VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0175-5
Math. Slovaca **63** (2013), No. 6, 1347–1360

# HENSTOCK-KURZWEIL-PETTIS INTEGRAL AND WEAK TOPOLOGIES IN NONLINEAR INTEGRAL EQUATIONS ON TIME SCALES

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(*Communicated by Christian Pötzsche*)

ABSTRACT. The goal of the present work is to give an existence result for a nonlinear integral equation on time scales by considering the Banach space endowed with its weak topology. More precisely, we obtain the existence of weakly continuous solutions for an integral equation that has on the right hand side the sum of two operators, one of them continuous while the other one satisfies a partial continuity condition and some integrability (in a nonabsolute sense) assumptions.



## 1. Introduction

As it is well known, the mathematical modeling has been developed in two parallel directions: one of them consisted in considering the observed phenomena to be a continuous phenomena and the other one was to consider it a discrete process. As many examples show, a very usual situation is that of hybrid processes, for which none of the two approaches fits. For this kind of processes, the most appropriate way to study is the time scales theory that works on general closed sets of real numbers (not only on real intervals or on discrete sets, as in the previously described theories).

2010 Mathematics Subject Classification: Primary 46G10, 45G10; Secondary 26A39, 47H10, 39A12.
Keywords: Henstock-Kurzweil-Pettis integral, nonlinear integral equation, time scales, fixed point theorem.
This paper was supported by the project "Progress and development through post-doctoral research and innovation in engineering and applied sciences – PRiDE – Contract no. POSDRU/89/1.5/S/57083", project co-funded from European Social Fund through Sectorial Operational Program Human Resources 2007-2013.

Introduced in 1988 by S. Hilger in his PhD thesis (see also [25]), it received a lot of attention in the last decade; thus, the delta-measure was introduced by Guseinov [23], then the Riemann and Lebesgue delta-integrals were studied by Bohner and Guseinov [3] in the 1-dimensional case and then generalized to the n-dimensional Euclidean space, the integration on curves in the time scales plane ([4]) and the Green's formula ([5]) were obtained by Bohner and Guseinov; even the Cauchy delta integral ([28]) or the weak delta Riemann-integral ([16]) with applications were discussed. Recently, the Henstock-Kurzweil $\Delta$ and $\nabla$-integrals were taken into consideration by Peterson and Thompson [34] on bounded time scales and also by Avsec, Bannish, Johnson and Meckler [2] on unbounded time scales. For Banach-valued functions, two $\Delta$-integrals of Henstock-type were presented in [15]; the most general kind of Henstock-type $\Delta$-integral is the $\Delta$-Henstock-Kurzweil-Pettis integral that was investigated by Sikorska-Nowak [36] and also by Cichoń [15].

In the setting of Henstock-Kurzweil-Pettis $\Delta$-integral we solve on time scales a nonlinear integral equation governed by the sum of two operators: a continuous operator and an integral one

$$x(t) = f(t, x(t)) + (\mathrm{HKP}) \int_0^t g(t, s, x(s))\, \Delta s.$$

For this purpose we apply a generalization of Krasnosel'skii fixed point theorem obtained in sequentially complete locally convex spaces by Vladimirescu [39].

Our main result continues the series of existence results obtained under weak topology assumptions (we refer to [18], [27], [38], [17] in Hilbert spaces, [14], [32], [31] in reflexive spaces or [8]). As far as we know, this is the first result yielding the existence of weakly continuous solutions for integral or differential problems involving Henstock-type integrals in general Banach spaces on time scales (but also on real intervals).

## 2. Notations and preliminary facts

We start with some basic elements of time scale theory; for a survey on this subject, we refer the reader to [1], [6], [7] and references therein.

A time scale $\mathbb{T}$ is a nonempty closed set of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$ (for example $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$, where $q > 1$). For two points $a, b$ in $\mathbb{T}$, we denote by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$ the time scales interval. The key elements in developing the time scales theory are:

**Definition 2.1.** The forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T}: \ s > t\}$, respectively $\rho(t) = \sup\{s \in \mathbb{T}: \ s < t\}$. Also, $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum $M$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum $m$).

A point $t \in \mathbb{T}$ is called right dense, right scattered, left dense, left scattered, dense, respectively isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively. When considering $\sigma$ one obtains the $\Delta$ part of the theory, while $\rho$ is used for the $\nabla$ part. We will be concerned only with the $\Delta$-theory, the other one could be obtained in a quite similar way.

Let $X$ be a Banach space with norm $\|\cdot\|$, $X_w$ the space endowed with the weak topology and denote by $C(\mathbb{T}, X_w)$ the space of continuous functions on $\mathbb{T}$ taking values in $X_w$, while $\|\cdot\|_C$ stands for the supremum norm. Denote also, for some positive $R$, by

$$B_R = \big\{x \in C(\mathbb{T}, X_w): \ \|x(s)\| \le R \ \text{ for all } \ s \in \mathbb{T}\big\}.$$

**Definition 2.2.** Let $f\colon \mathbb{T} \to X$ and $t \in \mathbb{T}$. Then the $\Delta$-derivative $f^{\Delta}(t)$ is the element of $X$ (if it exists) with the property that for any $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ such that

$$\big\|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\big\| \le \varepsilon|\sigma(t) - s| \qquad \text{for all } \ s \in U.$$

**Remark 1.** Note that the time scale calculus allows the unification (and also a generalization) of treatment of differential and difference equations since, in particular,

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

We denote by $\mu_{\Delta}$ the Lebesgue measure on $\mathbb{T}$ (for its definition and properties we refer the reader to [9]). For properties of Riemann $\Delta$-integral we refer to [23] and for Lebesgue integral on time scales see [3], [6], [7] or [23].

Concerning the Henstock-type integrals, as in the case where $\mathbb{T} = \mathbb{R}$ (see [10]), two different vector-valued integrals of Henstock-type where introduced in literature.

In order to recall these notions, let $\delta = (\delta_L, \delta_R)$ be a $\Delta$-gauge, that is a pair of positive functions such that $\delta_L(t) > 0$ on $(a, b]$, $\delta_R(t) > 0$ and $\delta_R(t) \ge \sigma(t) - t$ on $[a, b)$. A partition $\mathcal{D} = \big\{[x_{i-1}, x_i]_{\mathbb{T}};\, \xi_i : \ i = 1, 2, \dots, n\big\}$ of $[a, b]_{\mathbb{T}}$ is $\delta$-fine whenever:

$$\xi_i \in [x_{i-1}, x_i] \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)], \qquad 1 \le i \le n.$$

The Cousin's Lemma for time scale domains ([34: Lemma 1.9]) yields that such a partition exists for arbitrary positive pair of functions.

**Definition 2.3.** ([15])

i) A function $f\colon [a,b]_{\mathbb{T}} \to X$ is Henstock-$\Delta$-integrable on $[a,b]_{\mathbb{T}}$ if there exists a function $F\colon [a,b]_{\mathbb{T}} \to X$ satisfying the following property: given $\varepsilon > 0$, there exists a $\Delta$-gauge $\delta$ on $[a,b]_{\mathbb{T}}$ such that for every $\delta$-fine division $\mathcal{D} = \big\{[x_{i-1},x_i]_{\mathbb{T}}, \xi_i\big\}$ of $[a,b]_{\mathbb{T}}$, we have

$$\Big\| \sum_{i=1}^{n} \big(f(\xi_i)\mu_\Delta([x_{i-1},x_i]_{\mathbb{T}}) - (F(x_i) - F(x_{i-1}))\big)\Big\| < \varepsilon.$$

Then denote $F(t)$ by (H) $\int\limits_a^t f(s)\,\Delta s$ and call it the Henstock-$\Delta$-integral of $f$ on $[a,t]_{\mathbb{T}}$.

ii) A function $f\colon [a,b]_{\mathbb{T}} \to X$ is Henstock-Lebesgue-$\Delta$-integrable on $[a,b]_{\mathbb{T}}$ if there exists a function $F\colon [a,b]_{\mathbb{T}} \to X$ satisfying the following property: given $\varepsilon > 0$, there exists a $\Delta$-gauge $\delta$ on $[a,b]_{\mathbb{T}}$ such that for every $\delta$-fine division $\mathcal{D} = \big\{[x_{i-1},x_i]_{\mathbb{T}}, \xi_i\big\}$ of $[a,b]_{\mathbb{T}}$,

$$\sum_{i=1}^{n} \| f(\xi_i)\mu_\Delta([x_{i-1},x_i]_{\mathbb{T}}) - (F(x_i) - F(x_{i-1}))\| < \varepsilon.$$

Then $F(t)$ is denoted by (HL) $\int\limits_a^t f(s)\,\Delta s$ and it is called the Henstock-Lebesgue-$\Delta$-integral of $f$ on $[a,t]_{\mathbb{T}}$.

It can be easily seen that the Henstock-Lebesgue-$\Delta$-integrability implies the $\Delta$-integrability in Henstock sense; the converse implication doesn't hold even on real intervals (in fact the two concepts are equivalent only in finite dimensional Banach spaces, see [10]). The main difference between them is that the primitive in the sense of Henstock-Lebesgue-$\Delta$-integral is continuous and $\mu_\Delta$-a.e. differentiable, while the primitive in the sense of Henstock-$\Delta$-integral is continuous, but in general is not differentiable.

We will focus our attention on the Henstock-Kurzweil-Pettis $\Delta$-integral that was first considered in literature in [15] (some basic facts on it) as well as in [36], where applications were given.

**Definition 2.4.** A function $f$ on $[a,b]_{\mathbb{T}}$ is HKP-$\Delta$-integrable if:

(1) for any $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-$\Delta$-integrable;

(2) there exists $g\colon [a,b]_{\mathbb{T}} \to X$ such that for every $t$ and $x^* \in X^*$,

$$\langle x^*, g(t)\rangle = (\mathrm{HK}) \int\limits_a^t \langle x^*, f(s)\rangle\ \Delta s.$$

We denote $g(t) = (\mathrm{HKP})\int_a^t f(s)\,\Delta s$.

As in the classical case of real intervals,

**Lemma 2.1.** ([15: Theorem 3.4]) *If $f\colon [a,b]_{\mathbb{T}} \to X$ is HKP-$\Delta$-integrable then its primitive $t \mapsto (\mathrm{HKP})\int_a^t f(s)\,\Delta s$ is weakly continuous on $[a,b]_{\mathbb{T}}$.*

Denote through our work by $\mathcal{HKP}([a,b]_{\mathbb{T}}, X)$ the space of all HKP-$\Delta$-integrable functions provided with the topology given by the seminorms:

$$\|f\|_A^{x^*} = \sup_{t\in[a,b]_{\mathbb{T}}} \left| (\mathrm{HK}) \int_a^t \langle x^*, f(s)\rangle \ \Delta s \right|$$

for each $x^* \in X^*$ (we call it the weak-Alexiewicz topology).

Let us note that in the particular case when $\mathbb{T} = \mathbb{R}$, there are many papers containing applications to differential or integral problems of Henstock, Henstock-Lebesgue or Henstock-Kurzweil-Pettis integral (see Kurzweil and Schwabik [29], [30], Chew and Flordelija [11], [12], Federson and Táboas [20], Di Piazza and Satco [19], Heikkilä, Kumpulainen and Seikkala [24] or Cichoń [13]), while the general setting of time scale domains has not been sufficiently investigated yet.

## 3. An existence result with weak topologies for nonlinear integral equations on time scales

The main tool in our proof is an extension of classical Krasnosel'skii fixed point theorem that we recall below:

**Theorem 3.1.** *Let $S$ be a nonempty bounded closed convex subset of a Banach space $X$ and $A, B\colon S \to X$ satisfy:*

(i) *there exists $L \in [0,1)$ such that $\|Ax - Ay\| \le L\|x - y\|$, for all $x, y \in S$;*

(ii) *$B$ is continuous and $B(S)$ is relatively compact;*

(iii) *$Ax + By \in S$ for every $x, y \in S$.*

*Then the operator $A + B$ has a fixed point in $S$.*

This challenging result has found many applications and has since then been generalized in many directions. One of them was to generalize the properties of the space $X$; thus, there are several studies presenting Krasnosel'skii-type fixed point results in locally convex spaces. Another direction was suggested by the difficulty to check the condition (iii) in the classical result. Thus, various sufficient conditions to ensure the discussed hypothesis were obtained (for a survey on this topic, see [33]).

In the present paper, we will apply a generalization of Krasnosel'skii fixed point theorem given by Vladimirescu [39] for locally convex space.

**Theorem 3.2.** ([39: Theorem 2.1]) *Let $X$ be a sequentially complete locally convex Hausdorff topological vector space and $\{|\cdot|_\alpha\}_{\alpha\in\Lambda}$ the family of seminorms defining its topology. Suppose that the operators $A, B\colon X \to X$ satisfy:*

a) *for every $\alpha \in \Lambda$, there exists $L_\alpha \in [0,1)$ such that*

$$|Ax - Ay|_\alpha \le L_\alpha |x-y|_\alpha, \qquad x, y \in X;$$

b) *$B$ is a compact operator, i.e. it is continuous and maps bounded sets into relatively compact sets;*

c) *for every $\alpha\in\Lambda$, there exists a continuous increasing function $\phi_\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $x \in X$:*

$$|Bx|_\alpha \le \phi_\alpha(|x|_\alpha).$$

*If in addition*

$$\limsup_{t\to\infty} \frac{\phi_\alpha(t)}{t} < 1 - L_\alpha,$$

*then the operator $A + B$ possesses a fixed point.*

The following notions are usually involved in convergence results, let us recall them for the sake of completeness (we refer to [36] and for the particular case when $\mathbb{T} = \mathbb{R}$ to [30] or [22]):

**Definition 3.1.**

i) A function $F\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous in the restricted sense (briefly, $AC_*$) on $E \subset [a,b]_{\mathbb{T}}$ if, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, whenever $\big\{[c_i,d_i]_{\mathbb{T}} :\ 1 \le i \le N\big\}$ is a finite collection of non-overlapping intervals that have endpoints in $E$ and satisfy $\sum_{i=1}^{N} \mu_\Delta([c_i,d_i]_{\mathbb{T}}) < \eta_\varepsilon$, one has $\sum_{i=1}^{N} \operatorname{osc}(F,[c_i,d_i]_{\mathbb{T}}) < \varepsilon$;

ii) $F$ is said to be generalized absolutely continuous in the restricted sense (briefly, $ACG_*$) if it is continuous and the whole interval can be written as a countable union of sets on each of which $F$ is $AC_*$;

iii) A family of real functions is uniformly $ACG_*$ if one can write the unit interval as a countable union of sets on each of which the family is uniformly $AC_*$ (i.e. the above mentioned $\eta_\varepsilon$ is the same for all elements of the family);

iv) A collection of $X$-valued functions $\mathcal{F}$ is weakly uniformly $ACG_*$ if for every $x^* \in X^*$ the family of real functions $\big\{\langle x^*, f\rangle :\ f \in \mathcal{F}\big\}$ is uniformly $ACG_*$.

**Theorem 3.3.** *Let $f\colon \mathbb{T} \times X_w \to X_w$ and $g\colon \mathbb{T} \times \mathbb{T} \times X_w \to X_w$ satisfy the following conditions:*

i) *$f$ is continuous;*

ii) *for each $x^* \in X^*$, there exists $L_{x^*} \in [0,1)$ such that, whenever $x_1, x_2$ are in $C(\mathbb{T}, X_w)$,*

$$\|\langle x^*, f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot))\rangle\|_C \le L_{x^*} \|\langle x^*, x_1 - x_2\rangle\|_C;$$

iii) *for each $t, s \in \mathbb{T}$, $x \in X_w \mapsto g(t,s,x) \in X_w$ is continuous;*

iv) *for every $x \in C(\mathbb{T}, X_w)$ and every $t \in \mathbb{T}$, the function $g(t, \cdot, x(\cdot))$ is HKP-$\Delta$-integrable;*

v) *for every $R > 0$ and $t \in \mathbb{T}$, the set of HKP-primitives*

$$\left\{ (\mathrm{HKP}) \int_0^{\cdot} g(t, s, x(s))\, \Delta s : \; x \in B_R \right\} \subset C(\mathbb{T}, X_w)$$

*is equicontinuous, weakly uniformly $ACG_*$ and pointwisely relatively weakly compact;*

vi) *for each $R > 0$, the map $t \in \mathbb{T} \mapsto g(t, \cdot, x(\cdot))$ is continuous with respect to the weak-Alexiewicz topology, uniformly in $x \in B_R$;*

vii) *for each $x^* \in X^*$, there exists a continuous increasing function $\phi_{x^*}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\sup_{t\in\mathbb{T}} \left| \int_0^t \langle x^*, g(t,s,x(s))\rangle\, \Delta s \right| \le \phi_{x^*}(\|\langle x^*, x\rangle\|_C), \qquad \textit{for all} \quad x \in C(\mathbb{T}, X_w)$$

*and*

$$\limsup_{u\to\infty} \frac{\phi_{x^*}(u)}{u} < 1 - L_{x^*}.$$

*Then the integral equation*

$$x(t) = f(t, x(t)) + (\mathrm{HKP}) \int_0^t g(t, s, x(s))\, \Delta s$$

*possesses a weakly continuous solution.*

Proof. Define the operators $A$ and $B$ on the locally convex space $C(\mathbb{T}, X_w)$ endowed with the topology of weak uniform convergence, given by the seminorms

$$\|x\|_{x^*} = \|\, \langle x^*, x\rangle \,\|_C$$

(that is complete, see [21]) by:

$$Ax(t) = f(t, x(t)) \qquad \text{and} \qquad Bx(t) = (\mathrm{HKP}) \int_0^t g(t, s, x(s))\, \Delta s.$$

Hypothesis ii) ensures that the operator $A$ satisfies the first condition in Theorem 3.2.

Let us now prove that $B$ is a compact operator.

First of all, for an arbitrary bounded subset $M \subset C(\mathbb{T}, X_w)$ there exists $R > 0$ such that $M \subset B_R$. We will now prove that conditions v) and vi) involve the equicontinuity of set $\left\{(\mathrm{HKP})\int_0^{\cdot} g(\cdot, s, x(s))\,\Delta s : \ x \in B_R\right\} \subset C(\mathbb{T}, X_w)$.

To this purpose, fix $R > 0$ and $t_0 \in \mathbb{T}$. Then, for every $\varepsilon > 0$ and $x^* \in X^*$, by hypothesis vi), one can find $\eta_{R,x^*,\varepsilon} > 0$ such that, whenever $t_1 \in \mathbb{T}$ satisfies $|t_0 - t_1| < \eta_{R,x^*,\varepsilon}$, one has

$$\|g(t_1, \cdot, x(\cdot)) - g(t_0, \cdot, x(\cdot))\|_A^{x^*} = \|\langle x^*, g(t_1, \cdot, x(\cdot)) - g(t_0, \cdot, x(\cdot))\rangle\|_A < \varepsilon$$

for all $x \in B_R$. Also, by assumption v), one can suppose that for the same $\eta_{R,x^*,\varepsilon} > 0$,

$$\left|\left\langle x^*, (\mathrm{HKP})\int_{t_0}^{t_1} g(t_0, s, x(s))\,\Delta s\right\rangle\right| < \varepsilon$$

for all $x \in B_R$ and $t_1 \in \mathbb{T}$ with $|t_0 - t_1| < \eta_{x^*,\varepsilon}$.

Then

$$\begin{aligned}
&\left|\left\langle x^*, (\mathrm{HKP})\int_0^{t_1} g(t_1, s, x(s))\,\Delta s - (\mathrm{HKP})\int_0^{t_0} g(t_0, s, x(s))\,\Delta s\right\rangle\right| \\
&\le \left|\left\langle x^*, (\mathrm{HKP})\int_0^{t_1} g(t_1, s, x(s)) - g(t_0, s, x(s))\,\Delta s\right\rangle\right| \\
&\qquad + \left|\left\langle x^*, (\mathrm{HKP})\int_{t_0}^{t_1} g(t_0, s, x(s))\,\Delta s\right\rangle\right| \\
&\le \|\langle x^*, g(t_1, \cdot, x(\cdot)) - g(t_0, \cdot, x(\cdot))\rangle\|_A + \left|\left\langle x^*, (\mathrm{HKP})\int_{t_0}^{t_1} g(t_0, s, x(s))\,\Delta s\right\rangle\right| < 2\varepsilon
\end{aligned}$$

for every $x \in B_R$ and $t_1 \in \mathbb{T}$ such that $|t_0 - t_1| < \eta_{x^*,\varepsilon}$.

Therefore, $\{Bx : \ x \in M\}$ is equicontinuous and for every $t \in \mathbb{T}$ the set $\{Bx(t) : \ x \in M\}$ is relatively compact in $X_w$. By Ascoli theorem it follows that $\{Bx : \ x \in M\} \subset C(\mathbb{T}, X_w)$ is relatively compact.

Let us now show that $B$ is continuous. Consider an arbitrary net $(x_i)_{i \in I} \subset C(\mathbb{T}, X_w)$ convergent to $x \in C(\mathbb{T}, X_w)$ and fix $t \in \mathbb{T}$. Then for each $s \in \mathbb{T}$ the net $g(t, s, x_i(s))$ weakly converges to $g(t, s, x(s))$. Besides, by hypothesis v), for every $x^* \in X^*$, the set $\left\{\left\langle x^*, \int_0^{\cdot} g(t,s,x_i(s))\,\Delta s\right\rangle\right\}$ is uniformly $ACG_*$ and equicontinuous. Applying [36: Theorem 2.14] gives that

$$(\mathrm{HKP})\int_0^{\tau} g(t,s,x_i(s))\,\Delta s \to (\mathrm{HKP})\int_0^{\tau} g(t,s,x(s))\,\Delta s \qquad \text{for all} \quad \tau\in\mathbb{T}$$

with respect to the weak topology. In particular, this implies that for every $t \in \mathbb{T}$, $Bx_i(t) \to Bx(t)$ weakly.

On the other hand, as previously discussed, the set $\{Bx_i : i \in I\} \subset C(\mathbb{T}, X_w)$ is relatively compact, whence every subnet of $(Bx_i)_i$ possesses a subnet that converges in the topology of $C(\mathbb{T}, X_w)$ to some element of this space, that turns out to be $Bx$. Consequently, the whole $(Bx_i)_i$ converges in the topology of $C(\mathbb{T}, X_w)$ to $Bx$ and thus the continuity of $B$ is proved.

Finally, condition vii) yields that for each $x^* \in X^*$, there exists a continuous increasing function $\phi_{x^*}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned}\|Bx\|_{x^*} &= \sup_{t\in\mathbb{T}}\left|\int_0^t \langle x^*, g(t,s,x(s))\rangle\,\Delta s\right| \\ &\le \phi_{x^*}(\|\langle x^*, x\rangle\|_C) = \phi_{x^*}(x) \qquad \text{for all} \quad x\in C(\mathbb{T},X_w)\end{aligned}$$

with

$$\limsup_{u\to\infty}\frac{\phi_{x^*}(u)}{u} < 1 - L_{x^*}$$

and so, the hypothesis c) in the fixed point theorem is satisfied.

Consequently, the sum of operators $A$ and $B$ possesses fixed points, whence our equation has weakly continuous solutions. □

With another type of continuity condition on $g$, we can obtain:

**Theorem 3.4.** *If function $f\colon \mathbb{T}\times X_w \to X_w$ verifies hypothesis* i), ii) *in* Theorem 3.3 *and $g\colon \mathbb{T}\times\mathbb{T}\times X_w \to X_w$ satisfies* iv), vii) *and:*

- iii′) *for each $t \in \mathbb{T}$, $x \in C(\mathbb{T}, X_w) \mapsto g(t,\cdot,x(\cdot)) \in \mathcal{HKP}([a,b]_{\mathbb{T}}, X)$ is continuous with respect to the weak Alexiewicz topology;*
- v′) *for every $R > 0$, the set $\left\{(\mathrm{HKP})\int_0^{\cdot} g(\cdot,s,x(s))\,\Delta s :\ x\in B_R\right\} \subset C(\mathbb{T}, X_w)$ is equicontinuous and pointwisely relatively weakly compact,*

*then the integral equation*

$$x(t) = f(t,x(t)) + (\mathrm{HKP})\int_0^t g(t,s,x(s))\,\Delta s$$

*possesses a weakly continuous solution.*

Proof. In the proof of the main result, only the continuity of $B$ requires a special attention. Thus, whenever $(x_i)_{i\in I} \subset C(\mathbb{T}, X_w)$ is a net that converges to $x \in C(\mathbb{T}, X_w)$, by iii′),

$$\sup_{\tau\in\mathbb{T}}\left|\left\langle x^*, (\mathrm{HKP})\int_0^\tau g(t,s,x_i(s)) - g(t,s,x(s))\,\Delta s\right\rangle\right| \to 0,$$

for each $t \in \mathbb{T}$ and $x^* \in X^*$. Therefore, $Bx_i(t) = (\mathrm{HKP})\int_0^t g(t,s,x_i(s))\,\Delta s$ weakly tends to $Bx(t) = (\mathrm{HKP})\int_0^t g(t,s,x(s))\,\Delta s$ and the rest of the proof is the same. □

Another existence result under weak topology hypothesis can be obtained by assuming the function $g$ to be continuous with respect to the last two variables (in this case, the Pettis integral will be used):

**Theorem 3.5.** *Let $f\colon \mathbb{T}\times X_w \to X_w$ and $g\colon \mathbb{T}\times\mathbb{T}\times X_w \to X_w$ satisfy conditions* i), ii) *and* vii) *together with:*

iii″) *for each $t \in \mathbb{T}$, $(s,x) \in \mathbb{T}\times X_w \mapsto g(t,s,x) \in X_w$ is continuous;*

v′) *for every $R > 0$, the set $\left\{(\mathrm{P})\int_0^\cdot g(\cdot,s,x(s))\,\Delta s : \ x \in B_R\right\} \subset C(\mathbb{T}, X_w)$ is equicontinuous and pointwisely relatively weakly compact.*

*Then the integral equation*

$$x(t) = f(t,x(t)) + (\mathrm{P})\int_0^t g(t,s,x(s))\,\Delta s$$

*possesses a weakly continuous solution.*

Proof. By [13: Lemma 15], hypothesis iii″) implies that for every $x \in C(\mathbb{T}, X_w)$ and every $t \in \mathbb{T}$, the function $g(t,\cdot,x(\cdot))$ is Pettis-integrable.

Define again the operators $A, B\colon C(\mathbb{T}, X_w) \to C(\mathbb{T}, X_w)$. Only the proof of continuity of operator $B$ has to be modified.

The following Krasnosel'skii-type result can be used to overcome the difficulty arisen from the fact that in general Banach spaces balls are not (even weakly) compact and so, continuous functions are not uniformly continuous like in the finite dimensional case or in reflexive spaces:

**Lemma 3.1.** ([26], see also [37]) *Let $h\colon I\times X_w \to X_w$ be continuous, where $I$ is a compact real interval. Then for every $x^* \in X^*$, $\varepsilon > 0$ and $x \in C(I, X_w)$ there exists a weak neighborhood $U$ of $0$ in $X$ such that $|\langle x^*, h(t,x(t)) - h(t,w(t))\rangle| \le \varepsilon$ for all $t \in I$ and $w \in C(I, X_w)$ satisfying $x(s) - w(s) \in U$, for all $s \in I$.*

Consider an arbitrary net $(x_i)_{i\in I} \subset C(\mathbb{T}, X_w)$ convergent to $x \in C(\mathbb{T}, X_w)$. Then by Lemma 3.1, for each $x^* \in X^*$ and $\varepsilon > 0$ there exists $i_{x^*,\varepsilon,t} \in I$ such that for any $i \geq i_{x^*,\varepsilon,t}$ and all $s \in \mathbb{T}$,

$$-\varepsilon < \big\langle x^*, g(t,s,x_i(s)) - g(t,s,x(s))\big\rangle < \varepsilon.$$

It follows that

$$-\varepsilon t < \int_0^t \big\langle x^*, g(t,s,x_i(s)) - g(t,s,x(s))\big\rangle \, \Delta s < \varepsilon t$$

and so,

$$\left| \int_0^t \big\langle x^*, g(t,s,x_i(s)) - g(t,s,x(s))\big\rangle \, \Delta s \right| < \varepsilon t.$$

This can be written as

$$\left| \left\langle x^*, (\mathrm{HKP}) \int_0^t g(t,s,x_i(s)) - g(t,s,x(s)) \, \Delta s \right\rangle \right| < \varepsilon t$$

whence for every $t \in \mathbb{T}$, $Bx_i(t) \to Bx(t)$ with respect to the weak topology.

Again, as in the main theorem, every subnet of $(Bx_i)_i$ possesses a subnet that converges in the topology of $C(\mathbb{T}, X_w)$ to some element of this space, that has to be $Bx$. Consequently, the whole $(Bx_i)_i$ converges in the topology of $C(\mathbb{T}, X_w)$ to $Bx$ and thus the continuity of $B$ is proved. □

**Remark 2.** In the first part of the proof of the main theorem we showed, in fact, that under the assumption vi),
the equicontinuity of the set

$$\left\{ (\mathrm{HKP}) \int_0^{\cdot} g(\cdot, s, x(s)) \, \Delta s : \ x \in B_R \right\} \subset C(\mathbb{T}, X_w)$$

and the equicontinuity of

$$\left\{ (\mathrm{HKP}) \int_0^{\cdot} g(t, s, x(s)) \, \Delta s : \ x \in B_R \right\} \subset C(\mathbb{T}, X_w) \qquad \text{for all} \quad t \in \mathbb{T}$$

are two equivalent conditions.

**Remark 3.** Concerning the superpositional hypothesis iv) we want to refer the reader to the discussion on such type of condition for Pettis, Henstock or Henstock-Kurzweil-Pettis integral that can be found in [35] or [13]. Related to our main result, by [13: Lemma 17], condition iv) is fulfilled if in addition to hypothesis iii) we impose that for each $t \in \mathbb{T}$:

a) for every $y_0 \in X$, there exists $(\mathrm{HKP}) \int_0^t g(t, s, y_0) \, \Delta s$;

b) the set $\{\langle x^*, g(t,\cdot,y(\cdot))\rangle : \ y\colon \mathbb{T} \to X \text{ step function}, \ x^* \in B^*\}$ of real-valued functions is uniformly HK-$\Delta$-integrable.

To end, notice that, analogously to the real intervals situation (see [22: Theorems 13.16, 13.29]), the uniform HK-$\Delta$-integrability condition b) in Remark 3 can be achieved by assuming the equicontinuity and the uniform weak $ACG_*$ of the set of primitives.

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Received 12. 5. 2011
Accepted 20. 10. 2011

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