prove that (2.1) of Theorem 2.1 holds true. Since $g \in S^{m,l}_{p,q,s,\lambda}(\gamma,\theta)$, we find from (1.12) that

$$\mathrm{e}^{\mathrm{i}\theta}\frac{z\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)\right)'}{I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)} = \frac{1+(1-2\gamma)\omega}{1-\omega}\cos\theta+\mathrm{i}\sin\theta, \tag{2.5}$$

where ω is analytic in Δ, with $\omega(0)=0$ and

$$|\omega(z)| \leq |z| \qquad (z \in \Delta). \tag{2.6}$$

By using (1.10) in (2.5) and making simple calculations, we get

$$\begin{aligned}&\frac{I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)}{I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)}\\ &= \frac{[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}+[2\lambda(1-\gamma)\cos\theta-(p+l+(1-p)\lambda)\mathrm{e}^{\mathrm{i}\theta}]\omega}{\mathrm{e}^{\mathrm{i}\theta}(p+l)(1-\omega)},\end{aligned} \tag{2.7}$$

which, in view of (2.6), immediately yields the inequality

$$\begin{aligned}&|I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)|\\ &\leq \frac{|p+l|(1+|z|)}{|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}||z|}\\ &\quad\times|I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)|.\end{aligned} \tag{2.8}$$

Next, since $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)$ is majorized by $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)$ in Δ, thus from (1.3), we have

$$I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) = \varphi(z)I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z).$$

Differentiating it with respect to z and multiplying by z, we obtain

$$\begin{aligned}&z\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\right)'\\ &= z\varphi'(z)I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)+z\varphi(z)\left(I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)\right)'.\end{aligned} \tag{2.9}$$

Also, by using (1.10) in (2.9), we have

$$\begin{aligned}&I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)\\ &= \frac{\lambda z\varphi'(z)}{p+l}I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)+\varphi(z)I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z).\end{aligned} \tag{2.10}$$

Therefore, noting that the Schwarz function φ satisfies the inequality (see, e.g. Nehari [22])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in \Delta), \tag{2.11}$$

and making use of (2.8) and (2.11) in (2.10), we get

$$\begin{aligned} &|I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)| \\ &\leq \left(\frac{\lambda|z|(1-|\varphi(z)|^2)}{(1-|z|)(|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}||z|)}+|\varphi(z)|\right) \\ &\quad \times |I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)|, \end{aligned} \tag{2.12}$$

which, upon setting

$$|z| = r, \quad |\varphi(z)| = \rho \qquad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$\begin{aligned} &|I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)| \\ &\leq \frac{\Phi_1(\rho)}{(1-r)(|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|r)} \\ &\quad \times |I^{m+1,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)|, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \Phi_1(\rho) &= \lambda r(1-\rho^2)+(1-r) \\ &\quad \times (|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|r)\rho \end{aligned} \tag{2.14}$$

takes its maximum value at $\rho = 1$ with $r_1 = r_1(p,\lambda,l,\gamma,\theta)$ given by (2.3). Furthermore, if $0 \leq \delta \leq r_1(p,\lambda,l,\gamma,\theta)$, then the function $\Psi_1(\rho)$ defined by

$$\begin{aligned} \Psi_1(\rho) &= \lambda\delta(1-\rho^2)+(1-\delta) \\ &\quad \times (|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|\delta)\rho \end{aligned} \tag{2.15}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\begin{aligned} \Psi_1(\rho) &\leq \Psi_1(1) \\ &= (1-\delta)(|p+l+(1-p)\lambda|-|2\lambda(1-\gamma)\cos\theta-[p+l+(1-p)\lambda]\mathrm{e}^{\mathrm{i}\theta}|\delta) \\ &\qquad (0 \leq \delta \leq r_1(p,\lambda,l,\gamma,\theta)). \end{aligned}$$

Hence upon setting $\rho = 1$, in (2.15), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \leq r_1(p,\lambda,l,\gamma,\theta)$, where $r_1(p,\lambda,l,\gamma,\theta)$ is given by (2.3).

Now, we prove that (2.2) of Theorem 2.1 also holds true. By using (1.11) in (2.5) and making simple calculations, we obtain

$$\frac{I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)}{I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)} = \frac{(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}+[2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}]\omega}{\alpha_1\mathrm{e}^{\mathrm{i}\theta}(1-\omega)}. \tag{2.16}$$

In view of (2.6), (2.16) immediately yields the following inequality

$$|I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z)| \le \frac{|\alpha_1|(1+|z|)}{|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}||z|} \times |I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)|. \tag{2.17}$$

Also, by using (1.11) in (2.9), we have

$$I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)f(z) = \frac{1}{\alpha_1}z\varphi'(z)I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)g(z) + \varphi(z)I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z). \tag{2.18}$$

Therefore, noting that $\varphi(z)$ satisfies the inequality (2.11) and making use of (2.17) and (2.11) in (2.18), we see that

$$\begin{aligned} &|I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)f(z)| \\ &\le \left(\frac{|z|(1-|\varphi(z)|^2)}{(1-|z|)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}||z|)}+|\varphi(z)|\right) \qquad (2.19)\\ &\quad\times |I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)| \\ &= \frac{r(1-\rho^2)+(1-r)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|r)\rho}{(1-r)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|r)} \\ &\quad\times |I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)| \\ &= \frac{\Phi_2(\rho)}{(1-r)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|r)} \qquad (2.20)\\ &\quad\times |I^{m,l}_{p,q,s,\lambda}(\alpha_1+1,\beta_1)g(z)| \end{aligned}$$

$$(|z|=r,\quad |\varphi(z)|=\rho,\quad 0\le\rho\le 1,\quad z\in\Delta)$$

where the function $\Phi_2(\rho)$ defined by

$$\Phi_2(\rho)=r(1-\rho^2)+(1-r)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|r)\rho \tag{2.21}$$

takes its maximum value at $\rho=1$ with $r_2=r_2(p,\alpha_1,\gamma,\theta)$ given by (2.4). Furthermore, if $0\le\delta\le r_2(p,\alpha_1,\gamma,\theta)$, where $r_2(p,\alpha_1,\gamma,\theta)$ is given by (2.4), then the function

$$\Psi_2(\rho)=\delta(1-\rho^2)+(1-\delta)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|\delta)\rho \tag{2.22}$$

increases on the interval $0\le\rho\le 1$, so that $\Psi_2(\rho)$ does not exceed

$$\Psi_2(1)=(1-\delta)(|1+\alpha_1-p|-|2(1-\gamma)\cos\theta-(1+\alpha_1-p)\mathrm{e}^{\mathrm{i}\theta}|\delta)$$
$$(0\le\delta\le r_2(p,\alpha_1,\gamma,\theta)).$$

Therefore, from this fact, (2.20) gives the inequality (2.2). This completes the proof of Theorem 2.1. $\square$

As a special case of Theorem 2.1, when $p = 1$, we have:

Corollary 2.1. *Let the function $f \in A$ and suppose that $g \in S^{m,l}_{1,q,s,\lambda}(\gamma, \theta)$. If $I^{m,l}_{1,q,s,\lambda}(\alpha_1, \beta_1)f(z)$ is majorized by $I^{m,l}_{1,q,s,\lambda}(\alpha_1, \beta_1)g(z)$ in Δ, then*

(i)

$$\left|I^{m+1,l}_{1,q,s,\lambda}(\alpha_1, \beta_1)f(z)\right| \le \left|I^{m+1,l}_{1,q,s,\lambda}(\alpha_1, \beta_1)g(z)\right| \qquad (|z| \le r_3), \tag{2.23}$$

(ii)

$$\left|I^{m,l}_{1,q,s,\lambda}(\alpha_1 + 1, \beta_1)f(z)\right| \le \left|I^{m,l}_{1,q,s,\lambda}(\alpha_1 + 1, \beta_1)g(z)\right| \qquad (|z| \le r_4), \tag{2.24}$$

where

$$r_3 = r_3(\lambda, l, \gamma, \theta) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4|1+l||2\lambda(1-\gamma)\cos\theta - (1+l)e^{i\theta}|}}{2|2\lambda(1-\gamma)\cos\theta - (1+l)e^{i\theta}|} \tag{2.25}$$

$(\eta_1 = 2\lambda + |1+l| + |2\lambda(1-\gamma)\cos\theta - (1+l)e^{i\theta}|;\ \ \lambda, l \ge 0;\ \ 0 \le \gamma < 1;\ \ -\frac{\pi}{2} < \theta < \frac{\pi}{2})$

and

$$r_4 = r_4(\alpha_1, \gamma, \theta) = \frac{\eta_2 - \sqrt{\eta_2^2 - 4|\alpha_1||2(1-\gamma)\cos\theta - \alpha_1 e^{i\theta}|}}{2|2(1-\gamma)\cos\theta - \alpha_1 e^{i\theta}|} \tag{2.26}$$

$(\eta_2 = 2 + |\alpha_1| + |2(1-\gamma)\cos\theta - \alpha_1 e^{i\theta}|;\quad \alpha_1 \in C;\quad 0 \le \gamma < 1;\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2})$.

Setting $l = 0$ in Corollary 2.1, we get:

Corollary 2.2. *Let the function $f \in A$ and suppose that $g \in S^{m,0}_{1,q,s,\lambda}(\gamma, \theta)$. If $D^m_\lambda(\alpha_1, \beta_1)f(z)$ is majorized by $D^m_\lambda(\alpha_1, \beta_1)g(z)$ in Δ, then*

(i)

$$\left|D^{m+1}_\lambda(\alpha_1, \beta_1)f(z)\right| \le \left|D^{m+1}_\lambda(\alpha_1, \beta_1)g(z)\right| \qquad (|z| \le r_5),$$

(ii)

$$|D^m_\lambda(\alpha_1 + 1, \beta_1)f(z)| \le |D^m_\lambda(\alpha_1 + 1, \beta_1)g(z)| \qquad (|z| \le r_6),$$

where

$$r_5 = r_5(\lambda, \gamma, \theta) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4|2\lambda(1-\gamma)\cos\theta - e^{i\theta}|}}{2|2\lambda(1-\gamma)\cos\theta - e^{i\theta}|}$$

$(\eta_1 = 2\lambda + 1 + |2\lambda(1-\gamma)\cos\theta - e^{i\theta}|;\quad \lambda \ge 0;\quad 0 \le \gamma < 1;\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2})$

and

$$r_6 = r_6(\alpha_1, \gamma, \theta) = \frac{\eta_2 - \sqrt{\eta_2^2 - 4|\alpha_1||2(1-\gamma)\cos\theta - \alpha_1 \mathrm{e}^{\mathrm{i}\theta}|}}{2|2(1-\gamma)\cos\theta - \alpha_1 \mathrm{e}^{\mathrm{i}\theta}|}$$

$$(\eta_2 = 2 + |\alpha_1| + |2(1-\gamma)\cos\theta - \alpha_1 \mathrm{e}^{\mathrm{i}\theta}|; \quad \alpha_1 \in C; \quad 0 \le \gamma < 1; \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}).$$

Putting $m = \gamma = \theta = 0$ in Theorem 2.1, we obtain

Corollary 2.3. *Let the function $f \in A_p$ and suppose that $g \in S^{0,l}_{p,q,s,\lambda}(0,0)$. If $I^{0,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z)$ is majorized by $I^{0,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)g(z)$ in Δ, then*

(i)

$$\left|I^{1,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z)\right| \le \left|I^{1,l}_{p,q,s,\lambda}(\alpha_1, \beta_1)g(z)\right| \qquad (|z| \le r_7),$$

(ii)

$$\left|I^{0,l}_{p,q,s,\lambda}(\alpha_1 + 1, \beta_1)f(z)\right| \le \left|I^{0,l}_{p,q,s,\lambda}(\alpha_1 + 1, \beta_1)g(z)\right| \qquad (|z| \le r_8),$$

where

$$r_7 = r_7(\lambda, l) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4|1+l||2\lambda - l - 1|}}{2|2\lambda - l - 1|}$$

$$(\eta_1 = 2\lambda + |1+l| + |2\lambda - l - 1|; \quad \lambda, l \ge 0)$$

and

$$r_8 = r_8(\alpha_1) = \frac{\eta_2 - \sqrt{\eta_2^2 - 4|\alpha_1||2 - \alpha_1|}}{2|2 - \alpha_1|}$$

$$(\eta_2 = 2 + |\alpha_1| + |2 - \alpha_1|; \quad \alpha_1 \in C).$$

Taking $\lambda = 1$, $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = 1$ in Corollary 2.2, we obtain the result of Altintas et al. [17].

Corollary 2.4. *Let the function $f \in A$ and suppose that $g \in S^*((\gamma - 1)\mathrm{e}^{\mathrm{i}\theta}) = S^*_\theta(\gamma)$, where $0 \le \gamma < 1$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \le |g'(z)| \qquad (|z| \le r_9),$$

where

$$r_9 = r_9(\gamma, \theta) = \frac{\eta - \sqrt{\eta^2 - 4|2(1-\gamma)\cos\theta - \mathrm{e}^{\mathrm{i}\theta}|}}{2|2(1-\gamma)\cos\theta - \mathrm{e}^{\mathrm{i}\theta}|}$$

$$(\eta = 3 + |2(1-\gamma)\cos\theta - \mathrm{e}^{\mathrm{i}\theta}|; \quad 0 \le \gamma < 1; \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}).$$

Further putting $\theta = 0$ in Corollary 2.4, we get:

Corollary 2.5. *Let the function $f \in A$ and suppose that $g \in S^*(\gamma)$, where $0 \le \gamma < 1$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \le |g'(z)| \qquad (|z| \le r_{10}),$$

where

$$r_{10} = r_{10}(\gamma) = \frac{\eta - \sqrt{\eta^2 - 4|1-2\gamma|}}{2|1-2\gamma|} \qquad (\eta = 3 + |1-2\gamma|; \quad 0 \le \gamma < 1).$$

Also, putting $\gamma = 0$ in Corollary 2.5, we obtain the result of MacGregor [1].

Corollary 2.6. *Let the function $f \in A$ and suppose that $g \in S^*(0) = S^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \le |g'(z)| \qquad (|z| \le 2 - \sqrt{3}).$$

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