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ON EQUICONVERGENCE OF NUMBER SERIES

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ABSTRACT. In many classical tests for convergence of number series monotonicity of terms of series is a basic assumption. It was shown by Liflyand, Tikhonov and Zeltser that many of these tests are applicable not only to monotone sequences but also to those from a wider class, called weak monotone. Being more general this class still does not allow zeros and too much oscillation. In this paper we extend the class of weak monotonicity to include the mentioned cases and verify that the convergence tests considered by the mentioned authors still hold on this weaker assumption.

1. Introduction

In [3] it was proved that in many classical tests for convergence of number series the monotonicity assumption can be replaced by a much weaker one, called weak monotonicity.

Definition 1. A non-negative null sequence $\{a_k\}$ is called *weak monotone*, written WMS, if for some positive absolute constant C it satisfies

$$a_k \le C a_n \qquad \text{for any} \quad k \in [n, 2n]. \tag{1.1}$$

If $\{a_k\}$ is WMS we call the series $\sum\limits_k a_k$ WM.

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Definition 2. A non-negative function f defined on $(0, \infty)$ is called *weak monotone*, written WM, if

$$f(t) \le Cf(x) \qquad \text{for any} \quad t \in [x, 2x]. \tag{1.2}$$

Here we list several results from [3] concerning weak monotonicity.

Theorem 1.1. *Let $\{a_k\}$ be a* WMS*, and let $u_1 \le u_2 \le \dots$ be a sequence of integers such that $u_k \to \infty$ and $u_{k+1} = O(u_k)$. Then the series*

$$\sum_{k=1}^{\infty} a_k, \tag{1.3}$$

converges if and only if the series

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k} \tag{1.4}$$

converges.

The following result is a dual result of Theorem 1.1 for the class of lacunary sequences. An increasing sequence $\{u_k\}$ is called *lacunary* if $u_{k+1}/u_k \ge q > 1$. A more general class of sequences is that in which each sequence can be split into finitely-many lacunary sequences (see, e.g., [1: Introduction]). In this case we write $\{u_k\} \in \Lambda$.

Proposition 1.2. *Let $\{a_k\}$ be a non-negative* WMS*, and let a sequence $\{u_k\}$ be such that $\{u_k\} \in \Lambda$ and $u_{k+1} = O(u_k)$. Then the series* (1.3)*, and the series*

$$\sum_{k=1}^{\infty} u_k |\bar{\Delta} a_{u_k}| := \sum_{k=1}^{\infty} u_k |a_{u_k} - a_{u_{k+1}}|, \tag{1.5}$$

and

$$\sum_{k=1}^{\infty} u_k a_{u_k} \tag{1.6}$$

converge or diverge simultaneously.

Theorem 1.3. *Let $\{a_k\}$ be a* WMS*. If the series* (1.3) *converges, then $k a_k$ is a null sequence.*

Theorem 1.4. *Let f be a* WM *function. Then the series*

$$\sum_{k=1}^{\infty} f(k) \tag{1.7}$$

and the integral

$$\int_1^{\infty} f(t)\, \mathrm{d}t \tag{1.8}$$

converge or diverge simultaneously.

Being more general than monotonicity weak monotonicity is still too ordered and allows neither zeroes no too much oscillation. While we know that inserting zero terms or absolutely summable sequence into a series does not influence the convergence of the series. So we need to rework WM property in order to include these cases.

Definition 3. We call a non-negative null sequence $\{a_k\}$ *weak monotone type*, written WMS*, if there exist increasing index sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that

(i) $\{\alpha_k\} \cup \{\beta_k\} = \mathbb{N}$,
(ii) $\{\alpha_k\} \cap \{\beta_k\} = \emptyset$,
(iii) $\exists r \in \mathbb{N} : \ \beta_{k+r} > \beta_k + r$,
(iv) (a_{α_i}) is WMS,
(v) $a_{\beta_k} \le C a_{\alpha_{i(k)}}$, where $i(k) := \max\{i : \ \alpha_i < \beta_k\}$.

If $\{a_k\}$ is WMS* we call the series (1.3) WM*.

We can see that a WMS* sequence consists of two parts: one of them behaves nicely (as WMS), the other part can behave quite arbitrarily, i.e. contain any terms, except the assumption that its terms should not be much larger than the terms of the first part (condition (v)). The condition (iii) in the definition means that between indexes β_k and β_{k+r} there is at least one α_i. Note that (iii) implies $\alpha_{i(k)} \ge \beta_k - r$ as well as $\alpha_{i(k+r)} > \beta_k > \alpha_{i(k)}$.

Note that the choice of $\{\alpha_k\}$, $\{\beta_k\}$ is not unique. We can take for example $\alpha'_k = \alpha_{2k}$, $\{\beta'_k\} = \{\beta_k\} \cup \{\alpha_{2k-1}\}$.

The following example gives a WMS* sequence being not WMS:

Example 1.5. Set

$$a_{2k} = \frac{1}{k}, \qquad a_{2k-1} = \frac{|\sin k|}{k}.$$

Then $\{a_k\}$ is a WMS$^* \setminus$ WMS.

To introduce a counterpart for functions, we will assume that functions to be defined on $(0, \infty)$, are locally of bounded variation, and vanishing at infinity.

Let f be non-negative function such that there exists a non-negative increasing sequence $\{\gamma_k\}$ with $\gamma_0 = 0$ and $\gamma_k \to \infty$ satisfying (i') $f(t) \le Cf(x)$ for any $t \in [x, 2x] \cap A$ and $x \in A$, where $A = \bigcup_k (\gamma_{2k-1}, \gamma_{2k}]$,

(ii') $f(t) \le Cf(\gamma_{2k})$ for any $t \in (\gamma_{2k}, \gamma_{2k+1}]$,

(iii') $\delta_1 := \sup_k (\gamma_{2k+1} - \gamma_{2k}) < \infty$,

(iv') $\delta_2 := \inf_k (\gamma_{2k} - \gamma_{2k-1}) > 0$.

Figuratively speaking $f(t)$ behaves nicely (as WM) on A and can behave badly on $B = \bigcup_k (\gamma_{2k}, \gamma_{2k+1}]$. The size of "bad" sections is bounded (cf. (iii')) and the size of "good" sections is sufficiently large (cf. (iv')). Note that the choice of a sequence $\{\gamma_k\}$ is not unique. Take for example $\gamma'_{2k-1} = \gamma_{2k-1} + \delta_2/2$ and $\gamma'_{2k} = \gamma_{2k}$.

Our aim is to define a weak monotone type (WM*) property for a function f in such a way that setting $a_k = f(k)$, we would obtain $\{a_k\} \in$ WMS*. The above mentioned conditions do not guarantee it. Take for example $f(t)$ with $f(t) = 1/t$ for $t \in [k - 2/3; k - 1/3]$ ($k \in \mathbb{N}$) and $f(t) = 0$ otherwise. So we will revise the conditions slightly:

Definition 4. We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone type*, written WM*, if there exists a non-negative increasing index sequence $\{\gamma_k\}$ with $\gamma_0 = 0$ and $\gamma_k \to \infty$ satisfying the conditions (i')–(iii').

Note that the condition (iv') is automatically fulfilled for an increasing index sequence $\{\gamma_k\}$. Now for given $f \in$ WM* setting $a_k = f(k)$, we obtain $\{a_k\} \in$ WMS*. In this case $\{\alpha_k\} = A \cap \mathbb{N}$, $\{\beta_k\} = B \cap \mathbb{N}$ and for every $j \in \mathbb{N} \cap (\gamma_{2k}, \gamma_{2k+1}]$ we have $\alpha_{i(j)} = \gamma_{2k}$.

In the following by C_1, C_2 we denote absolute constants, that may be different in different occurrences.

2. Convergence tests for series

Our first aim is to extend Schlömilch Theorem (cf. Theorem 1.1) to WMS*. First we need some auxiliary results.

Lemma 2.1. *Let $\{a_k\}$ be a* WMS*, *let $\{t_k\}$ be an index sequence and $\{q_k\}$ a non-negative sequence. Set $b_{\alpha_k} := a_{\alpha_k}$ and $b_{\beta_k} := a_{\alpha_{i(k)}}$. Then $\{b_k\}$ is a* WMS *sequence and if the series*

$$\sum_k q_k b_{t_k}$$

converges then also the series

$$\sum_k q_k a_{t_k}$$

converges.

P r o o f. The first statement of the lemma is evident. The second statement follows in view of the relations

$$\begin{aligned}\sum_{k=1}^{\infty} q_k a_{t_k} &= \sum_{t_k\in\{\alpha_i\}} q_k a_{t_k} + \sum_{t_k\in\{\beta_i\}} q_k a_{t_k}\\ &\le \sum_{t_k\in\{\alpha_i\}} q_k b_{t_k} + C\sum_{t_k\in\{\beta_i\}} q_k b_{t_k} \le C\sum_{k=1}^{\infty} q_k b_{t_k}.\end{aligned}$$

□

Proposition 2.2. *Let* (1.3) *be a* WM* *convergent series and let* $u_1 \le u_2 \le \dots$ *be a sequence of integers such that* $u_k \to \infty$ *and* $u_{k+1} = O(u_k)$. *Then the series* (1.4) *converges.*

P r o o f. Taking in Lemma 2.1 $t_k := k$ and $q_k := 1$ we get that if the series

$$\sum_k b_k \tag{2.1}$$

converges, then (1.3) converges as well. Moreover since

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_{\alpha_k} + \sum_{k=1}^{\infty} a_{\alpha_{i(k)}} \le (r+1)\sum_{k=1}^{\infty} a_k,$$

the series (1.3) and (2.1) are equiconvergent. So by Theorem 1.1 the series (1.3) converges if and only if the series

$$\sum_{k=1}^{\infty} \Delta u_k b_{u_k} \tag{2.2}$$

converges. To finish the proof we need to show that the convergence of (2.2) implies the convergence of (1.4), but this follows from Lemma 2.1 with $t_k := u_k$ and $q_k := \Delta u_k$. □

Theorem 2.3. *Let* $\{a_k\}$ *be a* WMS* *and let* $u_1 \le u_2 \le \dots$ *be a sequence of integers such that* $u_k \to \infty$, $u_{k+1} = O(u_k)$,

$$\begin{aligned}\exists s\in\mathbb{N}\ \ \forall k\in\mathbb{N}: \Big(u_k\in\{\beta_i\} \implies \big[\exists j(k)\in\{k-s,\dots,k-1\}:\\ u_{j(k)}\in\{\alpha_i\}\ \ \&\ \ \Delta u_k = O(\Delta u_{j(k)})\big]\Big).\end{aligned} \tag{2.3}$$

Then the series (1.3) *converges if and only if the series* (1.4) *converges.*

P r o o f. We define WMS $\{b_k\}$ as in Lemma 2.1. In view of Proposition 2.2 and its proof the only thing which we need to show is that the convergence of (1.4) implies the convergence of (2.2). We have

$$\begin{aligned}\sum_{k=1}^{\infty} \Delta u_k b_{u_k} &= \sum_{u_k \in \{\alpha_i\}} \Delta u_k b_{u_k} + \sum_{u_k \in \{\beta_i\}} \Delta u_k b_{u_k} \\ &= \sum_{u_k \in \{\alpha_i\}} \Delta u_k a_{u_k} + \sum_{u_k \in \{\beta_i\}} \Delta u_k b_{u_k} = \text{(I)} + \text{(II)}.\end{aligned}$$

To estimate (II) we consider $u_k \in \{\beta_i\}$ and set $\alpha(k) := \max\{\alpha_i : \ \alpha_i < u_k\}$. Since $u_{i+1} = O(u_i)$ and $j(k) \in [k-s, k-1]$ then $u_k = O(u_{j(k)})$ as well as $\alpha(k) = O(u_{j(k)})$. So $b_{u_k} = a_{\alpha(k)} \le C_1 a_{u_{j(k)}}$ for some $C_1 > 0$. In view of $\Delta u_k = O(\Delta u_{j(k)})$ we have $\Delta u_k \le M \Delta u_{j(k)}$ for some $M > 0$. Hence

$$\text{(II)} \le MC_1 \sum_{u_k \in \{\beta_i\}} \Delta u_{j(k)} a_{u_{j(k)}} \le MC_1 s \sum_{u_k \in \{\alpha_i\}} \Delta u_k a_{u_k}.$$

Altogether

$$\sum_{k=1}^{\infty} \Delta u_k b_{u_k} \le MC_1(s+1) \sum_{k=1}^{\infty} \Delta u_k a_{u_k}$$

and we are done. □

The condition (2.3) consists of two parts: the first part agrees with the condition (iii) in the definition of WMS* and states that the number of β-s staying in a row in u_k-s can not be too large. The second part states that the factor Δu_k in (1.4) for $u_k \in \{\beta_i\}$ can not be too large. Both the parts are needed as the following examples demonstrate.

Examples 2.4.

1) To define $\{u_k\}$ for $2^i \le k < 2^{i+1}$ we set $\Delta u_k = 2^i$ $(i \in \mathbb{N})$, $u_1 = 1$, $u_2 = 2$. To define $\{a_k\}$ for $u_{2^i} \le k < u_{2^{i+1}}$ $(k \notin \{u_j\})$ we set $a_k = 4^{-i}$, for $j \notin \{2^i\}$ we set $a_{u_j} = 0$ and at last we set $a_{u_{2^i}} = 4^{-i}$. Then the series

$$\sum_{k=2}^{\infty} a_k = \sum_i 2^i \cdot 2^i \cdot \frac{1}{4^i} - \sum_i (2^i - 1) \cdot \frac{1}{4^i}$$

diverges while the series

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{4^i} = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

converges.

In this example $\{u_k\} \cap \{\beta_i\} = \{u_k\} \setminus \{u_{2^i}\}$. For $2^i < k < 2^{i+1}$ we have $j(k) = 2^i$ and $\Delta u_k / \Delta u_{j(k)} = 2^i/2^i = 1$. So the second part of the condition (2.3) is fulfilled but the first part is not.

2) Now we define $u_1 = 1$, $\Delta u_{2n-1} = 2^n$ and $\Delta u_{2n} = 1$. Then $u_{2n} = 2^n + u_{2n-1} > 2^n$. We set also $a_{u_{2n-1}} = 0$ and $a_k = 1/k$ for $k \notin \{u_{2n-1}\}$. Then the series (1.3) diverges while the series

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} a_{u_{2k}} < \sum_{k=1}^{\infty} \frac{1}{2^k}$$

converges.

In this case $\{u_k\} \cap \{\beta_i\} = \{u_{2k-1}\}$ and $j(2k-1) = 2k-2$, so the first part of the condition (2.3) is satisfied. In view of $\Delta u_{2k-1}/\Delta u_{j(2k-1)} = 2^k/1 = 2^k$ the second part of (2.3) does not hold.

As a corollary of Theorem 2.3 we obtain a generalization of the well-known Cauchy Condensation Theorem.

Corollary 2.5. *Let $\{a_k\}$ be a* WMS* *such that*

$$\exists s \in \mathbb{N}\ \forall k \in \mathbb{N}: \ \Big(2^k \in \{\beta_i\} \implies \big[\exists j(k) \in \{k-s, \ldots, k-1\}: \ 2^{j(k)} \in \{\alpha_i\}\big]\Big). \tag{2.4}$$

Then the series (1.3) *and*

$$\sum_{k=1}^{\infty} 2^k a_{2^k} \tag{2.5}$$

converge or diverge simultaneously.

P r o o f. We apply Theorem 2.3 with $u_k = 2^k$. The second part of the condition (2.3) follows in view of $\Delta u_k / \Delta u_{j(k)} = 2^k / 2^{j(k)} \leq 2^k/2^{k-s} = 2^s$. $\square$

The view of conditions imposed on $\{a_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$ in Definition 3 was dictated first of all by the needs of Cauchy Condensation Theorem (Corollary 2.5). In particular, we can not ask the number of β_k-s staying in a row in a sequence $\{a_k\}$ to unboundedly increase (cf. (iii)), no matter how slowly this number increases:

Proposition 2.6. *Let $\{\alpha_k\}$ and $\{\beta_k\}$ be increasing index sequences such that* (i) *and* (ii) *are fulfilled. Suppose*

$$u_k := \alpha_{k+1} - \alpha_k \to \infty \tag{2.6}$$

is such that the condition (2.4) *is satisfied. Then we can find a sequence $\{a_k\}$ fulfilling* (iv) *and* (v) *such that* (1.3) *converges, but* (2.5) *diverges.*

P r o o f. First of all we define an increasing index sequence $\{\nu_k\}$. Set $\nu_0 := 1$ and suppose that $\nu_1 < \cdots < \nu_n$ are already chosen. We choose $\nu_{n+1} > 2\nu_n$ such that $\alpha_{\nu_{n+1}} \in \{2^i | i \in \mathbb{N}\}$ and $u_k > 4^{n+1}$ for $k \geq \nu_{n+1}$, Now we define

$$a_k := \begin{cases} \left(\sqrt{\min\limits_{\nu_n \leq i < \nu_{n+1}} u_i \nu_{n+1}}\right)^{-1}, & \text{if } \alpha_{\nu_n} \leq k < \alpha_{\nu_{n+1}} \text{ and } k \in \{\alpha_i\} \cup \{2^i\}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that in view of (2.4) the number of β_i-s with $\alpha_{\nu_n} \leq i < \alpha_{\nu_{n+1}}$ and $\beta_i \in \{2^j\}$ does not exceed $s\nu_{n+1}$.

Now

$$\sum_{k=\alpha_{\nu_1}}^{\infty} a_k = \sum_n \sum_{k=\alpha_{\nu_n}}^{\alpha_{\nu_{n+1}-1}} a_k \leq \sum_n \frac{\nu_{n+1} + s\nu_{n+1}}{\sqrt{\min\limits_{\nu_n \leq i < \nu_{n+1}} u_i \nu_{n+1}}} \leq (s+1) \sum_n \frac{1}{2^n} < \infty.$$

On the other hand

$$\begin{aligned} \sum_k 2^k a_{2^k} &\geq \sum_n \sum_{\alpha_{\nu_n} \leq 2^k < \alpha_{\nu_{n+1}}} 2^k a_{2^k} = \sum_n a_{\alpha_{\nu_n}} \sum_{k=\nu_n}^{\nu_{n+1}-1} (u_k + 1) \\ &\geq \sum_n \frac{1}{\sqrt{\min\limits_{\nu_n \leq i < \nu_{n+1}} u_i \nu_{n+1}}} (\nu_{n+1} - \nu_n) \min_{\nu_n \leq i < \nu_{n+1}} u_i \\ &\geq \frac{1}{2} \sum_n \sqrt{\min_{\nu_n \leq i < \nu_{n+1}} u_i} \geq \sum_n 2^{n-1} = \infty \end{aligned}$$

□

In the following two propositions we relax assumptions for equiconvergence of the series (1.3), (1.5) and (1.6) (see Proposition 1.2).

Proposition 2.7. *Let $\{a_k\}$ be a non-negative sequence and let $\{u_k\} \in \Lambda$. Then the series* (1.5) *and* (1.6) *converge or diverge simultaneously.*

P r o o f. Revising the proof of [3: Proposition 5.1] we observe that the convergence of (1.5) implies the convergence of (1.6) for any non-negative sequence $\{a_k\}$ and $\{u_k\} \in \Lambda$ (without assumption of WMS property for $\{a_k\}$ and $u_{k+1} = O(u_k)$). To finish the proof we need to show the reverse implication. Suppose that the series (1.6) converges. Then in view of the inequalities

$$\sum_{k=1}^{\infty} u_k |\bar{\Delta} a_{u_k}| \leq \sum_{k=1}^{\infty} u_k a_{u_k} + \sum_{k=1}^{\infty} u_k a_{u_{k+1}} \leq \sum_{k=1}^{\infty} u_k a_{u_k} + \sum_{k=1}^{\infty} u_{k+1} a_{u_{k+1}}$$

the series (1.5) converges as well. □

Proposition 2.8. *Let $\{a_k\}$ be a non-negative* WMS**, and let a sequence $\{u_k\}$ be such that $\{u_k\} \in \Lambda$, $u_{k+1} = O(u_k)$ and*

$$\exists s \in \mathbb{N} \ \ \forall k \in \mathbb{N} : \ \Big(u_k \in \{\beta_i\} \implies \big[\exists j(k) \in \{k-s, \dots, k-1\} : \ u_{j(k)} \in \{\alpha_i\}\big]\Big) \tag{2.7}$$

Then the series (1.3)*, and the series* (1.5) *and* (1.6) *converge or diverge simultaneously.*

P r o o f. In view of Proposition 2.7 we need only to prove equiconvergence of (1.3) and (1.6). We define WMS $\{b_k\}$ as in Lemma 2.1. In the proof of Proposition 2.2 we showed that the series (1.3) and (2.1) are equiconvergent. By Proposition 1.2 the series (2.1) and

$$\sum_k u_k b_{u_k} \tag{2.8}$$

are equiconvergent. Now if we take $t_k := u_k$ and $q_k := u_k$ in Lemma 2.1, we get that the convergence of (2.8) implies the convergence of (1.6).

Now suppose that (1.6) converges. We have

$$\sum_{k=1}^{\infty} u_k b_{u_k} = \sum_{u_k \in \{\alpha_i\}} u_k a_{u_k} + \sum_{u_k \in \{\beta_i\}} u_k b_{u_k} = (\mathrm{I}) + (\mathrm{II}).$$

To estimate (II) we consider $u_k \in \{\beta_i\}$ and set $\alpha(k) := \max\{\alpha_i : \ \alpha_i < u_k\}$. Since $u_{i+1} = O(u_i)$ and $j(k) \in [k-s, k-1]$ then $\alpha(k) < u_k \le M u_{j(k)}$ for some $M > 0$. So $b_{u_k} = a_{\alpha(k)} \le C_1 a_{u_{j(k)}}$ for some $C_1 > 0$. Hence

$$(\mathrm{II}) \le MC_1 \sum_{u_k \in \{\beta_i\}} u_{j(k)} a_{u_{j(k)}} \le MC_1 s \sum_{u_k \in \{\alpha_i\}} u_k a_{u_k}.$$

Altogether

$$\sum_{k=1}^{\infty} u_k b_{u_k} \le MC_1(s+1) \sum_{k=1}^{\infty} u_k a_{u_k}$$

and we are done. $\square$

Observing the proof we see that the condition (2.7) is not needed to show the implication: (1.3) converges $\implies$ (1.6) converges. The following example demonstrates necessity of the condition (2.7) for the implication: (1.6) converges $\implies$ (1.3) converges.

Example 2.9. We set $u_k := 2^k$. To define $\{a_k\}$ for $k \notin \{u_j\}$ we set $a_k = 1/(k \ln k)$, for $j \notin \{2^i\}$ we set $a_{u_j} = 0$ and at last we set $a_{u_{2^i}} = 1/(u_{2^i} \ln u_{2^i})$. So for $2^i < k < 2^{i+1}$ we have $j(k) = 2^i$ and the condition (2.7) is not satisfied. Now the series (1.3) diverges, while the series (1.6) converges, since

$$\sum_{k=1}^{\infty} u_k a_{u_k} = \sum_{i=1}^{\infty} u_{2^i} \frac{1}{u_{2^i} \ln u_{2^i}} = \sum_{i=1}^{\infty} \frac{1}{2^i \ln 2} < \infty.$$

The following result gives generalization of the celebrated Maclaurin-Cauchy integral test for WM* functions.

Theorem 2.10. *Let f be a* WM* *function. Then the series* (1.7) *and the integral* (1.8) *converge or diverge simultaneously.*

P r o o f. Setting $g(t) := f(t)$ for $t \in A$ and $g(t) := f(\gamma_{2k})$ for $t \in B$ we get the WM function $g(t)$. So the series

$$\sum_{k=1}^{\infty} g(k) \tag{2.9}$$

converges if and only if the integral

$$\int_1^{\infty} g(t)\,\mathrm{d}t \tag{2.10}$$

converges. Also the series (1.7) and (2.9) converge simultaneously (cf. the proof of Proposition 2.2). To finish the proof we need to verify that the integrals (1.8) and (2.10) converge simultaneously.

Suppose that the integral (1.8) converges. Then also the integral (2.10) converges in view of

$$\begin{aligned}
\int_{\gamma_1}^{\infty} g(t)\,\mathrm{d}t &= \sum_k \Bigg(\int_{\gamma_{2k-1}}^{\gamma_{2k}} f(t)\,\mathrm{d}t + (\gamma_{2k+1} - \gamma_{2k}) f(\gamma_{2k}) \Bigg) \\
&\le \int_{\gamma_1}^{\infty} f(t)\,\mathrm{d}t + \sum_k \frac{\gamma_{2k+1} - \gamma_{2k}}{\gamma_{2k} - \varphi_k} (\gamma_{2k} - \varphi_k) f(\gamma_{2k}) \\
&\le \int_{\gamma_1}^{\infty} f(t)\,\mathrm{d}t + \sum_k \frac{\delta_1}{\delta_2} C_1 \int_{\varphi_k}^{\gamma_{2k}} f(t)\,\mathrm{d}t \le C_2 \int_{\gamma_1}^{\infty} f(t)\,\mathrm{d}t,
\end{aligned}$$

where

$$\varphi_k := \max\Big\{ \gamma_{2k-1}, \big[\tfrac{\gamma_{2k}}{2}\big] \Big\}.$$

Suppose now that the integral (2.10) converges. Then in view of

$$\begin{aligned}\int\limits_{\gamma_1}^{\infty} f(t)\,\mathrm{d}t &\le \sum_k \Bigg(\int\limits_{\gamma_{2k-1}}^{\gamma_{2k}} g(t)\,\mathrm{d}t + C \int\limits_{\gamma_{2k}}^{\gamma_{2k+1}} f(\gamma_{2k})\,\mathrm{d}t\Bigg)\\ &\le C\sum_k \Bigg(\int\limits_{\gamma_{2k-1}}^{\gamma_{2k}} g(t)\,\mathrm{d}t + \int\limits_{\gamma_{2k}}^{\gamma_{2k+1}} g(t)\,\mathrm{d}t\Bigg) = C\int\limits_{\gamma_1}^{\infty} g(t)\,\mathrm{d}t,\end{aligned}$$

the integral (1.8) converges as well. □

As a corollary of Theorem 2.10 we get generalization of the Ermakov type test.

Corollary 2.11. *Let f be a* WM* *function and let $\varphi(t)$ be a monotone increasing, positive function having a continuous derivative and satisfying $\varphi(t) > t$ for all t large enough.*

If for t large enough

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \le q < 1, \tag{2.11}$$

then the series (1.3) *converges, while if*

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \ge 1, \tag{2.12}$$

then the series (1.3) *diverges.*

Proof. Using only assumptions on functions $\varphi(t)$ we can show (cf. the proof of [3: Theorem 2.2]) that in the case (2.11) the integral (1.8) converges and in the case (2.12) diverges. □

3. Derived Cauchy series

Using methods of Theorem 2.3 we can generalize Abel-Olivier's kth term test:

Proposition 3.1. *Let $\{a_k\}$ be a* WMS*. *If series* (1.3) *converges, then ka_k is a null sequence.*

Proof. We define the sequence $\{b_k\}$ as in Proposition 2.2. Then the series $\sum\limits_k b_k$ converges. So by Theorem 1.3 $kb_k \to 0$. Now if $k = \alpha_s$, then $ka_k = kb_k \to 0$. If $k = \beta_s$, then

$$\beta_s a_{\beta_s} \le C\beta_s a_{\alpha_{i(s)}} \le C(\alpha_{i(s)} + r)a_{\alpha_{i(s)}} \to 0.$$

□

It turns out that making additional assumptions on terms of (1.3) we can improve the result of Proposition 3.1. Here we make use of ideas of the paper [2]. As well as in [2] we call the series (2.5) *the first Cauchy derived series* of (1.3). The first Cauchy derived series of (2.5) will be called *the second Cauchy derived series* of (1.3) and so on.

Theorem 3.2. *If* (1.3) *is a convergent* WM* *satisfying* (2.4) *and its first Cauchy derived series is also* WM**, then* $n \log n a_n \to 0$.

P r o o f. By Corollary 2.5 the series (2.5) converges, so by Proposition 3.1 for the first Cauchy derived series we have

$$\begin{aligned} n 2^n a_{2^n} &\to 0, \\ (\log 2) n 2^n a_{2^n} &\to 0, \\ 2^n \log 2^n a_{2^n} &\to 0. \end{aligned}$$

Now given $k \in \mathbb{N} \cap \{\alpha_i\}$ let $n \in \mathbb{N}$ be such that $2^{n-1} \le k < 2^n$, then if $2^{n-1} \in \{\alpha_i\}$ we get

$$\begin{aligned} 0 \le k \log k a_k &< C_1 2^n \log 2^n a_{2^{n-1}} \\ &= 2C_1 \left(\frac{n}{n-1} \right) 2^{n-1} \log 2^{n-1} a_{2^{n-1}} \to 0; \end{aligned}$$

if $2^{n-1} \in \{\beta_i\}$, then since $k = O(2^{j(n-1)})$ and $j(n-1) > n-1-s$ it follows that

$$\begin{aligned} 0 &\le k \log k a_k < C_2 2^n \log 2^n a_{2^{j(n-1)}} \\ &= 2^{n-j(n-1)} C_1 \left(\frac{n}{j(n-1)} \right) 2^{j(n-1)} \log 2^{j(n-1)} a_{2^{j(n-1)}} \to 0. \end{aligned}$$

For $k = \beta_n$ by (v) we get

$$\beta_n \log \beta_n a_{\beta_n} \le C_1 \alpha_{i(n)} \log \alpha_{i(n)} a_{\alpha_{i(n)}} \to 0,$$

so the statement of the theorem follows. $\square$

Note that on contrary to Proposition 3.1 we need to impose additional condition (2.4) in Theorem 3.2. In the following example all assumptions of Theorem 3.2 except (2.4) are fulfilled but $n \log n a_n \not\to 0$.

Example 3.3. We choose a monotone sequence c_n (cf. [2: Theorem 2a]) such that $\sum_n c_n$ converges and $n \log n c_n \not\to 0$. Set $a_{2^k} := 4^{-k}$ and $a_n := c_n$ for $n \notin \{2^k\}$. Then $\sum_n a_n$ is WM* and its first derived Cauchy series is WM but $n \log n a_n \not\to 0$.

Theorem 3.4. *If* (1.3) *is a convergent* WM* *satisfying* (2.4) *and the first* k *Cauchy derived series are also* WM*, *then* $n \log n \log_{[2]} n \dots \log_{[k]} n a_n \to 0$, *where* $\log_{[2]} n = \log\log n$, $\log_{[3]} n = \log\log\log n$ *etc.*

Proof. We will prove this statement by induction. Theorem 3.2 provides a basis for the induction for $k = 1$. Assume that the statement is true for $k - 1$. Since by the assumptions of the theorem the $(k-1)$st Cauchy derived series of the first Cauchy derived (2.5) is WM*, so by the hypotheses of the induction

$$n \log n \log_2 n \dots \log_{[k-1]} n(2^n a_{2^n}) \to 0. \tag{3.1}$$

Multiplying by $\log 2$ we get that

$$2^n \log 2^n \log n \log_{[2]} n \dots \log_{[k-1]} n a_{2^n} \to 0.$$

Adding the null sequence $2^n \log 2^n \log_{[2]} 2 \log_{[2]} n \dots \log_{[k-1]} n a_{2^n}$ we get that

$$2^n \log 2^n \log_{[2]} 2^n \log_{[2]} n \log_{[3]} n \dots \log_{[k-1]} n a_{2^n} \to 0.$$

Now since for any j with $3 \le j \le k$ we have

$$\log_{[j]} 2^n = \log_{[j-2]}(\log n + \log_{[2]} 2) \le \log_{[j-2]}(\log n + \log_{[2]} 10) = \log_{[j-1]} n$$

it follows that

$$2^n \log 2^n \log_{[2]} 2^n \dots \log_{[k]} 2^n a_{2^n} \to 0. \tag{3.2}$$

The rest of the proof follows that of Theorem 3.2. $\square$

Remark 3.5. Hamming tried to finish the proof of the corresponding result (Corollary to [2: Theorem 4]) for monotone series by multiplying (3.1) by the constant $\log 2 \log_{[2]} 2 \dots \log_{[k]} 2$. This idea does not work, since $\log_{[k]} 2$ is not defined for $k \ge 3$. In the case of $k = 2$, multiplying by $\log 2 \log_{[2]} 2$ gives $2^n \log 2^n \log(\log 2)^{\log n}$ but not $2^n \log 2^n \log_{[2]} 2^n$.

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