VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0183-5
Math. Slovaca **64** (2014), No. 1, 21–26

UPPER BOUNDS ON RELATIVE CLASS NUMBERS OF CYCLOTOMIC FIELDS

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. We explain how one can use the explicit formulas for the mean square values of L-functions which we established elsewhere to obtain explcit upper bounds on relative class numbers of cyclotomic number fields. As an example, we show that the relative class numbers of the cyclotomic fields of conductor $4p$, $p \geq 3$ a prime, are less than or equal to $8\sqrt{p}(p/16)^{(p-1)/2}$.

1. Introduction

Let $f \not\equiv 2 \pmod 4$ be the conductor of a cyclotomic field $K = \mathbf{Q}(\zeta_f)$. Let K^+ denote the maximal totally real subfield of K, i.e. K^+ is the subfield of K fixed by the complex conjugation. Let P_f be the group of primitive Dirichlet characters of conductors dividing f associated with K. Let X_f be the group of non necessarily primitive Dirichlet characters mod f. Hence P_f and X_f are both of order $\phi(f)$. Let P_f^- and X_f^- denote the subsets of the odd Dirichlet characters in P_f and X_f, respectively. Let $h_f^-, w_f, Q_f \in \{1,2\}$ denote the relative class number of K, the number of complex roots of unity in K, the Hasse unit index of K and d_f and d_f^+ denote the absolute values of the discriminants of K and K^+, respectively. We have

$$h_f^- = Q_f w_f \sqrt{d_f/d_f^+} \prod_{\chi \in P_f^-} \frac{1}{2\pi} L(1,\chi) \tag{1}$$

(e.g., see [Was]). Now, whereas there is no known simple formula for the mean square value $\sum_{\chi \in P_f^-} |L(1,\chi)|^2$ (see [Lou2]), there is the following simple mean

2010 Mathematics Subject Classification: Primary 11R18; Secondary 11R29, 11M06.
Keywords: cyclotomic field, class number, relative class number.

square value formula (see [Lou1: Théorème 2]):

$$S(1,f) := \frac{2}{\phi(f)} \sum_{\chi\in X_f^-} \left|\frac{1}{2\pi}L(1,\chi)\right|^2 = \frac{\phi(f)}{24f}\left(\prod_{p|f}\left(1+\frac{1}{p}\right)-\frac{3}{f}\right). \tag{2}$$

Assume that $f=p^m$ is a prime power. Then $Q_f=1$ (see [Was: Corollary 4.13]) and if $\tilde{\chi}\in X_f^-$ is the non necessarily primitive Dirichlet character mod f induced by $\chi\in P_f^-$, then $L(1,\chi)=L(1,\tilde{\chi})$. Hence, by (1), we have

$$h_f^- = Q_f w_f \sqrt{\frac{d_f}{d_f^+}} \prod_{\chi\in X_f^-} \frac{1}{2\pi}L(1,\chi) \le w_f\sqrt{d_f/d_f^+}\,(S(1,f))^{\phi(f)/4}, \tag{3}$$

and (2) yields, for example, the following explicit bound (see also [Lep], [Met1] and [Met2]):

$$h_p^- \le 2p\left(\frac{p}{24}\right)^{(p-1)/4} \qquad (p\ge 3) \tag{4}$$

(use $S(1,p)\le 1/24$, $w_p=2p$ and $d_p/d_p^+=p^{(p-1)/2}$). More generally, for $p\ge 3$ and $m\ge 1$ we obtain:

$$h_{p^m}^- \le 2p^m\left(\frac{p^{m-\frac{1}{p-1}+\frac{1}{\phi(p^m)}}}{24}\right)^{\phi(p^m)/4} \qquad (p\ge 3,\ \ m\ge 1) \tag{5}$$

(use $S(1,p^m)\le 1/24$, $w_{p^m}=2p^m$ and $d_{p^m}/d_{p^m}^+=\sqrt{pd_{p^m}}=p^{(p-1)/2}$, by [Was: Proposition 2.1, Lemma 4.19]).

2. The general case

Now, assume that f is not a prime power. Then $Q_f=2$ (see [Was: Corollary 4.13]) and

$$L(1,\chi)=\prod_{p|f}\left(1-\frac{\chi(p)}{p}\right)^{-1} L(1,\tilde{\chi}) \qquad (\chi\in P_f^-).$$

Hence, we have (compare with (3)):

$$h_f^- = \frac{Q_f w_f}{\Pi_f}\sqrt{\frac{d_f}{d_f^+}} \prod_{\chi\in X_f^-}\frac{1}{2\pi}L(1,\chi) \le \frac{2w_f}{\Pi_f}\sqrt{\frac{d_f}{d_f^+}}\,(S(1,f))^{\phi(f)/4}, \tag{6}$$

where

$$\Pi_f := \prod_{p|f}\prod_{\chi\in P_f^-}\left(1-\frac{\chi(p)}{p}\right).$$

Lemma 1. *For $p \geq 2$ a prime and $f > 2$, set*

$$\Pi(p,f) := \prod_{\chi \in X_f^-} \left(1 - \frac{\chi(p)}{p}\right),$$

If $f = \prod_{p|f} p^{e_p}$ and $f_p := f/p^{e_p}$, then

$$\Pi_f = \prod_{p|f} \Pi(p, f_p).$$

Assume that a prime $p \geq 2$ is coprime with $f > 2$. Let l be the order of p mod f*. Then,*

$$\Pi(p,f) = \begin{cases} (1+p^{-l/2})^{\phi(f)/l} & \text{if } l \text{ is even and } p^{l/2} \equiv -1 \pmod{f} \\ (1-p^{-l})^{\phi(f)/2l} & \text{otherwise} \end{cases}$$

and $\mathrm{e}^{-1/2l} \leq \Pi(p,f) \leq \mathrm{e}^{1/l}$*. Hence,* $\Pi(p,f) = 1 + O\Big(\frac{\log p}{\log f}\Big)$.

P r o o f. For the first assertion, notice that

$$\Pi_f = \prod_{p|f} \prod_{\substack{\chi \in P_f^- \\ \gcd(p, f_\chi)=1}} \left(1 - \frac{\chi(p)}{p}\right) = \prod_{p|f} \prod_{\chi \in X_{f_p}^-} \left(1 - \frac{\chi(p)}{p}\right),$$

where f_χ is the conductor of $\chi \in X_f^-$ For the last assertion, see [Lou1: Lemme (c)] and [Lou3: Lemma 9]. □

3. An example

Let $p \geq 3$ be an odd prime. Set $f = 4p$. Let $K = \mathbf{Q}(\zeta_f)$ be the cyclotomic field of conductor $f = 4p$. Here, $w_f = 4p$. Let χ_4 be the odd quadratic Dirichlet character of conductor 4. Let χ_p be any odd primitive Dirichlet character of conductor p and order $p-1$. Then, P_f is generated by χ_4 and χ_p, and

$P_f^- = \{\chi_4\} \cup \big\{\chi_p^k :\ 1 \leq k \leq p-2,\ k \text{ odd}\big\} \cup \big\{\chi_4\chi_p^k :\ 2 \leq k \leq p-3,\ k \text{ even}\big\}$.

Hence,

$$d_f/d_f^+ = \prod_{\chi \in P_f^-} f_\chi = 4 \times p^{(p-1)/2} \times (4p)^{(p-3)/2} = 2^{p-1}p^{p-2}.$$

We have $\Pi_f = \Pi(2,p)\Pi(p,4)$, $1 - \frac{1}{p} \leq \Pi(p,4) = 1 - \frac{\chi_4(p)}{p} = 1 - \big(\frac{-1}{p}\big)\frac{1}{p}$ (Legendre's symbol) and

$$S(1,4p) = \frac{1}{32}\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{2p}\right),$$

by (2). Using (6) and Lemma 1, we obtain:

$$h_{4p}^- \leq \frac{8\sqrt{p}(2p)^{(p-1)/2}}{\left(1-\left(\frac{-1}{p}\right)\frac{1}{p}\right)\Pi(2,p)}\left(S(1,4p)\right)^{(p-1)/2}$$

$$\leq \frac{8\sqrt{p}}{\left(1-\frac{1}{p}\right)\Pi(2,p)}\left(\left(1-\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)\frac{p}{16}\right)^{\frac{p-1}{2}}$$

$$\leq \frac{8\sqrt{p}}{\left(1-\frac{1}{p}\right)\mathrm{e}^{-\frac{\log 2}{2\log(p+1)}}}\left(\left(1-\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)\frac{p}{16}\right)^{\frac{p-1}{2}}$$

that is less than or equal to $8\sqrt{p}(p/16)^{(p-1)/2}$ for $p \geq 13$. Since $h_{4p}^- = 1$ for $p = 3, 5, 7$ and 11 (see the Tables in [Was]), we obtain:

Theorem 2. *For $p \geq 3$, it holds that*

$$h_{4p}^- \leq 8\sqrt{p}(p/16)^{(p-1)/2}.$$

4. Taking into account small primes

Now, fix $f_0 \geq 1$, a product of small distinct prime numbers $q \geq 2$. Let χ_0 be the trivial character mod f_0. Assume that f is coprime with f_0, and for $\chi \in X_f^-$, let $\chi_0\chi$ be the odd character mod $f_0 f$ induced by χ.

As in [Lou3: Lemma 2] where f was restricted to perfect prime powers, we have the following bound valid for any conductor f (compare with (6)):

$$h_f^- \leq \frac{Q_f w_f/\Pi_f}{\prod_{q|f_0}\Pi(q,f)}\left(S(f_0,f)\right)^{\phi(f)/4}, \tag{7}$$

where

$$S(f_0,f) := \frac{2}{\phi(f)}\sum_{\chi\in X_f^-}\left|\frac{1}{2\pi}L(1,\chi_0\chi)\right|^2.$$

To conclude, we apply this bound with $f_0 = 3$ to the case $f = 4p$, $p > 3$. We will use (see [Lou3: Proposition 8]): if 3 does not divide f, then

$$S(3,f) = \frac{1}{27}\prod_{p|f}\left(1-\frac{1}{p^2}\right) - \frac{1}{18f}\prod_{p|f}\left(1-\frac{1}{p}\right) + \frac{1}{54f}\left(\frac{f}{3}\right)\prod_{p|f}\left(1-\left(\frac{p}{3}\right)\frac{1}{p}\right). \tag{8}$$

Theorem 3. (Compare with Theorem 2). *For $p > 3$, it holds that*

$$h_{4p}^- \leq (8+o(1))\sqrt{p}\,(p/18)^{(p-1)/2}.$$

Proof. By (8), we have

$$S(3,4p) = \frac{1}{36}\left(1 - \frac{1-(\frac{p}{3})}{4p} - \frac{1}{p^2}\right) \leq \frac{1}{36},$$

$p > 3$. Using (7) with $f_0 = 3$ and $f = 4p$, $p > 3$, we obtain:

$$h_{4p}^- \leq \frac{8\sqrt{p}}{\left(1-\frac{1}{p}\right)\Pi(2,p)\Pi(3,4p)}\left(\frac{p}{18}\right)^{(p-1)/2}.$$

The desired result follows, by Lemma 1. □

5. Explicit bounds on $h_{p^m}^-/h_{p^{m-1}}^-$

Let $p \geq 3$ be an odd prime and $m \geq 2$ be a rational integer. Every $\chi \in X_{p^{m-1}}^-$ induces a character $\chi^* \in X_{p^m}^-$ such that $L(1,\chi^*) = L(1,\chi)$, i.e. we may and we will consider that $X_{p^{m-1}}^-$ is a subset of $X_{p^m}^-$. Set

$$\mu(p^m) = \#X_{p^m}^- - \#X_{p^{m-1}}^- = (\phi(p^m) - \phi(p^{m-1}))/2 = \frac{p^m}{2}\left(1-\frac{1}{p}\right)^2 \qquad (m \geq 2),$$

$$W_m := w_{p^m}/w_{p^{m-1}} = p \qquad (m \geq 1),$$

$$D_m := \frac{d_{p^m}/d_{p^m}^+}{d_{p^{m-1}}/d_{p^{m-1}}^+} = p^{m\mu(p^m)} \qquad (m \geq 2).$$

By (2), we have

$$S_m := \sum_{\chi \in X_{p^m}^-}\left|\frac{1}{2\pi}L(1,\chi)\right|^2 - \sum_{\chi \in X_{p^{m-1}}^-}\left|\frac{1}{2\pi}L(1,\chi)\right|^2 = \frac{1}{24}\left(1-\frac{1}{p^2}\right)\mu(p^m)$$

$$(m \geq 2).$$

The geometric mean being less than or equal to the arithmetic mean, we have

$$\frac{\prod_{\chi \in X_{p^m}^-}\frac{1}{2\pi}L(1,\chi)}{\prod_{\chi \in X_{p^{m-1}}^-}\frac{1}{2\pi}L(1,\chi)} = \prod_{\chi \in X_{p^m}^- \setminus X_{p^{m-1}}^-}\frac{1}{2\pi}L(1,\chi) \leq \left(\frac{S_m}{\mu(p^m)}\right)^{\mu(p^m)/2},$$

and, by (3), we have

$$h_{p^m}^-/h_{p^{m-1}}^- = W_m\sqrt{D_m}\frac{\prod_{\chi \in X_{p^m}^-}\frac{1}{2\pi}L(1,\chi)}{\prod_{\chi \in X_{p^{m-1}}^-}\frac{1}{2\pi}L(1,\chi)}.$$

Hence, we end up with the following bound:

Theorem 4. *For $p \geq 3$ a prime number and $m \geq 2$, it holds that*

$$h_{p^m}^- / h_{p^{m-1}}^- \leq p \left(\frac{p^m}{24} \left(1 - \frac{1}{p^2} \right) \right)^{\mu(p^m)/2}. \tag{9}$$

Notice that (9) and (4) imply (5) (inductively on $m \geq 1$).

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Received 26. 8. 2011
Accepted 19. 10. 2011

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