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PATH COMPONENTS OF THE SPACE OF GRADIENT VECTOR FIELDS ON THE TWO-DIMENSIONAL DISC

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ABSTRACT. We present a short proof that if two gradient maps on the two-dimensional disc have the same degree, then they are gradient homotopic.

Introduction

In 1990 A. Parusiński [4] proved a quite unexpected result. Namely, if two gradient vector fields on the unit disc D^n and nonvanishing in S^{n-1} are homotopic, then they are also gradient homotopic. Consequently, we can establish a bijection (induced by the inclusion) between the path-components of the space of gradient vector fields on D^n nonvanishing in S^{n-1} and those of the space of all continuous vector fields with the same boundary condition. Of course, the path-components of these function spaces are precisely the respective homotopy classes of maps.

The original proof of Parusiński's Theorem is by induction on n i.e. the dimension of the space. In the first step for $n = 2$ he shows that if two gradient fields on D^2 nonvanishing in S^1 have the same degree different from 1, then they are gradient homotopic. However, in case the degree is 1 he proves only that any gradient field is gradient homotopic either to the identity or to the minus identity, but he omits the case of gradient homotopy between the identity and minus identity. So in our paper we also fill this small gap by providing two independent proofs of that fact (see Step 2 in the proof of Main Lemma).

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The main goal of our paper is to present a new proof of Parusiński's Theorem in the case of the plane ($n = 2$). In our approach we want to emphasize strongly the geometric aspects of the proof. We also hope that our viewpoint sheds some new light on the following question: Is the mentioned inclusion of the gradient vector fields into all continuous vector fields a homotopy equivalence?

The important thing is to understand the difference between the homotopy and otopy relations. In [1] we prove Parusiński's theorem for gradient otopy classes. Although the results presented here and in [1] are partially similar, they are formally independent. Furthermore, the otopy relation is more flexible, since it relates maps with not necessarily the same domain (so called *local maps*), but it is also weaker. As a consequence, the methods used here differ essentially from those in [1].

It may be worth pointing out that even though Parusiński's result does not hold for equivariant maps, his techniques may be still used to study homotopy classes of gradient equivariant maps (see [2,3]).

The organization of the paper is as follows. Section 1 contains some preliminaries. Section 2 presents so called Parusiński's Trick which allows to replace gradient homotopy classes of gradient vector fields on two dimensional disc by homotopy classes of pairs of functions on S^1 without common zeros and one of which has integral over S^1 equal to zero. Section 3 contains some additional facts needed in the proof of the main lemma. In Section 4 our main result is stated and proved. Section 5 presents an application of the main theorem.

1. Preliminaries

Let $I = [0, 1]$. We will denote by D^n the unit disc in $\mathbb{R}^n$ and by S^{n-1} its boundary. Maps on S^1 will be identified with 2π-periodic maps on $[0, 2\pi]$. We will consider continuous maps on D^n with no zeros on its boundary and their homotopies i.e. continuous maps $h\colon D^n \times I \to \mathbb{R}^n$ nonvanishing on $S^{n-1} \times I$. Recall that a map f is called *gradient* if there is a C^1-function $\varphi\colon D^n \to \mathbb{R}$ such that $f = \nabla\varphi$. Similarly, we say that a homotopy $h\colon D^n \times I \to \mathbb{R}^n$ is *gradient* if $h(x, t) = \nabla_x \xi(x, t)$ for some continuous function ξ that is C^1 with respect to x. Of course, we still assume that f (resp. h) has no zeros on S^{n-1} (resp. $S^{n-1} \times I$).

2. Parusiński's trick in the plane

Let $\mathcal{V}$ denote the space of all continuous maps on the unit disc in $\mathbb{R}^2$ nonvanishing on its boundary i.e.

$$\mathcal{V} = C^0\big(D^2, S^1; \mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}\big)$$

and $\mathcal{G}$ denote the subspace of $\mathcal{V}$ consisting of gradient maps i.e.

$$\mathcal{G} = \{ f \in \mathcal{V} \mid f \text{ is gradient} \}.$$

Both spaces are equipped with natural compact-open topology. It is well-known that homotopy classes (path-components) of $\mathcal{V}$ are classified by the topological degree. Let us denote the path-component consisting of maps of degree k by $\mathcal{V}_k$. Furthermore, let $\mathcal{G}_k = \mathcal{G} \cap \mathcal{V}_k$.

Let $\vec{\tau}$ (resp. $\vec{\nu}$) denote a unit tangent (resp. normal) vector field on S^1. For $v \in \mathcal{V}$ there exist two (unique) functions $f, g\colon S^1 \to \mathbb{R}$, non vanishing simultaneously such that

$$v\restriction_{S^1} = f \cdot \vec{\tau} + g \cdot \vec{\nu}.$$

Thus, we obtain the natural mapping (called *tilde map*)

$$\mathcal{V} \ni v \mapsto \widetilde{v} = (f, g) \in \widetilde{\mathcal{V}},$$

where $\widetilde{\mathcal{V}} := C^0(S^1; \mathbb{R}^2 \setminus \{0\})$. Let $\widetilde{\mathcal{V}}_k := \{\widetilde{v} \in \widetilde{\mathcal{V}} \mid \deg \widetilde{v} = k\}$. It is easily seen that if $v \in \mathcal{V}_k$, then $\widetilde{v} \in \widetilde{\mathcal{V}}_{k-1}$. Set

$$\begin{aligned}\widetilde{\mathcal{G}} &:= \big\{(f,g) \in \widetilde{\mathcal{V}} \mid \big(\exists\, \eta\colon S^1 \xrightarrow{C^1} \mathbb{R}\big)(f = \eta')\big\}\\ &= \Big\{(f,g) \in \widetilde{\mathcal{V}} \mid \int_0^{2\pi} f(s)\,\mathrm{d}s = 0\Big\}.\end{aligned}$$

Clearly, if $v \in \mathcal{G}$, then $\widetilde{v} \in \widetilde{\mathcal{G}}$. Write $\widetilde{\mathcal{G}}_k = \widetilde{\mathcal{G}} \cap \widetilde{\mathcal{V}}_k$.

Example 1. For $v_0(x,y) = (x,y)$ and $v_1(x,y) = (-x,-y)$ we have $\widetilde{v}_0 = (f_0, g_0) = (0,1)$ and $\widetilde{v}_1 = (f_1, g_1) = (0,-1)$.

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \widetilde{\mathcal{G}} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{V} & \longrightarrow & \widetilde{\mathcal{V}}, \end{array} \tag{2.1}$$

where the vertical arrows are inclusions and the horizontal ones correspond to the above tilde map. If we replace all function spaces in Diagram (2.1) by their sets of path-components i.e. the sets of respective homotopy classes in these spaces, we obtain the following commutative diagram of induced maps

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & \widetilde{G} \\ \Big\downarrow & & \Big\downarrow \\ V & \longrightarrow & \widetilde{V}, \end{array} \tag{2.2}$$

Notice that the map $\Phi\colon G \to \widetilde{G}$ (induced by the tilde map) is given by the formula

$$[v] = [\nabla\varphi] \mapsto \left[\left(\frac{\partial\varphi}{\partial\theta}, \frac{\partial\varphi}{\partial r}\right)\Big|_{S^1}\right].$$

Lemma 2.1 (**Parusiński's Trick in the Plane**). *The maps induced by the tilde map (the horizontal arrows) in* Diagram (2.2) *are bijections.*

Proof. Let $(f,g) = (\eta', g) \in \widetilde{\mathcal{G}}$ represent a homotopy class in $\widetilde{G}$. If g is not smooth we can always choose a smooth function g_1 satisfying $\operatorname{sgn} g_1(s) = \operatorname{sgn} g(s)$ for all s such that $f(s) = 0$ and define a potential on D^2 by

$$\xi(\theta, r) = \varrho(r)[\eta(\theta) + (r-1)g_1(\theta)],$$

where $\varrho\colon I \to I$ is a smooth function equal to 0 near 0 and equal to 1 near 1. Note that for any two choices of such g_1, we obtain the same homotopy class of $\nabla\xi$ in G. To see this, it is enough to use the straight-line homotopy of potentials. In consequence, the assignment $[(f,g)] \mapsto [\nabla\xi]$ defines a map $\Psi\colon \widetilde{G} \to G$ which is inverse to Φ. The bijectivity of the lower arrow is due to the fact that the homotopy classes in V and $\widetilde{V}$ are enumerated by the topological degree. $\square$

Remark 1. As we stated before, the main result of this paper is the bijectivity of $G \to V$ in Diagram 2.2. We will prove it in Section 4 using the above lemma and Main Lemma, which says that $\widetilde{G} \to \widetilde{V}$ is a bijection.

3. Retraction

Consider a function $f\colon [0, 2\pi] \to \mathbb{R}$ such that $f(0) = f(2\pi)$. We will make use of the following natural notation for functions

$$f^+ := \max\{f, 0\}, \qquad f^- := \min\{f, 0\}$$

and their integrals

$$c^+(f) := \int_0^{2\pi} f^+(s)\,\mathrm{d}s,$$

$$c^-(f) := -\int_0^{2\pi} f^-(s)\,\mathrm{d}s,$$

$$c(f) := c^+(f) - c^-(f) = \int_0^{2\pi} f(s)\,\mathrm{d}s.$$

Recall that $\widetilde{\mathcal{V}}_k = \bigl\{\widetilde{v}\colon S^1 \to \mathbb{R}^2\backslash\{0\} \mid \deg \widetilde{v} = k\bigr\}$. By definition, $\widetilde{\mathcal{G}}_k = \bigl\{\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}_k \mid c(f) = 0\bigr\}$. Set

$$\begin{aligned}
\widetilde{\mathcal{V}}_k^{\#} &= \bigl\{\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}_k \mid c^+(f) > 0 \ \text{and}\ c^-(f) > 0\bigr\},\\
\widetilde{\mathcal{G}}_k^{\#} &= \bigl\{\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}_k \mid f \equiv 0\bigr\},\\
\widetilde{\mathcal{V}}_k^{*} &= \widetilde{\mathcal{V}}_k^{\#} \cup \widetilde{\mathcal{G}}_k^{\#}.
\end{aligned}$$

Note that $\widetilde{\mathcal{V}}_k^* = \{\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}_k \mid c^+(f) = 0 \equiv c^-(f) = 0\}$.

Proposition 3.1. *Let $\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}$. If $c^+(f) = 0$ or $c^-(f) = 0$, then $\deg \widetilde{v} = 0$.*

P r o o f. Suppose that $\deg \widetilde{v} \neq 0$. Then the image of the curve $\widetilde{v}$ intersects both the positive and negative x-axes, so there are $s_0, s_1 \in [0, 2\pi]$ such that $\widetilde{v}(s_0) = (f(s_0), 0)$, $\widetilde{v}(s_1) = (f(s_1), 0)$ and $f(s_0) < 0 < f(s_1)$. This clearly forces $c^+(f) > 0$ and $c^-(f) > 0$. □

By the above Proposition, for $k \neq 0$ we have

$$\widetilde{\mathcal{V}}_k = \widetilde{\mathcal{V}}_k^{\#},$$

but $\widetilde{\mathcal{V}}_0^*$ is a proper subset of $\widetilde{\mathcal{V}}_0$.

Let $R\colon \widetilde{\mathcal{V}}_k^* \to \widetilde{\mathcal{G}}_k$ be given by

$$R(\widetilde{v}) = R(f,g) := \begin{cases} \left(\frac{\min\{c^-(f), c^+(f)\}}{c^+(f)} f^+ + \frac{\min\{c^-(f), c^+(f)\}}{c^-(f)} f^-, g\right) & \text{if } \widetilde{v} \in \widetilde{\mathcal{V}}_k^{\#}, \\ \widetilde{v} & \text{if } \widetilde{v} \in \widetilde{\mathcal{G}}_k^{\#}. \end{cases}$$

Proposition 3.2. *R is a retraction.*

P r o o f. For $k \neq 0$ the continuity of R is immediate, so let $k = 0$. Note that R is obviously continuous on the open subset $\widetilde{\mathcal{V}}_0^{\#} \subset \widetilde{\mathcal{V}}_0^*$. It remains to check the continuity at points in $\widetilde{\mathcal{G}}_0^{\#}$. Since for $\widetilde{v} = (f,g) \in \widetilde{\mathcal{V}}_0^{\#}$ and $s \in [0, 2\pi]$

$$\left| \frac{\min\{c^-(f), c^+(f)\}}{c^+(f)} f^+(s) + \frac{\min\{c^-(f), c^+(f)\}}{c^-(f)} f^-(s) \right| \leq |f(s)|,$$

it follows that for $\widetilde{w} \in \widetilde{\mathcal{G}}_0^{\#}$ and $\widetilde{v} \in \widetilde{\mathcal{V}}_0^{\#}$

$$d(\widetilde{w}, \widetilde{v}) \geq d(\widetilde{w}, R\widetilde{v}) = d(R\widetilde{w}, R\widetilde{v}),$$

where d denotes the supremum metric, i.e. $d(a,b) = \sup\limits_{s \in [0.2\pi]} \|a(s) - b(s)\|$. The last inequality evidently implies the continuity of R at $\widetilde{w} \in \widetilde{\mathcal{G}}_0^{\#}$. □

Observe that R composed with the inclusion is homotopic to $\mathrm{Id}_{\widetilde{\mathcal{V}}_k^*}$ simply via the straight-line homotopy. Moreover, for $k \neq 0$ the retraction R is defined on the whole $\widetilde{\mathcal{V}}_k$. As a consequence we get the following result.

Lemma 3.1. *For $k \neq 0$ the inclusion $\widetilde{\mathcal{G}}_k \hookrightarrow \widetilde{\mathcal{V}}_k$ is a homotopy equivalence.*

4. Main result

Let us formulate the main result of this paper.

Main Theorem. *For each $k \in \mathbb{Z}$ any two maps in $\mathcal{G}_k$ are gradient homotopic.*

Remark 2. In other words, the left vertical arrow in Diagram (2.2) (the map induced by the inclusion $\mathcal{G} \hookrightarrow \mathcal{V}$) is an injection. In fact, it is also a surjection, since

- $\widetilde{v}_k(s) = (\cos(ks), \sin(ks)) \in \widetilde{\mathcal{G}}_k$ for each $k \in \mathbb{Z}$,
- Diagram (2.2) commutes,
- the horizontal arrows in Diagram (2.2) are bijections.

Consequently, all maps in Diagram (2.2) are bijections.

By Lemma 2.1 our Main Theorem is an immediate consequence of the following result.

Main Lemma. *Any two maps in $\widetilde{\mathcal{G}}_k$ are homotopic in $\widetilde{\mathcal{G}}_k$, i.e. if $\widetilde{v}_0, \widetilde{v}_1 \in \widetilde{\mathcal{G}}_k$, then there exists a homotopy $h\colon S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$ such that*

- *$h_0 = \widetilde{v}_0$ and $h_1 = \widetilde{v}_1$,*
- *$h_t \in \widetilde{\mathcal{G}}_k$ for all $t \in I$.*

Proof. For $k \neq 0$ it follows easily from Lemma 3.1, so assume that $\widetilde{v} = (f, g) \in \widetilde{\mathcal{G}}_0$. Observe that the proof will be completed by showing the following two facts

(1) $\widetilde{v}$ is homotopic in $\widetilde{\mathcal{G}}_0$ either to $(0,1)$ or to $(0,-1)$ (treated as constant maps),

(2) $(0,1)$ is homotopic in $\widetilde{\mathcal{G}}_0$ to $(0,-1)$.

Proof of Fact (1):
It will be convenient to write maps in polar coordinates as $(\theta(\alpha), r(\alpha))$ or (θ, r) for short. We use the standard transformation from polar to Cartesian coordinates $\Pi(\theta, r) := (r\cos\theta, r\sin\theta)$, which is a local diffeomorphism. Let $\widetilde{v} = (f, g) = \Pi(\theta, r) \in \widetilde{\mathcal{G}}_0$. By the definition of $\widetilde{\mathcal{G}}_0$ there is a lift $\varphi\colon [0, 2\pi] \to \mathbb{R}$ of θ (i.e. $\theta = p \circ \varphi$ with $p(x) = x \pmod{2\pi}$) with $\varphi(0) = \varphi(2\pi)$ such that for some $k \in \mathbb{Z}$ either $\varphi \equiv \pi/2 + k\pi$ or there are $x_1, x_2 \in [0, 2\pi]$ satisfying $\varphi(x_1) > \pi/2 + k\pi$ and $\varphi(x_2) < \pi/2 + k\pi$. Of course, in both cases there is $x_0 \in [0, 2\pi]$ such that $\varphi(x_0) = \pi/2 + k\pi$. Observe that homotopy

$$h_t := R\Pi\big(p \circ k_t, (1-t)r + t\big),$$

where R is the retraction from previous section and

$$k_t(x) := (1-t)\varphi(x) + t\left(\frac{\pi}{2} + k\pi\right),$$

connects $\widetilde{v}$ either to $(0,1)$ or to $(0,-1)$, which depends on the partity of k, through maps from $\widetilde{\mathcal{G}}_0$.

Proof of Fact (2):
Let $\widetilde{v}_0 = (0,1)$ and $\widetilde{v}_1 = (0,-1)$. We show that $\widetilde{v}_0$ and $\widetilde{v}_1$ are homotopic in $\widetilde{\mathcal{G}}_0$. To do that we only need to control that for every $t \in I$ our homotopy $\widetilde{v}_t = (f_t, g_t)$ satisfies the following three conditions:

- $f_t(0) = f_t(2\pi)$ and $g_t(0) = g_t(2\pi)$,
- $f_t^{-1}(0) \cap g_t^{-1}(0) = \emptyset$,
- $\int\limits_0^{2\pi} f_t \,\mathrm{d}s = 0$.

We present two different approaches to construct such a homotopy.

First Method:
The desired homotopy is shown below in Figure 1. As you can observe, on each step of homotopy we modify only one chosen function (either f_t or g_t) while the other one is fixed (doing that alternately).

Second Method:
We will make use of an auxiliary function $\gamma\colon [0,2\pi] \to \mathbb{R}$ given by

$$\gamma(s) := \begin{cases} 2s & \text{if } \ s \in \left[0, \frac{\pi}{2}\right], \\ 2\pi - 2s & \text{if } \ s \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ 2s - 4\pi & \text{if } \ s \in \left[\frac{3\pi}{2}, 2\pi\right]. \end{cases}$$

For each $t \in [0,1]$ the image of the map $\widetilde{v}_t = (f_t, g_t)$ is a closed path in $\mathbb{R}^2 \setminus \{0\}$ and the components f_t and g_t can be regarded as the coordinates of a moving point. On Figures 2-4 the continuous line represents the image of (f_t, g_t) for given t. For $t = 0$ we begin with a motionless point at $(0,1)$. Then we stretch this initial path uniformly (see Figure 2) via the homotopy

$$\widetilde{v}_t(s) := \big(\sin\left(3t\gamma(s)\right), \cos\left(3t\gamma(s)\right)\big) \qquad \text{for} \quad t \in [0, 1/3].$$

Next we rotate the whole path to the right (see Figure 3) by means of the homotopy

$$\widetilde{v}_t(s) := \big(\sin\left(\gamma(s) + (3t-1)\pi\right), \cos\left(\gamma(s) + (3t-1)\pi\right)\big) \qquad \text{for} \quad t \in [1/3, 2/3].$$

Finally we collapse the path to the single point $(0,-1)$ for $t = 1$ (see Figure 4) via

$$\widetilde{v}_t(s) := \big(\sin\left((3-3t)\gamma(s) + \pi\right), \cos\left((3-3t)\gamma(s) + \pi\right)\big) \qquad \text{for} \quad t \in [2/3, 1].$$

It is easily seen that $\int\limits_0^{2\pi} f_t(s)\,\mathrm{d}s = 0$ for all $t \in [0,1]$ and the path (f_t, g_t) is contained in the unit circle, so it never passes through the origin. $\square$

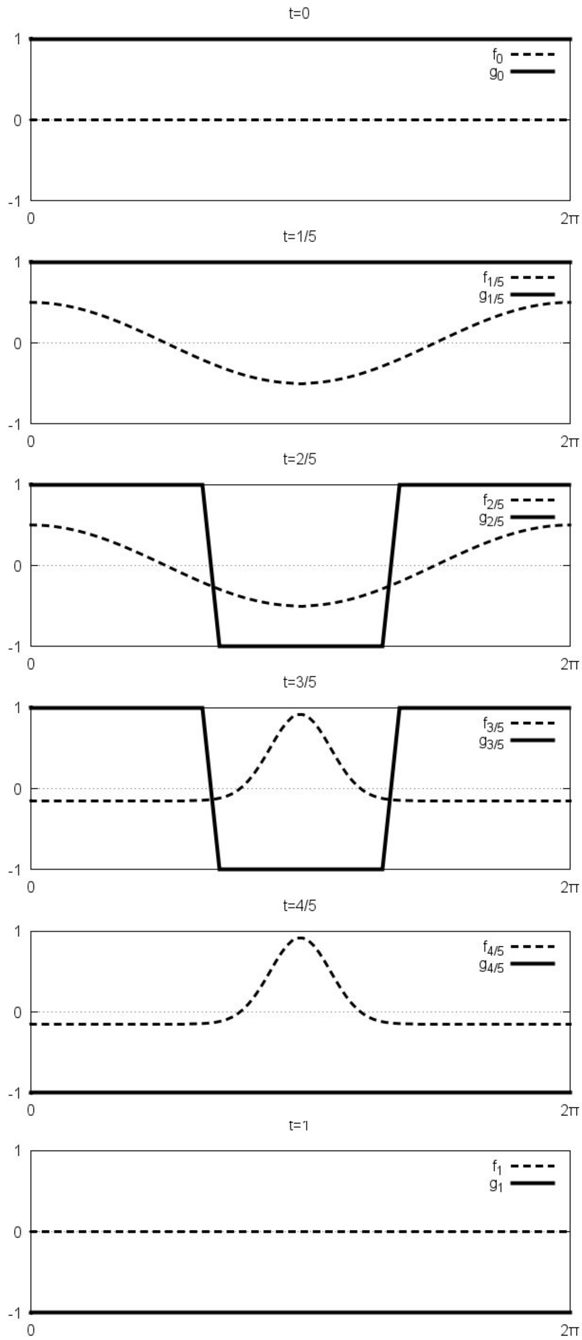

FIGURE 1. Homotopy $\widetilde{v}_t = (f_t, g_t)$ in $\widetilde{\mathcal{G}}$ – first method

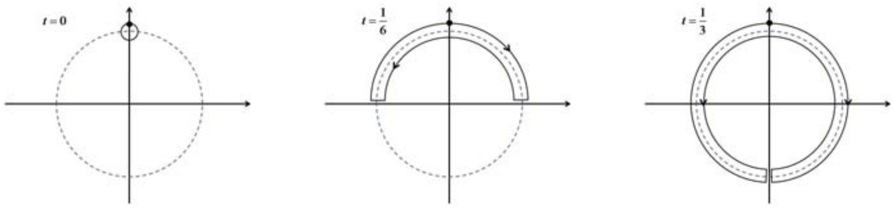

FIGURE 2. $t \in [0, 1/3]$

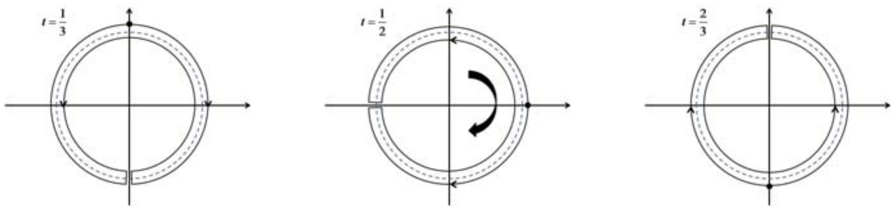

FIGURE 3. $t \in [1/3, 2/3]$

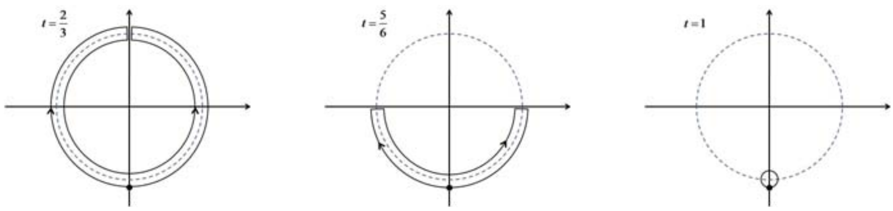

FIGURE 4. $t \in [2/3, 1]$

COROLLARY 4.1. *The identity and minus identity on the unit disc in the plane are gradient homotopic.*

Remark 3. The above result is a homotopy analogue of [1: Proposition 3.4], but it does not follow easily from it. Moreover, although [1: Proposition 3.4] has a very simple (a few lines) proof, the shortest proof of Corollary 4.1 that we know requires Parusiński's trick in the plane and Fact 2 from the previous proof, so it is definitely longer and more complicated. We believe it is not a coincidence, but it shows the real difference between the otopy and homotopy theories.

5. Application of main result

Let us mention a consequence of Main Theorem that is a homotopy version of [1: Proposition 3.5]. Let R denote the reflection $\mathrm{R}(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$.

Theorem 5.1. *If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map represented by a nonsingular symmetric matrix, then $A\restriction_{D^n}$ is gradient homotopic either to $\mathrm{Id}\restriction_{D^n}$, if $\det A > 0$, or to $\mathrm{R}\restriction_{D^n}$, if $\det A < 0$.*

P r o o f. The proof is almost identical to that of [1: Proposition 3.5]. The only difference is that in the last step we have to use Corollary 4.1 instead of [1: Proposition 3.4]. □

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