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A NOTE ON λ-COMPACT SPACES

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ABSTRACT. We extend the well-known and important fact that
"a topological space X is compact if and only if every ideal in $C(X)$ is fixed",
to more general topological spaces. Some interesting consequences are also observed. In particular, the maximality of compact Hausdorff spaces with respect to the property of compactness is generalized and the topological spaces with this generalized property are characterized.

1. Introduction

First, we recall a definition.

Definition 1.1. If R is a commutative ring, then an ideal I of R is $\mathfrak{a}$-generated, where $\mathfrak{a}$ is a cardinal number, if it admits a generating set S with $|S| \leq \mathfrak{a}$. The least element in the set of cardinal numbers of all generating sets of I is denoted by $g(I)$.

We recall that if $g(I) < \mathfrak{b}$ for each ideal I of R, where $\mathfrak{b}$ is the least regular cardinal with this property, then R is called $\mathfrak{b}$-Noetherian, see [8] for more details and the fact that X is an infinite space if and only if $C(X)$, the ring of continuous real valued functions on the topological space X, is $\mathfrak{b}$-Noetherian, where $\mathfrak{b} > \aleph_1$. We can easily see that for each $x \in X$, $g(O_x) \leq \mathfrak{b}$, where O_x denotes the set of all f in $C(X)$ for which $Z(f)$ is a neighborhood of x, see [5: 4I], if and only if x has a base of neighborhoods whose cardinality is less than or equal to $\mathfrak{b}$ (i.e., the character of x is less than or equal to $\mathfrak{b}$), which is a generalization of [5: Ex. 4I.4].

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In [8: Corollary 1.5], the classical Hilbert basis lemma (i.e., a ring R is Noetherian if and only if $R[x]$ is Noetherian) is naturally extended to all rings, by showing that for any ring R, R and $R[x]$ are $\mathfrak{a}$-Noetherian for the same regular cardinal number $\mathfrak{a}$ (note, every ring is $\mathfrak{a}$-Noetherian for some regular cardinal number $\mathfrak{a}$ and $\aleph_\circ$-Noetherian rings R are just the Noetherian rings, i.e., every ideal in R is finitely generated). In this note, motivated by the latter result we aim to extend the result of the abstract, naturally, to all topological spaces. Consequently, the known fact that whenever X, Y are compact (realcompact) spaces, then X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic is extended naturally to all topological spaces. Finally, in a series of remarks and comments we extend and slightly strengthen some useful and well-known properties of compact (Lindelöf) spaces to more general spaces. In particular, in the final result of this note, the well-known fact that compact Hausdorff topological spaces are maximal with respect to the property of compactness, is extended to general topological spaces.

We also need the following definition.

Definition 1.2. A topological space X (not necessarily Hausdorff) is said to be λ-compact whenever each open cover of X can be reduced to an open cover of X whose cardinality is less than λ, where λ is the least infinite cardinal number with this property.

We should remind the reader that our notion of λ-compactness coincides with the concept of Lindelöf$^+$ cardinal of a topological space, see [10]. In fact when a space X is λ-compact, then λ is the least cardinal such that X is a finally λ-compact space, see [9], [10]. The reader is referred to [10], where it is shown that the notion of ordinal compactness is much more applicable than cardinal compactness. Moreover, in [10] it is also rightly noted that the concept of λ-compactness is more convenient, since it distinguishes between compactness and Lindelöfness. We note that compact spaces (respectively, Lindelöf non-compact spaces) are $\aleph_\circ$-compact (respectively, $\aleph_1$-compact spaces) and in general every topological space X is λ-compact for some infinite cardinal number λ, see the concept of Lindelöf number in [2: p. 193], see also [6: Definition 1.8]. We also observe that given any infinite cardinal number λ there exists a completely regular space Y which is λ-compact and if $\lambda \geq \aleph_1$ is regular, then Y is a P-space too (note, there are no infinite compact P-spaces). In this note, as it is usual in the context of $C(X)$ whenever we deal with $C(X)$ the topological space X is always assumed to be a completely regular Hausdorff space, unless otherwise mentioned. We should also emphasize that an ideal I of a ring R is always a proper ideal, see [4] for possible undefined terms and notations. A well-known and useful result in the context of $C(X)$ asserts that a topological space X is compact if and only if each ideal I of $C(X)$ is fixed, i.e., $\bigcap_{f\in I} Z(f) \neq \varnothing$, see [5: Theorem 4.11]. Noting that in any topological space X if I is a finitely generated ideal of $C(X)$ then I is fixed, for $I = (f_1, f_2, \dots, f_n)$ implies that $\bigcap_{f\in I} Z(f) = \bigcap_{i=1}^{n} Z(f_i) = Z(f_1^2 + f_2^2 + \cdots + f_n^2) \neq \varnothing$ (note, $f_1^2 + f_2^2 + \cdots + f_n^2 \in I$ is not invertible). We can now recast the former result by saying that X is compact if and only if every ideal I of $C(X)$ with $g(I) \geq \aleph_\circ$ is fixed whenever every subideal A of I with $g(A) < \aleph_\circ$ is fixed (note, by the above comment the fact that A is fixed is always valid in $C(X)$). Motivated by this restatement of the above classical result we define an ideal I of $C(X)$ with $g(I) \geq \lambda$ to be λ-fixed, where λ is an infinite cardinal number, whenever each subideal A of I with $g(A) < \lambda$, is fixed. Consequently, X is compact if and only if every $\aleph_\circ$-fixed ideal is fixed.

We are now able to present the following general theorem.

Theorem 1.3. *X is a λ-compact space if and only if every λ-fixed ideal in $C(X)$ is fixed and λ is the least infinite cardinal number with this property.*

Proof. First, by our restatement of the result in the abstract, we may assume without loss of generality, that $\lambda > \aleph_\circ$ (i.e., X is not compact). Now let X be a λ-compact space and I be an ideal of $C(X)$ with $g(I) \geq \lambda$ which is λ-fixed. Then we are to show that I is fixed. Put $I = \sum_{k\in K} f_k C(X)$, where $|K| = g(I)$. Clearly, $\bigcap_{f\in I} Z(f) = \bigcap_{k\in K} Z(f_k)$. Hence it suffices to show that $\bigcap_{k\in K} Z(f_k) \neq \varnothing$. We claim that $\bigcap_{k\in K} Z(f_k) = \varnothing$ leads us to a contradiction. To see this, since X is λ-compact we infer that $\bigcap_{k\in T} Z(f_k) = \varnothing$, for some $T \subseteq K$ with $|T| < \lambda$. This means that the ideal $A = \sum_{k\in T} f_k C(X) \subseteq I$ is free (i.e., not fixed) which is impossible by our assumption (note, $g(A) \leq |T| < \lambda$). Finally, it remains to be shown that λ is the least infinite cardinal number with this property. Clearly, we have already proved that there exists a least infinite cardinal number $\alpha \leq \lambda$ such that every α-fixed ideal of $C(X)$, is fixed. We must show that $\alpha = \lambda$. To this end, it suffices to prove the converse. Hence suppose that every λ-fixed ideal of $C(X)$ is fixed and λ is the least infinite cardinal number with this property and we must show that X is λ-compact. Let $X = \bigcup_{f\in S} (X \setminus Z(f))$, where $|S| \geq \lambda$, then $\varnothing = \bigcap_{f\in S} Z(f)$. Put $I = \sum_{f\in S} fC(X)$ and note that it is free. We may suppose, without loss of generality that for at least one of the open covers of the above form for X, the corresponding ideal I is a proper ideal in $C(X)$, for otherwise $1 \in I$ implies that $1 = f_1 g_1 + \cdots + f_n g_n$, where $f_1, \ldots, f_n \in S$, $g_1, \ldots, g_n \in C(X)$, which implies that $\bigcap_{i=1}^{n} Z(f_i) = \varnothing$, i.e., $X = \bigcup_{i=1}^{n} (X \setminus Z(f_i))$, which means that X is compact and therefore we must have $\lambda = \aleph_\circ$ (note, we are using our restatement of the theorem above, in this case) which is absurd, i.e., we are done. Now let the ideal I be proper and free and consider two cases. First, let $g(I)$ be less than λ, hence we can put $I = \sum_{k \in K} f_k C(X)$, where $|K| = g(I)$ is less than λ. Clearly, there exists a subset L of S with $|L| \le |K| < \lambda$ which generates I, hence we may assume that K is a subset of S. Consequently, this implies that $\bigcap_{f \in I} Z(f) = \bigcap_{k \in K} Z(f_k) = \varnothing$, i.e., X is γ-compact for some infinite cardinal number $\gamma \le \lambda$. But, this implies that X is λ-compact, for we shall shortly see that $\gamma < \lambda$ leads us to a contradiction, and hence we are through. In the second case, let us assume that $g(I) \ge \lambda$, hence inasmuch as I is free, we infer that it is not λ-fixed and therefore, by our assumption, there must exist a subideal A of I with $g(A) < \lambda$ such that A is free. Put $A = \sum_{l \in L} h_l C(X)$, where $|L| < \lambda$ (note, L is an infinite set by the comment preceding the theorem). But, for each $l \in L$, we have $h_l \in I$, i.e., $h_l \in (f_1, \ldots, f_r)$, for some $f_1, \ldots, f_r \in S$. This shows that there exists a subset $S' \subset S$ such that $A \subseteq \sum_{f \in S'} f C(X)$ and $|S'| \le |L| < \lambda$. Clearly, $\bigcap_{f \in A} Z(f) \supseteq \bigcap_{f \in S'} Z(f)$, hence $\bigcap_{f \in S'} Z(f) = \varnothing$ (note, A is free), which implies that $X = \bigcup_{f \in S'} (X \setminus Z(f))$, $|S'| < \lambda$. This shows that X is γ-compact for some infinite cardinal number $\gamma \le \lambda$. We claim that $\gamma = \lambda$, for otherwise if $\gamma < \lambda$, then by what we have already proved in the first part, that there exists an infinite cardinal number $\beta \le \gamma$ such that every β-fixed ideal I of $C(X)$, is fixed and we arrive at a contradiction by the minimality of λ. □

Next, we recall that if X is any space and I is a free ideal in $C(X)$, then $g(I) \ge \aleph_\circ$, hence each maximal ideal M of $C(X)$ containing I is $\aleph_\circ$-fixed. Again motivated by this trivial fact, let us say that a space X has λ-free property, where λ is an infinite cardinal number, if whenever I is a free ideal in $C(X)$ with $g(I) \ge \gamma \le \lambda$, where γ is a cardinal number, then any maximal ideal M of $C(X)$ containing I is γ-fixed (note, we have already observed that any space X has $\aleph_\circ$-free property).

Now we are able to fully generalize [4: Theorem 4.11].

Corollary 1.4. *Let X be a topological space with λ-free property. Then the following statements are equivalent.*

(1) *X is λ-compact.*

(2) *Every λ-fixed ideal in $C(X)$ is fixed and λ is the least infinite cardinal number with this property.*

(3) *Every λ-fixed prime ideal P of $C(X)$ is fixed and λ is the least infinite cardinal number with this property.*

(4) *Every λ-fixed maximal ideal M of $C(X)$ is fixed and λ is the least infinite cardinal number with this property.*

Proof.

(1) $\Longrightarrow$ (2) It is evident by the above theorem.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) We note trivially that every λ-fixed prime (maximal) ideal in $C(X)$ is fixed. Hence it remains to be shown that λ is the least infinite cardinal number in parts (3) and (4). Clearly, it suffices to show that it is the least cardinal number in part (4). Now let $\gamma \leq \lambda$ be the least cardinal number such that each γ-fixed maximal ideal in $C(X)$ is fixed, then we claim that each γ-fixed ideal in $C(X)$, is fixed and then by (2) we must have $\gamma = \lambda$. Hence let I be a γ-fixed ideal in $C(X)$. We show that if I is free we obtain a contradiction. Now let I be free and hence by the λ-free property of X every maximal ideal of $C(X)$ containing I is γ-fixed, which must be fixed by our assumption, hence I is fixed which is absurd.

(4) $\Longrightarrow$ (1) By the previous theorem and (4) it suffices to show that every λ-fixed ideal of $C(X)$ is fixed. To this end, let I be a λ-fixed ideal of $C(X)$. We are to show that I is fixed. If not, let M be a maximal ideal of $C(X)$ containing I, then by the λ-free property of X, M is λ-fixed. But by (4) M is fixed, a fortiori, I is fixed, which is absurd. $\square$

It is well-known that if $\mathrm{Max}(C(X))$, the set of maximal ideals of $C(X)$, is considered as a topological space with the Zariski topology (it is called hull-kernel topology and also Stone topology in [4]), then its subspace $\mathrm{FMax}(C(X))$, the set of fixed maximal ideals in $C(X)$, is dense in it and in fact X, βX are homeomorphic with $\mathrm{FMax}(C(X))$ and $\mathrm{Max}(C(X))$, respectively, see [4: p 105]. It is manifest that if X, Y are any two topological spaces, then every ring isomorphism between $C(X)$ and $C(Y)$ takes real maximal ideals to real maximal ideals. But, real maximal ideals are fixed in compact (realcompact) spaces, hence we immediately obtain the well-known fact which says if X and Y are two compact (realcompact) spaces, then they are homeomorphic if and only if $C(X) \cong C(Y)$. Next, we aim to extend this fact to all topological spaces. Let us first define an isomorphism $\varphi\colon C(X) \to C(Y)$ to be λ-fixed isomorphism if whenever M is a fixed maximal ideal in $C(X)$ $(C(Y))$, then for any subideal $I \subseteq M$ of M

with $g(I) < \lambda$, $\varphi(I)$ ($\varphi^{-1}(I)$) is fixed (note, if there is such an isomorphism between $C(X)$ and $C(Y)$, we say that X and Y are λ-fixedly isomorphic). Clearly, every isomorphism $\varphi\colon C(X) \to C(Y)$ is $\aleph_\circ$-fixed isomorphism. It is also interesting to note that if X and Y are compact (realcompact) spaces, then any isomorphism between $C(X)$ and $C(Y)$ is λ-fixed isomorphism for every cardinal number λ. We should remind the reader that if $\vartheta\colon X \to Y$ is a homeomorphism between two spaces X and Y, then it is manifest that the natural isomorphism $\varphi\colon C(X) \to C(Y)$, where $\varphi(f) = f\vartheta^{-1}$ for all $f \in C(X)$, is λ-fixed isomorphism for any cardinal number λ.

The following result, which is a generalization of the previous fact concerning compact (realcompact) spaces, is now in order.

Theorem 1.5. *Two λ-compact spaces X and Y are homeomorphic if and only if they are λ-fixedly isomorphic.*

Proof. Let $\varphi\colon C(X) \to C(Y)$ be a λ-fixed isomorphism and M be a fixed maximal ideal of $C(X)$. If $g(M) < \lambda$, then $\varphi(M)$ is fixed by our assumption and if $g(M) \geq \lambda$, then clearly M is λ-fixed. But in the latter case $g(\varphi(M)) \geq \lambda$ and for any subideal A of $\varphi(M)$ with $g(A) < \lambda$, there exists a subideal I of M with $g(I) < \lambda$ such that $\varphi(I) = A$. Since φ is λ-fixed isomorphism, we infer that A is fixed, hence $\varphi(M)$ is λ-fixed. Consequently, in view of Theorem 1.3, $\varphi(M)$ is fixed. This shows that $\varphi(\mathrm{FMax}(C(X))) = \mathrm{FMax}(C(Y))$ and since $\mathrm{Max}(C(X))$ endowed with the Zariski topology is algebraic invariant, we infer that $\mathrm{FMax}(C(X))$ and $\mathrm{FMax}(C(Y))$ are homeomorphic. Thus by the comment preceding the theorem X and Y are homeomorphic too. The converse is evident, by the comment preceding the theorem. $\square$

Clearly, if X is any space and M is a real maximal ideal in $C(X)$ with $g(M) < \aleph_1$, then M is fixed and it is taken to a fixed maximal ideal under any isomorphism (note, M is generated by an idempotent, see [4: Theorem 5.14] and for a more general result see [1: Proposition 3.1, Corollary 3.5]). Now in view of Theorem 1.3, we observe that if X is a Lindelöf space (i.e., a λ-compact space with $\lambda \leq \aleph_1$) then the canonical mapping $f \to f^{\upsilon}$ of $C(X)$ onto $C(\upsilon X)$ is $\aleph_1$-fixed isomorphism, see [4: Remark 8.8] (note, we are not using the fact that X is realcompact) and therefore the realcompactness of X follows from the following more general result.

Corollary 1.6. *Let X be a λ-compact space such that the canonical mapping between $C(X)$ and $C(\upsilon X)$ is λ-fixed isomorphism. Then X is realcompact.*

Proof. Consider the isomorphism $\varphi\colon C(X) \to C(\upsilon X)$, $\varphi(f) = f^{\upsilon}$, see [5: Remark 8.8]. Clearly, $\varphi(\mathrm{FMax}(C(X))) \subseteq \mathrm{FMax}(C(\upsilon X))$ and we aim to show that the equality holds. Now let M be a fixed maximal ideal of $C(\upsilon X)$ with $g(M) < \lambda$, then $\varphi^{-1}(M)$ is a fixed ideal by our hypothesis. Finally, let M be a fixed maximal ideal of $C(\upsilon X)$ with $g(M) \geq \lambda$, hence $g(\varphi^{-1}(M)) \geq \lambda$ which implies that $\varphi^{-1}(M)$ is λ-fixed, by our hypothesis. Now in view of Theorem 1.3, we infer that $\varphi^{-1}(M)$ is fixed. Thus, we have already proved that the equality that we were after, holds, i.e., $\varphi(\mathrm{FMax}(C(X)) = \mathrm{FMax}(C(\upsilon X))$ which in turn implies that X and υX are homeomorphic, hence X is realcompact. $\square$

It is well-known that a discrete space X is a realcompact space if and only if its cardinal is nonmeasurable, see [5: Theorem 12.2]. In [1: Corollary 5.11], it is observed that a discrete space X is realcompact if and only if υX is first countable, see also [2: p. 240].

We also have the following fact.

Corollary 1.7. *Let X be a discrete space with $|X| = \lambda$. Then λ is nonmeasurable if and only if the canonical mapping $f \to f^{\upsilon}$ of $C(X)$ onto $C(\upsilon X)$ is λ^{+}-fixed isomorphism, where λ^{+} is the least cardinal number greater than λ.*

Although every infinite discrete space of cardinality λ is λ^{+}-compact. But for the sake of completeness we show that for any infinite cardinal number λ there exists a non-discrete completely regular (even, normal) Hausdorff space which is λ-compact. In what follows, we shall remove the blanket assumption of complete regularity on topological spaces, but still every space is Hausdorff, unless otherwise mentioned. The next example, which is a nondiscrete P-space, is more general than the one given in [4: 4N].

Example 1.8. Just take (X, τ) to be a discrete space with $|X| = \lambda$ and similar to the one-point compactification construction of X define $Y = X^{*} = X \cup \{a\}$, $a \notin X$, $\tau^{*} = \tau \cup \big\{G_a \subseteq Y :\ a \in G_a,\ \ |Y \setminus G_a| < \lambda\big\}$. It is clear that (Y, τ^{*}) is a Hausdorff λ-compact space. To see that this space is completely regular (in fact, normal), we just recall that a Hausdorff space whose set of nonisolated points is finite, is normal. Moreover, if $\lambda \geq \aleph_1$ is a regular cardinal number, one can show that for any cardinal number $\gamma < \lambda$, the intersection $\bigcap\limits_{i \in I} G_i$, where every G_i is open in Y and $|I| \leq \gamma$, is open (note, let us for the sake of brevity, call a space with the latter property a P_{λ}-space, where λ is any cardinal, regular or not).

Example 1.9. Let $Z = \bigoplus\limits_{i \in I} X_i$ be the free sum of the mutually disjoint spaces X_i, where $|I| < \lambda$, λ a regular cardinal number, and each X_i is a homeomorphic copy of the space Y in the above example. Then it is manifest that Z is a λ-compact

Hausdorff normal space and the set of its nonisolated points has cardinality $|I|$. Moreover, Z is a P_λ-space.

Considering Y, Z as prototypes of λ-compact spaces with these properties, we conclude this note with the following interesting remarks.

Remark 1.10. Since every closed subspace of a λ-compact space is γ-compact for some cardinal number $\gamma \leq \lambda$, we infer that the topology τ^* in Example 1.8, is the only topology containing τ which makes Y a λ-compact space.

It is well-known that every compact Hausdorff space is maximal compact. The next remark is a generalization of this fact.

Remark 1.11. Let (X, τ) be a λ-compact Hausdorff space which is also a P_λ-space. It follows that τ is a maximal λ-compact topology on X.

We know that every compact (i.e., $\aleph_\circ$-compact) subspace of a Hausdorff space X (note, X is λ-compact for some $\lambda \geq \aleph_\circ$, and X is always $P_{\aleph_\circ}$-space) is closed. To extend this fact to γ-compact subspaces, we may simply say, as we observed in Remark 1.11, that every γ-compact subspace of a P_γ-space X, is closed. The next remark which is also a natural generalization of this fact, gives us more insight.

Remark 1.12. Let for a fixed cardinal number γ, X be a P_γ-space. Then every β-compact subspace of X, where $\beta \leq \gamma$, is closed. Moreover, if X is a λ-compact space, where λ is a regular cardinal number, and every γ-compact subspace of X,where $\gamma \leq \lambda$, is closed, then X is a P_λ-space.

It is well-known that every countable subset of a P-space (i.e., $P_{\aleph_1}$-space) is closed and discrete, see [4: 4K.1]. The previous remark leads us to a stronger result, namely, every Lindelöf subspace of a P-space is closed and conversely, whenever every Lindelöf subspace of a Lindelöf space X is closed, then X must be a P-space. As a consequence of the previous remark, we also infer that if X is a P_λ-space, then every subset A of X whose cardinality is less than λ is closed and discrete and moreover if X is completely regular, then A is C-embedded in X, which is a generalization of [4: 4K.2], see also [4: 3L.1].

Let us recall that no infinite compact space X (i.e., $\aleph_\circ$-compact with $|X| \geq \aleph_\circ$) is a P-space (i.e., $P_{\aleph_1}$-space). The next remark is an extension of this fact to λ-compact spaces.

Remark 1.13. No λ-compact space X with $|X| \geq \lambda$ can be a P_{λ^+}-space.

In contrast with the previous remark, we have the following immediate fact.

Remark 1.14. Let X be a space with $|X| = \lambda$. Then X is discrete if and only if it is λ^+-compact, P_{λ^+}-space.

The next remark shows that certain γ-compact subspaces in maximal topologies are closed.

Remark 1.15. If (X, τ) is a maximal λ-compact space (not necessarily Hausdorff), then each β-compact subspace of X, where $\beta \leq \lambda$, is closed. To see this, let Y be a nonclosed β-compact subspace of X, where $\beta \leq \lambda$, and seek a contradiction. Let us consider the topology τ' on X generated by $\tau \cup \{Y^c\}$, where Y^c is the complement of Y in X. We claim that (X, τ') is also λ-compact and since it is clearly finer than (X, τ) we get the desired contradiction. Hence let $\{G_i : i \in I\}$ be an open cover of (X, τ'), where $|I| \geq \lambda$. Clearly each G_i is of the form $G_i = (Y^c \cap H_i) \cup E_i$, where H_i and E_i are in τ. We also note that $Y \subseteq \bigcup\limits_{i \in I} E_i$ and since Y is β-compact, we infer that $Y \subseteq \bigcup\limits_{i \in K \subseteq I} E_i$, where $|K| < \beta \leq \lambda$. Put $G = \bigcup\limits_{i \in K} E_i$ and note that G^c as a closed subset of (X, τ) is γ-compact for some $\gamma \leq \lambda$. Since $G^c \subseteq Y^c$, we infer that $G^c \cap G_i = (G^c \cap H_i) \cup (G^c \cap E_i) = G^c \cap (H_i \cup E_i)$. But $G^c \subseteq \bigcup\limits_{i \in I} G_i$ and the latter equalities imply that $G^c \subseteq \bigcup\limits_{i \in I} (H_i \cup E_i)$ and since G^c is γ-compact in (X, τ) we have $G^c \subseteq \bigcup\limits_{i \in L} (H_i \cup E_i)$, where $L \subseteq I$ and $|L| < \gamma \leq \lambda$. It follows that $G^c \subseteq \bigcup\limits_{i \in L} ((G^c \cap H_i) \cup E_i) \subseteq \bigcup\limits_{i \in L} ((Y^c \cap H_i) \cup E_i) = \bigcup\limits_{i \in L} G_i$. Consequently, $X = G \cup G^c = (\bigcup\limits_{i \in K} E_i) \cup (\bigcup\limits_{i \in L} G_i) = (\bigcup\limits_{i \in K} G_i) \cup (\bigcup\limits_{i \in L} G_i)$ and since $|L \cup K| < \lambda$, we infer that (X, τ') is μ-compact for some $\mu \leq \lambda$. Noting that τ' is finer than τ we must have $\lambda \leq \mu$, hence we are done.

In the light of Remarks 1.11, 1.12, and 1.15, we can now present our final result in this article which generalizes [11: Theorem 2, Theorem 3].

Theorem 1.16. *Let X be a λ-compact Hausdorff space, where λ is a regular cardinal number. Then the following statements are equivalent.*

(1) *X is P_λ-space.*

(2) *X is a maximal λ-compact space.*

(3) *Every γ-compact subspace of X, where $\gamma \leq \lambda$, is closed.*

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