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OSCILLATION CRITERIA FOR FOURTH-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Some new criteria for the oscillation of all solutions of certain fourth-order functional differential equations are established.

1. Introduction

This paper is concerned with some criteria for the oscillation of all solutions of fourth-order functional differential equations of the type

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)\right)^{\alpha}\right)+q(t)f\big(x[g(t)]\big)=0 \tag{1.1}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)\right)^{\alpha}\right)=q(t)f\big(x[g(t)]\big)+p(t)h\big(x[\sigma(t)]\big), \tag{1.2}$$

where the following conditions are always assumed to hold:

(i) α is the ratio of two positive odd integers;

(ii) $a, p, q \in \mathrm{C}([t_0,\infty),(0,\infty))$, and either

$$\int^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s<\infty, \tag{1.3}$$

or

$$\int^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s=\infty; \tag{1.4}$$

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(iii) $g, \sigma \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $g(t) < t$, $\sigma(t) > t$, $g'(t) \geq 0$ and $\sigma'(t) \geq 0$ for $t \geq t_0 \geq 0$, and $\lim_{t\to\infty} g(t) = \infty$;

(iv) $f, h \in \mathrm{C}(\mathbb{R},\mathbb{R})$, $xf(x) > 0$, $xh(x) > 0$, $f'(x) \geq 0$ and $h'(x) \geq 0$ for $x \neq 0$, and

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \qquad \text{for} \quad xy > 0 \tag{1.5}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \qquad \text{for} \quad xy > 0. \tag{1.6}$$

By a solution of equation (1.1) (respectively (1.2)) is meant a function $x\colon [T_x,\infty) \to \mathbb{R}$, $T_x \geq t_0$ such that x, x', x'', and $(a(x'')^\alpha)'$ are continuously differentiable and satisfy equation (1.1) (respectively (1.2)) on $[T_x,\infty)$. Our attention will be restricted to these solutions x of equation (1.1) and (1.2) which satisfy $\sup\{x(t) : \ t \geq T\} > 0$ for any $T \geq T_x$. Such a solution is said to be *oscillatory* if it has a sequence of zeros tending to infinity and *nonoscillatory* otherwise.

In the last three decades there has been an increasing interest in studying oscillation and nonoscillation of solutions of functional differential equations. Most of the work on this subject, however, has been restricted to first and second-order equations as well as equations of type (1.1) and (1.2) when $\alpha = 1$ and other higher-order equations. For recent contributions, we refer to [1–12]. It appears that little is known regarding the oscillation of equation (1.1) and (1.2). Therefore, our main goal is to establish some new criteria for the oscillation of all solutions of equation (1.1) and/or solutions x with the property that $\frac{x(t)}{t^2} \to 0$ as $t \to \infty$. Also, we present some new results for the oscillation of all solutions of equation (1.2) and/or solutions x with the property that $\frac{x(t)}{t} \to 0$ as $t \to \infty$. Most of the results of this paper are established via comparison with first and second-order equations whose oscillatory characters are known. The obtained results extend, improve and unify many of the existing results that appeared in the literature for equations related to equations (1.1) and (1.2).

2. Oscillation of equation (1.1)

In this section, we shall establish sufficient conditions for the oscillation of equation (1.1). For $t \geq t_0$, we let

$$A[t,t_0] = \int_{t_0}^{t} \int_{t_0}^{s} (ua^{-1}(u))^{1/\alpha} \, \mathrm{d}u \, \mathrm{d}s$$

and

$$m(t) = \int_t^\infty a^{-1/\alpha}(s)\,\mathrm{d}s.$$

We begin with the following result.

Theorem 2.1. *Let conditions* (i)–(iv) *and* (1.3) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \ge t_0$. *If the first-order delay equation*

$$y'(t) + cq(t)f\big(A[g(t),t_0]\big)f\big(y^{1/\alpha}[g(t)]\big) = 0 \tag{2.1}$$

for any constant $c \in (0,1)$, *all bounded solutions of the second-order delay equation*

$$z''(t) - \overline{c}q(t)f(g(t))f\big([\xi(t)-g(t)]a^{-1/\alpha}[\xi(t)]\big)f\big(z^{1/\alpha}[\xi(t)]\big) = 0 \tag{2.2}$$

for any constant $\overline{c} \in (0,1)$, *and the second-order delay equation*

$$w''(t) + q(t)f\big(\xi(t)-g(t)\big)f\big(m[\xi(t)]\big)f\big(w^{1/\alpha}[\xi(t)]\big) = 0 \tag{2.3}$$

are oscillatory, and

$$\int^\infty \left(\frac{1}{a(s)}\int_{t_0}^{s}\int_{t_0}^{v} q(u)f(g(u))f\big(m[g(u)]\big)\,\mathrm{d}u\,\mathrm{d}v\right)^{1/\alpha}\mathrm{d}s = \infty, \tag{2.4}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \ge t_0 \ge 0$. Hence $(a(x'')^\alpha)'$ is decreasing on $[t_0,\infty)$. There exists $t_1 \ge t_0$ such that one of the four possibilities

$$\begin{array}{lllll}
\text{(I)} & (a(x'')^\alpha)' > 0, & x'' > 0 \text{ and } x' > 0 & \text{on} & [t_1,\infty);\\
\text{(II)} & (a(x'')^\alpha)' > 0, & x'' < 0 \text{ and } x' > 0 & \text{on} & [t_1,\infty);\\
\text{(III)} & (a(x'')^\alpha)' > 0, & x'' > 0 \text{ and } x' < 0 & \text{on} & [t_1,\infty);\\
\text{(IV)} & (a(x'')^\alpha)' < 0, & x'' < 0 \text{ and } x' > 0 & \text{on} & [t_1,\infty)
\end{array}$$

may occur, while the other four possible cases

$$\begin{array}{llll}
(a(x'')^\alpha)' > 0, & x'' < 0 \text{ and } x' < 0 & \text{on} & [t_1,\infty);\\
(a(x'')^\alpha)' < 0, & x'' > 0 \text{ and } x' > 0 & \text{on} & [t_1,\infty);\\
(a(x'')^\alpha)' < 0, & x'' > 0 \text{ and } x' < 0 & \text{on} & [t_1,\infty);\\
(a(x'')^\alpha)' < 0, & x'' < 0 \text{ and } x' < 0 & \text{on} & [t_1,\infty)
\end{array}$$

are obviously disregarded. Now we consider Cases (I)–(IV).

Case (I). Let $k \in (0,1)$ and $t_2 = t_1/(1-k)$. Define $y = a(x'')^\alpha$. Then

$$y(t) = y(t_1) + \int_{t_1}^{t} y'(\tau)\,\mathrm{d}\tau \geq \int_{t_1}^{t} y'(\tau)\,\mathrm{d}\tau \geq y'(t)(t-t_1) = y'(t)t\left(1-\frac{t_1}{t}\right) \geq kty'(t)$$

for $t \geq t_2$. Thus

$$x''(t) \geq k_1 \left(\frac{t}{a(t)}\right)^{1/\alpha} (y'(t))^{1/\alpha} \qquad \text{for} \quad t \geq t_2,$$

where $k_1 = k^{1/\alpha}$. Integrating the above inequality twice, we have

$$x(t) \geq k_1 A[t, t_2](y'(t))^{1/\alpha} \qquad \text{for} \quad t \geq t_2$$

so that there exists $t_3 \geq t_2$ such that

$$x[g(t)] \geq k_1 A\big[g(t), t_2\big]\big(y'[g(t)]\big)^{1/\alpha} \qquad \text{for} \quad t \geq t_3. \tag{2.5}$$

Using (2.5) and (1.5) in equation (1.1), we find

$$z'(t) + cq(t)f\big(A[g(t), t_2]\big)f\big(z^{1/\alpha}[g(t)]\big) \leq 0 \qquad \text{for} \quad t \geq t_3, \tag{2.6}$$

where $c = f(k_1)$ and $z = y'$. Integrating (2.6) from t to u with $u \geq t \geq t_3$ and letting $u \to \infty$, we have

$$z(t) \geq c \int_{t}^{\infty} q(s)f\big(A[g(s), t_2]\big)f\big(z^{1/\alpha}[g(s)]\big)\,\mathrm{d}s.$$

The function z is obviously nonincreasing on $[t_3, \infty)$. Hence, by [12: Theorem 1], we conclude that there exists a positive solution y of equation (2.1) with $\lim_{t\to\infty} y(t) = 0$, which is a contradiction.

Case (II). Let $k \in (0,1)$ and $t_2 = t_1/(1-k)$. Then

$$x(t) = x(t_1) + \int_{t_1}^{t} x'(\tau)\,\mathrm{d}\tau \geq x'(t)(t-t_1) = x'(t)t\left(1-\frac{t_1}{t}\right) \geq ktx'(t)$$

for $t \geq t_2$ so that there exists $t_3 \geq t_2$ such that

$$x[g(t)] \geq kg(t)x'[g(t)] \qquad \text{for} \quad t \geq t_3. \tag{2.7}$$

Using (2.7) and (1.5) in equation (1.1), one can easily find

$$(a(y')^\alpha)''(t) + \overline{c}q(t)f(g(t))f(y[g(t)]) \leq 0 \qquad \text{for} \quad t \geq t_3, \tag{2.8}$$

where $\overline{c} = f(k)$ and $y = x'$. Clearly, we see that

$$y > 0, \quad y' < 0 \quad \text{and} \quad (a(y')^\alpha)' > 0 \qquad \text{on} \quad [t_3, \infty).$$

Now, for $t \geq s \geq t_3$, we obtain

$$y(s) = y(t) - \int_s^t y'(\tau)\,\mathrm{d}\tau \geq (t-s)(-y'(t)).$$

Replacing s and t by $g(t)$ and $\xi(t)$, respectively, there exists $t_4 \geq t_3$ such that

$$y[g(t)] \geq (\xi(t) - g(t))(-y'[\xi(t)]) = \frac{\xi(t) - g(t)}{a^{1/\alpha}[\xi(t)]}\big(-a[\xi(t)](y'[\xi(t)])^\alpha\big)^{1/\alpha} \tag{2.9}$$

for $t \geq t_4$. Using (2.9) and (1.5) in inequality (2.8), we get

$$z''(t) \geq \overline{c}q(t)f(g(t))f\big([\xi(t) - g(t)]a^{-1/\alpha}[\xi(t)]\big)f\big(z^{1/\alpha}[\xi(t)]\big) \qquad \text{for} \quad t \geq t_4,$$

where $z = -a(y')^\alpha$. Clearly, we see that $z > 0$ and $z' < 0$ on $[t_4, \infty)$. By a known result, see [8: Theorem 2.3.3], we arrive at the desired contradiction with (2.2).

Case (III). From the monotonicity of x', we get that for $t \geq s \geq t_1$,

$$x(s) = x(t) - \int_s^t x'(\tau)\,\mathrm{d}\tau \geq (t-s)(-x'(t)).$$

Replacing s and t by $g(t)$ and $\xi(t)$, respectively, there exists $t_2 \geq t_1$ such that

$$x[g(t)] \geq (\xi(t) - g(t))(-x'[\xi(t)]) \qquad \text{for} \quad t \geq t_2. \tag{2.10}$$

Using (2.10) and (1.5) in equation (1.1), we get

$$(a(y')^\alpha)''(t) - q(t)f([\xi(t) - g(t)])f(y[\xi(t)]) \geq 0 \qquad \text{for} \quad t \geq t_2, \tag{2.11}$$

where $y = -x'$. Clearly, we see that

$$(a(y')^\alpha)' < 0, \quad y' < 0 \quad \text{and} \quad y > 0 \qquad \text{on} \quad [t_2, \infty).$$

Thus for $s \geq t \geq t_2$, we get

$$-a(s)(y'(s))^\alpha \geq -a(t)(y'(t))^\alpha,$$

i.e.,

$$-\,y'(s) \geq \big(-a^{1/\alpha}(t)y'(t)\big)a^{-1/\alpha}(s). \tag{2.12}$$

Integrating (2.12) from t to u with $u \geq t \geq t_2$ and letting $u \to \infty$, we obtain

$$y(t) \geq \left(\int_t^\infty a^{-1/\alpha}(s)\,\mathrm{d}s\right) Z^{1/\alpha}(t) = m(t)Z^{1/\alpha}(t) \qquad \text{for} \quad t \geq t_2,$$

where $Z = -a(y')^\alpha$. Replacing t by $\xi(t)$, there exists $t_3 \geq t_2$ such that

$$y[\xi(t)] \geq m[\xi(t)]Z^{1/\alpha}[\xi(t)] \qquad \text{for} \quad t \geq t_3. \tag{2.13}$$

Using (2.13) and (1.5) in (2.11), we have

$$Z''(t) + q(t)f(m[\xi(t)])f(\xi(t) - g(t))f(Z^{1/\alpha}[\xi(t)]) \le 0 \qquad \text{for} \quad t \ge t_3.$$

By a well-known result, see [7, 8, 11], we arrive at the desired contradiction with (2.3).

Case (IV). Proceeding as in Case (II) above, we get the inequality (2.8) for $t \ge t_3$. Clearly, we see that

$$(a(y')^\alpha)' < 0, \quad y' < 0 \quad \text{and} \quad y > 0 \qquad \text{on} \quad [t_3, \infty),$$

where $y = x'$. As in the proof of Case (III) above, we obtain (2.12) for $t \ge t_3$. It is easy to see that there exists $t_4 \ge t_3$ such that

$$y(g(t)) \ge bm[g(t)] \qquad \text{for} \quad t \ge t_4 \tag{2.14}$$

for some constant $b > 0$. Using (2.14) and (1.5) in (2.8), we obtain

$$-(a(y')^\alpha)''(t) \ge \overline{b}q(t)f(g(t))f(m[g(t)]) \qquad \text{for} \quad t \ge t_4, \tag{2.15}$$

where $\overline{b} = \overline{c}f(b)$. Integrating (2.15) twice from t_4 to t, we get

$$-y'(t) \ge (\overline{b})^{1/\alpha} \left(\frac{1}{a(t)} \int_{t_4}^{t} \int_{t_4}^{u} q(s)f(g(s))f(m[g(s)])\,\mathrm{d}s\,\mathrm{d}u \right)^{1/\alpha}.$$

Once again, integrating the above inequality from t_4 to t, we get

$$\begin{aligned}
\infty > y(t_4) &\ge y(t_4) - y(t) \\
&\ge (\overline{b})^{1/\alpha} \int_{t_4}^{t} \left(\frac{1}{a(v)} \int_{t_4}^{v} \int_{t_4}^{u} q(s)f(g(s))f(m[g(s)])\,\mathrm{d}s\,\mathrm{d}u \right)^{1/\alpha} \mathrm{d}v \\
&\to \infty \qquad \text{as} \quad t \to \infty,
\end{aligned}$$

which is a contradiction with (2.4). This completes the proof. □

When condition (1.4) holds, we see that Cases (III) and (IV) in the proof of Theorem 2.1 are disregarded. In this case, we have the following result.

Theorem 2.2. *Let conditions* (i)–(iv) *and* (1.4) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0, \infty), \mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $\ge t_0$. *If the first-order delay equation* (2.1) *and all bounded solutions of equation* (2.2) *are oscillatory, then equation* (1.1) *is oscillatory.*

Next we are concerned with the oscillation of all bounded solutions of equation (1.1). In fact, we present the following results.

Theorem 2.3. *Let conditions* (i)–(iv) *and* (1.3) *and* (2.4) *hold, and suppose that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \ge t_0$. *If all bounded solutions of equation* (2.2) *and all solutions of equation* (2.3) *are oscillatory, then all bounded solutions of equation* (1.1) *are oscillatory.*

Proof. Let x be a bounded nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \ge t_0 \ge 0$. We only consider Cases (II), (III) and (IV) from the proof of Theorem 2.1. The rest of the proof is similar as in these cases and hence is omitted. □

When condition (1.4) holds, we see from the proof of Theorem 2.1 that Case (II) holds, and so we obtain the following result.

Theorem 2.4. *Let conditions* (i)–(iv) *and* (1.4) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \ge t_0$. *If all bounded solutions of equation* (2.2) *are oscillatory, then all bounded solutions of equation* (1.1) *are oscillatory.*

Remark 2.5. Since our results of this paper are obtained via comparison with first and second-order equations whose oscillatory characters are known, we see that many sufficient conditions for the oscillation of equation (1.1) may be drawn from the known oscillation results in [8]. The details are left to the reader.

Next, we shall study the monotone and oscillatory behavior of the solutions x of equation (1.1) with the property that $\lim\limits_{t\to\infty} \frac{x(t)}{t^2} = 0$. In what follows, we assume

$$\lim_{t\to\infty} \frac{t^2}{\int\limits_{t_0}^{t}\int\limits_{t_0}^{s} \left(\frac{u}{a(u)}\right)^{1/\alpha} \mathrm{d}u\,\mathrm{d}s} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{t^2}{\int\limits_{t_0}^{t}\int\limits_{t_0}^{s} a^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s} < \infty. \tag{2.16}$$

First, we present the following interesting result.

Theorem 2.6. *Let conditions* (i)–(iv), (1.4) *and* (2.16) *hold. If x is a nontrivial positive solution of equation* (1.1) *such that* $\frac{x(t)}{t^2} \to 0$ *as* $t \to \infty$, *then*

$$x > 0, \quad x' > 0, \quad x'' < 0 \quad \textit{and} \quad (a(x'')^\alpha)' > 0 \qquad \textit{on} \quad [t_0,\infty) \tag{2.17}$$

and

$$a(t)(x''(t))^\alpha \to 0 \quad \textit{and} \quad (a(x'')^\alpha)'(t) \to 0 \qquad \textit{monotonically as} \quad t \to \infty.$$

Proof. Let x be a nontrivial positive solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0 \geq 0$. Integrating equation (1.1) from t_0 to $t \geq t_0$, we get

$$(a(x'')^\alpha)'(t) = (a(x'')^\alpha)'(t_0) - \int_{t_0}^{t} q(s) f(x[g(s)])\, \mathrm{d}s.$$

We claim that $(a(x'')^\alpha)'(t_0) > 0$. To prove this, assume the contrary. Then $(a(x'')^\alpha)'$ is nonpositive and nonincreasing on $[t_0, \infty)$, and for some $t_1 > t_0$, we find

$$(a(x'')^\alpha)'(t_1) = (a(x'')^\alpha)'(t_0) - \int_{t_0}^{t_1} q(s) f(x[g(s)])\, \mathrm{d}s < 0,$$

i.e.,

$$(a(x'')^\alpha)'(t) \leq (a(x'')^\alpha)'(t_1) < 0 \qquad \text{for} \quad t \in [t_1, \infty).$$

Consequently,

$$a(t)(x''(t))^\alpha \to -\infty \qquad \text{as} \quad t \to \infty,$$

irrespective of $a(t_0)(x''(t_0))^\alpha$. This in turn implies $x'(t) \to -\infty$ as $t \to \infty$ and hence $x(t) \to -\infty$ as $t \to \infty$, contrary to the hypothesis that $x > 0$ on $[t_0, \infty)$. This contradiction proves $(a(x'')^\alpha)'(t_0) > 0$. Since t_0 is arbitrary, we conclude $(a(x'')^\alpha)' > 0$ on $[t_0, \infty)$. It is now easy to see that $(a(x'')^\alpha)'(t) \to 0$ as $t \to \infty$. If this was not the case, there would exist a constant $c > 0$ such that

$$(a(x'')^\alpha)' > c \qquad \text{on} \quad [t_0, \infty).$$

However, this implies

$$x(t) > c^{1/\alpha} \int_{t_0}^{t} \int_{t_0}^{s} \left(\frac{u}{a(u)} \right)^{1/\alpha} \mathrm{d}u\, \mathrm{d}s \qquad \text{for} \quad t \geq t_0,$$

contradicting with (2.16) the asymptotic behavior $\frac{x(t)}{t^2} \to 0$ as $t \to \infty$. Next, we shall prove that $x'' < 0$ on $[t_0, \infty)$. If $a(t_0)(x''(t_0))^\alpha \geq 0$, then $a(x'')^\alpha \geq 0$ on $[t_0, \infty)$ and there would exist constants $b > 0$ and $T_1 > t_0$ such that $a(x'')^\alpha > b$ on $[T_1, \infty)$. However, this again leads to the contradiction that

$$x(t) > b^{1/\alpha} \int_{T_1}^{t} \int_{T_1}^{s} a^{-1/\alpha}(u)\, \mathrm{d}u\, \mathrm{d}s \qquad \text{for} \quad t \geq T_1.$$

Thus $a(t_0)(x''(t_0))^\alpha < 0$ and $a(t)(x''(t))^\alpha < 0$ since t_0 is arbitrary. Moreover, we must have $a(t)(x''(t))^\alpha \to 0$ as $t \to \infty$, for otherwise we would again be led to the contradiction that $x(t) \to 0$ as $t \to \infty$. Continuing this process, we deduce that $x' > 0$ on $[t_0, \infty)$. This completes the proof. □

In order to characterize the behavior of solutions, we may reformulate Theorem 2.6 as follows.

Corollary 2.7. *Let conditions* (i)–(iv) *and* (2.16) *hold. Let x be a nontrivial solution of equation* (1.1) *such that* $\frac{x(t)}{t^2} \to 0$ *as* $t \to \infty$. *Then either*

(I_1) *x is oscillatory on* $[t_0, \infty)$*; or else*

(I_2) $x' > 0$ *[< 0] on* $[t_1, \infty)$ *for some* $t_1 \geq t_0$ *and x [$-x$] satisfies the inequalities in* (2.17) *of* Theorem 2.6. *In particular x [$-x$] increases [decreases] monotonically on* $[t_0, \infty)$.

If x is a nontrivial solution of equation (1.1) such that $x(t) \to 0$ as $t \to \infty$, it cannot satisfy the inequalities in (2.17) of Theorem 2.6. Thus, we conclude by Corollary 2.7 that x is oscillatory.

Next we shall establish some sufficient conditions for the oscillation of every solution x of equation (1.1) such that $\frac{x(t)}{t^2} \to 0$ as $t \to \infty$. We let

$$Q(t) = \left(\frac{1}{a(t)} \int_t^\infty (s-t) q(s) \, \mathrm{d}s \right)^{1/\alpha}.$$

First we present the following comparison result.

Theorem 2.8. *Let conditions* (i)–(iv) *and* (2.16) *hold. If the equation*

$$y''(t) + Q(t) f^{1/\alpha}(y[g(t)]) = 0 \tag{2.18}$$

is oscillatory, then every solution x of equation (1.1) *with* $\lim_{t\to\infty} \frac{x(t)}{t^2} = 0$ *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1) such that $x(t) > 0$ for $t \geq t_0 \geq 0$ and $\lim_{t\to\infty} \frac{x(t)}{t^2} = 0$. By Theorem 2.6, x satisfies inequalities (2.17). Integrating equation (1.1) twice from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$-x''(t) \geq \left(\frac{1}{a(t)} \int_t^\infty (s-t) q(s) f(x[g(s)]) \, \mathrm{d}s \right)^{1/\alpha} \geq Q(t) f^{1/\alpha}(x[g(t)]) \tag{2.19}$$

for $t \geq t_0$. Integrating (2.19) from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$x'(t) \geq \int_t^\infty Q(s) f^{1/\alpha}(x[g(s)]) \, \mathrm{d}s.$$

Integrating this inequality from t_0 to $t \ge t_0$, we have

$$x(t) \ge x(t_0) + \int_{t_0}^{t} \int_{s}^{\infty} Q(u) f^{1/\alpha}(x[g(u)])\, \mathrm{d}u\, \mathrm{d}s.$$

Now we define a sequence of functions $\{y_m\}_{m\in\mathbb{N}_0}$ by

$$y_0(t) = x(t),$$
$$y_{m+1}(t) = x(t_0) + \int_{t_0}^{t} \int_{s}^{\infty} Q(u) f^{1/\alpha}(y_m[g(u)])\, \mathrm{d}u\, \mathrm{d}s, \qquad m \in \mathbb{N}_0, \quad t \ge t_0.$$

It is easy to check that $\{y_m\}_{m\in\mathbb{N}_0}$ is well defined as a nonincreasing sequence and satisfies

$$x(t_0) \le y_m(t) \le x(t) \qquad \text{for} \quad t \ge t_0, \quad m \in \mathbb{N}_0.$$

Hence there exists a function y on $[t_0, \infty)$ such that

$$\lim_{m\to\infty} y_m(t) = y(t) \qquad \text{for} \quad t \ge t_0$$

and

$$x(t_0) \le y(t) \le x(t) \qquad \text{for} \quad t \ge t_0.$$

From the Lebesgue dominated convergence theorem it follows that

$$y(t) = x(t_0) + \int_{t_0}^{t} \int_{s}^{\infty} Q(u) f^{1/\alpha}(y[g(u)])\, \mathrm{d}u\, \mathrm{d}s \qquad \text{for} \quad t \ge t_0.$$

Differentiating this equation, we conclude that x is nonoscillatory, which contradicts the hypothesis. This completes the proof. □

The following result is immediate.

Theorem 2.9. *Let conditions* (i)–(iv) *and* (2.16) *hold. Then every solution x of equation* (1.1) *with* $\lim_{t\to\infty} \frac{x(t)}{t^2} = 0$ *is oscillatory if one of the following conditions holds:*

(II$_1$) $\displaystyle\int^{\pm\infty} f^{-1/\alpha}(u)\, \mathrm{d}u < \infty$ *and* $\displaystyle\int^{\infty} g'(s) \int_{s}^{\infty} Q(u)\, \mathrm{d}u\, \mathrm{d}s = \infty.$

(II$_2$) $\displaystyle\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} \int_{s}^{\infty} Q(\tau)\, \mathrm{d}\tau\, \mathrm{d}s > 0.$

(II$_3$) $\displaystyle\int_{\pm 0} f^{-1/\alpha}(u)\,\mathrm{d}u < \infty$ *and* $\displaystyle\int^{\infty} Q(u) f^{1/\alpha}(g(u))\,\mathrm{d}u = \infty.$

We note that Theorems 2.6–2.9 remain valid if the statement "x is a solution of equation (1.1) with $\lim_{r\to\infty} \frac{x(t)}{x^2} = 0$" is replaced by "bounded solution".

From the proof of Case (II) in Theorem 2.1, we obtain the following result.

Theorem 2.10. *Let conditions* (i)–(iv) *and* (2.16) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \ge t_0$. *If all bounded solutions of equation* (2.2) *are oscillatory, then every solution* x *of equation* (1.1) *with* $\lim \frac{x(t)}{t^2} = 0$ *is oscillatory.*

The following result is immediate.

Theorem 2.11. *Let conditions* (i)–(iv) *and* (2.16) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \ge t_0$. *Then every solution* x *of equation* (1.1) *with* $\lim_{t\to\infty} \frac{x(t)}{t^2} = 0$ *is oscillatory if one of the following conditions holds:*

(O$_1$) $f(x) = x^{\alpha}$, *and either*

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} q(s) g^{\alpha}(s)(\xi(s) - g(s))^{\alpha} a^{-1}(\xi(s))(\xi(t) - \xi(s))\,\mathrm{d}s > 1$$

or

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} \int_{u}^{t} q(s) g^{\alpha}(s)(\xi(s) - g(s))^{\alpha} a^{-1}(\xi(s))\,\mathrm{d}s\,\mathrm{d}u > 1.$$

(O$_2$) $f(x) = x^{\beta}$, β *is the ratio of positive odd integers,* $0 < \beta < \alpha$, *and either*

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} \int_{u}^{t} q(s) g^{\beta}(s)(\xi(s) - g(s))^{\beta} a^{-\beta/\alpha}(\xi(s))\,\mathrm{d}s\,\mathrm{d}u > 1$$

or

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} q(s) g^{\beta}(s)(\xi(s) - g(s))^{\beta} a^{-\beta/\alpha}(\xi(s))(\xi(t) - \xi(s))\,\mathrm{d}s > 0.$$

3. Oscillation of equation (1.2)

In this section, we shall investigate the oscillatory behavior of all solutions of equation (1.2) and obtain the following result.

Theorem 3.1. *Let conditions* (i)–(iv) *and* (1.3) *hold, and assume that there exist nondecreasing functions* $\xi, \zeta \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *and* $\sigma(t) > \zeta(t) > t$ *for* $t \ge t_0$*. If all unbounded solutions of the second-order advanced equation*

$$y''(t) - p(t)h\left(\int_{\zeta(t)}^{\sigma(t)} [\sigma(t)-s]a^{-1/\alpha}(s)\,\mathrm{d}s\right) h(y^{1/\alpha}[\zeta(t)]) = 0, \tag{3.1}$$

all bounded solutions of the second-order delay equations

$$z''(t) - cq(t)f\left(\int_{t_1}^{g(t)} sa^{-1/\alpha}(s)\,\mathrm{d}s\right) f(z^{1/\alpha}[g(t)]) = 0 \tag{3.2}$$

for any $t_1 \ge t_0$ *and any constant* $c \in (0,1)$*, and*

$$w''(t) - q(t)f\left(\int_{g(t)}^{\xi(t)} [s-g(t)]a^{-1/\alpha}(s)\,\mathrm{d}s\right) f(w^{1/\alpha}[\xi(t)]) = 0, \tag{3.3}$$

and all solutions of the second-order equation

$$v''(t) + \overline{c}q(t)f(g(t))f(m[g(t)])f(v^{1/\alpha}[g(t)]) = 0 \tag{3.4}$$

are oscillatory, and

$$\int^{\infty}\left(\frac{1}{a(s)}\int_{t_0}^{s}\int_{t_0}^{u} q(\tau)f(\xi(\tau)-g(\tau))f(m[g(\tau)])\,\mathrm{d}\tau\,\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s = \infty, \tag{3.5}$$

then equation (1.2) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.2), say $x(t) > 0$, $x[g(t)] > 0$ and $x[\sigma(t)] > 0$ for $t \ge t_0$. Since $(a(x'')^{\alpha})'' \ge 0$ on $[t_0,\infty)$, there exists $t_1 \ge t_0$ such that one of the following five possibilities holds:

(I)	$(a(x'')^{\alpha})' > 0$,	$x'' > 0$	and $x' > 0$	on	$[t_1,\infty)$;
(II)	$(a(x'')^{\alpha})' < 0$,	$x'' > 0$	and $x' > 0$	on	$[t_1,\infty)$;
(III)	$(a(x'')^{\alpha})' < 0$,	$x'' > 0$	and $x' < 0$	on	$[t_1,\infty)$;
(IV)	$(a(x'')^{\alpha})' < 0$,	$x'' < 0$	and $x' > 0$	on	$[t_1,\infty)$;
(V)	$(a(x'')^{\alpha})' > 0$,	$x'' < 0$	and $x' > 0$	on	$[t_1,\infty)$.

We may note that the three cases

$$\begin{array}{llll} (a(x'')^\alpha)' > 0, & x'' > 0 \quad \text{and} \quad x' < 0 & \text{on} & [t_1, \infty); \\ (a(x'')^\alpha)' > 0, & x'' < 0 \quad \text{and} \quad x' < 0 & \text{on} & [t_1, \infty); \\ (a(x'')^\alpha)' < 0, & x'' < 0 \quad \text{and} \quad x' < 0 & \text{on} & [t_1, \infty) \end{array}$$

are obviously disregarded. Now we consider Cases (I)–(V).

Case (I). By Taylor's expansion, it is easy to see that there exists $t_2 \geq t_1$ such that

$$\begin{aligned} x[\sigma(t)] &\geq \int_{\zeta(t)}^{\sigma(t)} [\sigma(t) - s] x''(s) \,\mathrm{d}s \\ &= \int_{\zeta(t)}^{\sigma(t)} [\sigma(t) - s] a^{-1/\alpha}(s) \left(a(s)(x''(s))^\alpha\right)^{1/\alpha} \mathrm{d}s \\ &\geq \left(\int_{\zeta(t)}^{\sigma(t)} [\sigma(t) - s] a^{-1/\alpha}(s) \,\mathrm{d}s \right) y^{1/\alpha}(\zeta(t)) \qquad \text{for} \quad t \geq t_2, \end{aligned} \tag{3.6}$$

where $y = a(x'')^\alpha$. Using (3.6) and (1.6) in equation (1.2), we find

$$y''(t) \geq p(t) h \left(\int_{\zeta(t)}^{\sigma(t)} [\sigma(t) - s] a^{-1/\alpha}(s) \,\mathrm{d}s \right) h(y^{1/\alpha}(\zeta(t))) \qquad \text{for} \quad t \geq t_2.$$

By applying the known results, see [7, 8], we arrive at the desired contradiction with (3.1).

Case (II). Let $k \in (0, 1)$ and $t_2 = t_1/(1 - k)$. Then

$$x'(t) = x'(t_1) + \int_{t_1}^{t} x''(\tau) \,\mathrm{d}\tau \geq (t - t_1) x''(t) \geq k t x''(t) \qquad \text{for} \quad t \geq t_2.$$

Integrating this inequality from t_2 to t, we get

$$x(t) \geq k \int_{t_2}^{t} s a^{-1/\alpha}(s) \left(a(s)(x''(s))^\alpha\right)^{1/\alpha} \mathrm{d}s \geq k \left(\int_{t_2}^{t} s a^{-1/\alpha}(s) \,\mathrm{d}s \right) z^{1/\alpha}(t),$$

where $z = a(x'')^\alpha$, and thus there exists $t_3 \geq t_2$ such that

$$x[g(t)] \geq k \left(\int_{t_2}^{g(t)} s a^{-1/\alpha}(s) \,\mathrm{d}s \right) z^{1/\alpha}[g(t)] \qquad \text{for} \quad t \geq t_3. \tag{3.7}$$

Using (3.7) and (1.5) in equation (1.2), we have

$$z''(t) \geq f(k)q(t)f\left(\int_{t_2}^{g(t)} sa^{-1/\alpha}(s)\,\mathrm{d}s\right) f(z^{1/\alpha}[g(t)]) \qquad \text{for} \quad t \geq t_3.$$

Clearly $z > 0$ and $z' < 0$ on $[t_3, \infty)$. By a known result, see [7,8], we arrive at the desired contradiction with (3.2).

Case (III). By Taylor series, one can easily see that there exists $t_2 \geq t_1$ such that

$$\begin{aligned} x[g(t)] &\geq \int_{g(t)}^{\xi(t)} [s-g(t)]x''(s)\,\mathrm{d}s \\ &= \int_{g(t)}^{\xi(t)} [s-g(t)]a^{-1/\alpha}(s)\left(a(s)(x''(s))^{\alpha}\right)^{1/\alpha}\,\mathrm{d}s \\ &\geq \left(\int_{g(t)}^{\xi(t)} [s-g(t)]a^{-1/\alpha}(s)\,\mathrm{d}s\right) w^{1/\alpha}[\xi(t)] \qquad \text{for} \quad t \geq t_2, \end{aligned} \tag{3.8}$$

where $w = a(x'')^{\alpha}$. Using (3.8) and (1.5) in equation (1.2), we get

$$w''(t) \geq q(t)f\left(\int_{g(t)}^{\xi(t)} [s-g(t)]a^{-1/\alpha}(s)\,\mathrm{d}s\right) f(w^{1/\alpha}[\xi(t)]) \qquad \text{for} \quad t \geq t_2.$$

Now, similar as in Case (II) above, we arrive at a contradiction with (3.3).

Case (IV). Let $k \in (0,1)$ and $t_2 = t_1/(1-k)$. Then

$$x(t) = x(t_1) + \int_{t_1}^{t} x'(\tau)\,\mathrm{d}\tau \geq x'(t)(t-t_1) = x'(t)t\left(1 - \frac{t_1}{t}\right) \geq ktx'(t)$$

for $t \geq t_2$ so that there exists $t_3 \geq t_2$ such that

$$x[g(t)] \geq kg(t)x'[g(t)] \qquad \text{for} \quad t \geq t_3. \tag{3.9}$$

Using (3.9) and (1.5) in equation (1.2), we get

$$(a(y')^{\alpha})''(t) \geq f(k)q(t)f(g(t))f\big(y[g(t)]\big) \qquad \text{for} \quad t \geq t_3, \tag{3.10}$$

where $y = x'$. Clearly, we see that

$$y > 0, \quad y' < 0 \quad \text{and} \quad \big(a(y')^{\alpha}\big)' < 0 \qquad \text{on} \quad [t_3, \infty).$$

Proceeding as in Case (III) of Theorem 2.1, we obtain (2.13) for $t \geq t_4$, and hence, by using (2.13) and (1.5) in (3.10), we get

$$Z''(t) + f(k)q(t)f(g(t))f\big(m[g(t)]\big)f\big(Z^{1/\alpha}[g(t)]\big) \leq 0 \qquad \text{for} \quad t \geq t_4.$$

The rest of the proof is similar to that of Case (III) in Theorem 2.1.

Case (V). Proceeding as in the proof of Case (III) of Theorem 2.1, we obtain (2.10) for $t \geq t_2$. Using (2.10) and (1.5) in equation (1.2), we have

$$\big(a(y')^{\alpha}\big)''(t) + q(t)f\big(\xi(t) - g(t)\big)f\big(y[\xi(t)]\big) \leq 0 \qquad \text{for} \quad t \geq t_2,$$

where $y = -x'$. The rest of the proof is exactly the same as that of Case (IV) in Theorem 2.1 and hence is omitted. This completes the proof. □

When condition (1.4) holds, we see from the proof of Theorem 3.1 that Cases (IV) and (V) are disregarded, and so we present the following result.

Theorem 3.2. *Let conditions* (i)–(iv) *and* (1.4) *hold, and assume that there exist nondecreasing functions* $\xi, \zeta \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *and* $\sigma(t) > \zeta(t) > t$ *for* $t \geq t_0$. *If all unbounded solutions of equation* (3.1) *and all bounded solutions of equations* (3.2) *and* (3.3) *are oscillatory, then equation* (1.2) *is oscillatory.*

When condition (1.3) holds and we are concerned with bounded oscillation of equation (1.2), we see from the proof of Theorem 3.1 that only Cases (III), (IV) and (V) are to be considered, and so we give the following result.

Theorem 3.3. *Let conditions* (i)–(iv), (1.3) *and* (3.5) *hold, and suppose that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \geq t_0$. *If all bounded solutions of equation* (3.3) *and all solutions of equation* (3.4) *are oscillatory, then all bounded solutions of equation* (1.2) *are oscillatory.*

When condition (1.4) holds, we see from the proof of Theorem 3.1 that Case (III) is the only case to be considered. Thus, we have the following result.

Theorem 3.4. *Let conditions* (i)–(iv) *and* (1.4) *hold, and assume that there exists a nondecreasing function* $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \geq t_0$. *If all bounded solutions of equation* (3.3) *are oscillatory, then all bounded solutions of equation* (1.2) *are oscillatory.*

In Theorem 3.3 and 3.4, we observe that the function h in equation (1.2) may be disregarded. We also note that equations (3.2) and (3.3) may be combined in one, namely

$$v'' - P(t)f\big(v^{1/\alpha}[\xi(t)]\big) = 0,$$

where

$$P(t) = \min\left\{cq(t)f\left(\int\limits_{t_1}^{g(t)} sa^{-1/\alpha}(s)\,\mathrm{d}s\right),\ q(t)f\left(\int\limits_{g(t)}^{\xi(t)} [s-g(t)]a^{-1/\alpha}(s)\,\mathrm{d}s\right)\right\}$$

for any constant $c \in (0,1)$.

Next, we shall consider equation (1.2) with $h \equiv 0$, i.e.,

$$\big(a(x'')^\alpha\big)''(t) = q(t)f\big(x[g(t)]\big). \tag{3.11}$$

We also let

$$\lim_{t\to\infty} \frac{t}{\int\limits_{t_0}^{t}\int\limits_{t_0}^{s}\left(\frac{u}{a(u)}\right)^{1/\alpha}\mathrm{d}u\,\mathrm{d}s} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{t}{\int\limits_{t_0}^{t}\int\limits_{t_0}^{s} a^{1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s} < \infty \tag{3.12}$$

In a somewhat similar fashion to Theorem 2.6, we can prove the next result.

Theorem 3.5. *Let conditions* (i)–(iv), (1.4) *and* (3.12) *hold. If x is a nontrivial solution of equation* (3.11) *such that $x(t) > 0$ on $[t_0, \infty)$ and $\frac{x(t)}{t} \to 0$ as $t \to \infty$, then*

$$x > 0, \quad x' < 0, \quad x'' > 0 \quad \text{and} \quad \big(a(x'')^\alpha\big)' < 0 \qquad \text{on} \quad [t_0,\infty), \tag{3.13}$$

and $x'(t) \to 0$, $a(t)(x''(t))^\alpha \to 0$ and $(a(x'')^\alpha)'(t) \to 0$ monotonically as $t \to \infty$.

The following corollary is immediate.

Corollary 3.6. *Let conditions* (i)–(iv), (1.4) *and* (3.12) *hold. Suppose x is a nontrivial solution of equation* (3.11) *such that $x(t) > 0$ and $\lim\limits_{t\to\infty}\frac{x(t)}{t} = 0$. If x violates any of the inequalities in* (3.13)*, then x is oscillatory.*

From the proof of Case (III) of Theorem 3.1, we obtain the following result.

Theorem 3.7. *Let conditions* (i)–(iv), (1.4) *and* (3.12) *hold, and assume that there exists a nondecreasing function $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ such that $g(t) < \xi(t) < t$ for $t \ge t_0$. If all bounded solutions of equation* (3.3) *are oscillatory, then every solution x of equation* (3.11) *with $\lim\limits_{t\to\infty}\frac{x(t)}{t} = 0$ is oscillatory.*

The following result is immediate.

Theorem 3.8. *Let conditions* (i)–(iv), (1.4) *and* (3.12) *hold, and assume that there exists a nondecreasing function $\xi \in \mathrm{C}([t_0,\infty),\mathbb{R})$ such that $g(t) < \xi(t) < t$ for $t \ge t_0$. Then every solution x of equation* (3.11) *with $\lim\limits_{t\to\infty}\frac{x(t)}{t} = 0$ is oscillatory if one of the following conditions holds:*

(A_1) *$f(x) = x^\alpha$, and either*

$$\limsup_{t\to\infty} \int\limits_{\xi(t)}^{t} q(s) \left(\int\limits_{g(s)}^{\xi(t)} (u - g(s)) a^{-1/\alpha}(u) \, \mathrm{d}u \right)^{\alpha} (\xi(t) - \xi(s)) \, \mathrm{d}s > 1$$

or

$$\limsup_{t\to\infty} \int\limits_{\xi(t)}^{t} \int\limits_{s}^{t} q(u) \left(\int\limits_{g(u)}^{\xi(u)} (\tau - g(u)) a^{-1/\alpha}(\tau) \, \mathrm{d}\tau \right)^{\alpha} \mathrm{d}u \, \mathrm{d}s > 1.$$

(A_2) *$f(x) = x^\beta$, β is the ratio of positive odd integers, $0 < \beta < \alpha$, and either*

$$\limsup_{t\to\infty} \int\limits_{\xi(t)}^{t} q(s) \left(\int\limits_{g(s)}^{\xi(s)} (u - g(s)) a^{-1/\alpha}(u) \, \mathrm{d}u \right)^{\beta} (\xi(t) - \xi(s)) \, \mathrm{d}s > 0$$

or

$$\limsup_{t\to\infty} \int\limits_{\xi(t)}^{t} \int\limits_{s}^{t} q(u) \left(\int\limits_{g(u)}^{\xi(u)} (\tau - g(u)) a^{-1/\alpha}(\tau) \, \mathrm{d}\tau \right)^{\beta} \mathrm{d}u \, \mathrm{d}s > 0.$$

Remark 3.9.

1. The results of this paper are presented in a form which is essentially new. The obtained results can be extended to fourth-order dynamic equations of the form

$$(a(x^{\Delta\Delta})^{\alpha})^{\Delta\Delta}(t) + \delta q(t) f(x(g(t))) = 0,$$

where $\delta = \pm 1$ and a, q are real-valued positive and rd-continuous functions on a time scale $\mathbb{T} \subseteq \mathbb{R}$ and $g\colon \mathbb{T} \to \mathbb{T}$ is an rd-continuous and nondecreasing function satisfying $\lim\limits_{t\to\infty} g(t) = \infty$. The details are left to the reader.

2. The results of this paper can be extended to neutral equations of the form

$$(a(x + p(x \circ \tau))'')''(t) + \delta q(t) f(x(g(t))) = 0,$$

where $\delta = \pm 1$, $q \in \mathrm{C}([t_0, \infty), \mathbb{R}^+)$, $p, \tau, g \in \mathrm{C}([t_0, \infty), \mathbb{R})$ and $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$. The details are left to the reader.

REFERENCES

[1] AGARWAL, R. P.—GRACE, S. R.: *The oscillation of higher-order differential equations with deviating arguments*, Comput. Math. Appl. **38** (1999), 185–199.

[2] AGARWAL, R. P.—GRACE, S. R.—KIGURADZE, I.—O'REGAN, D.: *Oscillation of functional differential equations*, Math. Comput. Modelling **41** (2005), 417–461.

[3] AGARWAL, R. P.—GRACE, S. R.—O'REGAN, D.: *Oscillation of certain fourth order functional differential equations*, Ukraïn. Mat. Zh. **59** (2007), 291–313.

[4] AGARWAL, R. P.—GRACE, S. R.—WONG, P. J. Y.: *On the bounded oscillation of certain fourth order functional differential equations*, Nonlinear Dyn. Syst. Theory **5** (2005), 215–227.

[5] AGARWAL, R. P.—GRACE, S. R.—O'REGAN, D.: *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 2000.

[6] AGARWAL, R. P.—GRACE, S. R.—O'REGAN, D.: *Oscillation criteria for certain nth order differential equations with deviating arguments*, J. Math. Anal. Appl. **262** (2001), 601–622.

[7] AGARWAL, R. P.—GRACE, S. R.—O'REGAN, D.: *Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic Publishers, Dordrecht, 2002.

[8] AGARWAL, R. P.—GRACE, S. R.—O'REGAN, D.: *Oscillation Theory for Second Order Dynamic Equations*. Ser. Math. Anal. Appl. 5, Taylor & Francis Ltd., London, 2003.

[9] AGARWAL, R. P.—O'REGAN, D.: *Nonlinear generalized quasi-variational inequalities: a fixed point approach*, Math. Inequal. Appl. **6** (2003), 133–143.

[10] GRACE, S. R.: *Oscillation theorems for nth-order differential equations with deviating arguments*, J. Math. Anal. Appl. **101** (1984), 268–296.

[11] GRACE, S. R.—LALLI, B. S.: *A comparison theorem for general nonlinear ordinary differential equations*, J. Math. Anal. Appl. **120** (1986), 39–43.

[12] PHILOS, C. G.: *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays*, Arch. Math. (Basel) **36** (1981), 168–178.

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