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# STABILITY ANALYSIS OF DIFFERENTIAL EQUATIONS WITH MAXIMUM

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ABSTRACT. In this paper, we investigate stability of the zero solution of differential equations with maximum by using Lyapunov functions and Razumikhin techniques. Sufficient conditions for stability, uniform stability and asymptotic stability of the zero solution of such equations are found.



## 1. Introduction

It is well-known that the systems with aftereffect, with time lag or with delay are of great theoretical interest and form an important class as regards their applications. Such systems are frequently encountered as mathematical modes of most dynamical process in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Especially, since 1960s different classes of delay differential equations have been studied by many authors (see for example [1, 5, 6, 8–11, 18, 20] and the references listed there).

As it is also known, the investigation of qualitative properties of solutions, in particular, the stability of solutions is a very important problem in the theory and applications of the differential equations. The most efficient tool for the study of the stability of a given nonlinear system is provided by Lyapunov's second method [13]. Its application to systems with delay has been developed in two directions:

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1. The first direction use of Lyapunov functions and the Razumikhin technique [1, 9, 12], [17]–[21].

2. The second method is Lyapunov-Krasovskii functional method [5, 9, 10].

One special class of functional differential equations that has many applications in the mathematical simulation of some systems with automatic regulation [16] are differential equations with maximum. Recently, Bainov and Vulov [3, 22] have formulated some mathematical models of differential equations with maximum and interesting results about the stability using a modification of Lyapunov's first method were obtained; theorems for existence, uniqueness and continuability are proved in [2,3,7,11,14]; some oscillation results are obtained in [4]; the averaging method is justified in [15]. However, so far the problem for the stability of the solutions of such equations by means of Lyapunov-Razumikhin method have not been considered.

The present paper deals with a system of differential equations with maximum and we determine conditions of the Lyapunov-Razumikhin type which guarantee the stability, uniform stability and asymptotic stability of the zero solution of the system under consideration. Three examples illustrating the results obtained are given.

## 2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with norm $|\cdot|$; $\Omega$ be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0, \infty)$; $\mathbb{R} = (-\infty, \infty)$; $t_0 \in \mathbb{R}_+$; $\tau > 0$. Let $x\colon [t-\tau, t] \to \Omega$, $x = (x_1, x_2, \ldots, x_n)$. We denote

$$\max_{s\in[t-\tau,t]} x(s) = \Big(\max_{s\in[t-\tau,t]} x_1(s), \max_{s\in[t-\tau,t]} x_2(s), \ldots, \max_{s\in[t-\tau,t]} x_n(s)\Big).$$

Consider the following system of differential equations with maximum

$$\dot{x}(t) = f\Big(t, x(t), \max_{s\in[t-\tau,t]} x(s)\Big), \qquad t > t_0, \tag{2.1}$$

where $f\colon [t_0, \infty) \times \Omega \times \Omega \to \mathbb{R}^n$.

Let $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$. Denote by $x(t) = x(t; t_0, \varphi_0)$, $x \in \Omega$ the solution of system (2.1) satisfying the initial condition:

$$x(t; t_0, \varphi_0) = \varphi_0(t - t_0), \qquad t_0 - \tau \le t \le t_0, \tag{2.2}$$

and by $J^+(t_0, \varphi_0)$ — the maximal interval of type $[t_0, \beta)$ in which the solution $x(t; t_0, \varphi_0)$ is defined.

Introduce the following notation:

$\|\phi\| = \max_{t\in[t_0-\tau,t_0]} |\phi(t-t_0)|$ is the norm of the function $\phi \in C\big[[-\tau,0],\Omega\big]$.

The following assumptions will be needed throughout the paper:

H2.1. $f \in C\big[[t_0,\infty)\times\Omega\times\Omega,\mathbb{R}^n\big]$.

H2.2. The function $f$ is Lipschitz continuous with respect its second and third arguments in $[t_0,\infty)\times\Omega\times\Omega$, uniformly on $t\in[t_0,\infty)$, i.e the function $f(t,x,y)$ satisfies the condition

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \le L(|x_1-x_2|+|y_1-y_2|),$$

for $(t,x_1,y_1)$, $(t,x_2,y_2)\in[t_0,\infty)\times\Omega\times\Omega$, $L=\text{const}>0$.

We give the following assertion which establishes the existence, uniqueness and continuability of solutions of (2.1) ([2,3,11,14]).

**Lemma 2.1.** *Let condition* H2.1 *holds. Then for each* $(t_0,\varphi_0)\in\mathbb{R}\times C[[-\tau,0],\Omega]$*:*

1. *There exists a solution* $x(t)=x(t;t_0,\varphi_0)$ *of the initial value problem* (2.1), (2.2) *defined on* $J^+(t_0,\varphi_0)$.
2. $J^+(t_0,\varphi_0)=[t_0,\infty)$.
3. *If, moreover, condition* H2.2 *is met then the solution* $x(t;t_0,\varphi_0)$ *is unique.*

We shall investigate the stability of the zero solution $x(t)\equiv 0$ of system (2.1). That is why the following condition will be used:

H2.3. $f(t,0,0)=0,\quad t\in[t_0,\infty)$.

We shall use the following definitions of Lyapunov like stability of the zero solution of (2.1).

**Definition 2.1.** The zero solution $x(t)\equiv 0$ of system (2.1) is said to be:

a) *stable*, if

$$\big(\forall t_0\in\mathbb{R}_+\big)\big(\forall\varepsilon>0\big)\big(\exists\delta=\delta(t_0,\varepsilon)>0\big)\big(\forall\varphi_0\in C\big[[-\tau,0],\Omega\big]:\ \|\varphi_0\|<\delta\big)$$
$$(\forall t\ge t_0):\ |x(t;t_0,\varphi_0)|<\varepsilon;$$

b) *uniformly stable*, if the number $\delta$ in a) is independent of $t_0\in\mathbb{R}$;

c) *uniformly attractive*, if

$$\big(\exists\lambda>0\big)\big(\forall\varepsilon>0\big)\big(\exists T=T(\varepsilon)>0\big)\big(\forall t_0\in\mathbb{R}_+\big)$$
$$\big(\forall\varphi_0\in C\big[[-\tau,0],\Omega\big]:\ \|\varphi_0\|<\lambda\big)\big(\forall t\ge t_0+T\big):\ |x(t;t_0,\varphi_0)|<\varepsilon;$$

d) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attractive.

Define the following classes of functions:

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : \ a(u) \ \text{ is strictly increasing and such that } \ a(0) = 0\};$$

$$V_0 = \big\{V : [t_0, \infty) \times \Omega \to \mathbb{R}_+ : \ V \in C\big[[t_0, \infty) \times \Omega, \mathbb{R}_+\big],$$
$$V(t, 0) = 0, \quad t \in [t_0, \infty), \quad V \ \text{ is locally Lipschitzian in } \ x \in \Omega\big\}.$$

**Definition 2.2.** Given a function $V \in V_0$. For $t \geq t_0$ and $\phi \in C\big[[t-\tau, t], \Omega\big]$ the *upper right-hand derivative of* $V$ *with respect to system* (2.1) is defined by

$$D^+V(t, \phi(t))$$
$$= \lim_{h \to 0^+} \sup \frac{1}{h}\big[V(t+h, \phi(t) + hf(t, \phi(t), \max_{s \in [-\tau, 0]} \phi(t+s))) - V(t, \phi(t))\big].$$

Note that in Definition 2.2, $D^+V(t, \phi(t))$ is a functional whereas $V$ is a function.

**Lemma 2.2.** ([12]) *Assume that:*

1. *Conditions* H2.1–H2.3 *hold.*
2. *The function* $g : [t_0, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous in* $[t_0, \infty) \times \mathbb{R}_+$.
3. *The maximal solution* $u^+(t; t_0, u_0)$ *of the scalar problem*

$$\begin{cases} \dot{u}(t) = g(t, u(t)), & t > t_0, \\ u(t_0) = u_0 \geq 0 \end{cases}$$

   *is defined in the interval* $[t_0, \infty)$.
4. *The function* $V \in V_0$ *is such that* $V(t_0, \varphi_0(0)) \leq u_0$, *and the inequality*

$$D^+V(t, \phi(t)) \leq g(t, V(t, \phi(t)))$$

   *is valid for any* $t \in [t_0, \infty)$ *and any function* $\phi \in C\big[[t-\tau, t], \mathbb{R}^n\big]$ *for which* $V(t+s, \phi(t+s)) \leq V(t, \phi(t))$, $s \in [-\tau, 0]$.

*Then*

$$V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, u_0), \quad t \in [t_0, \infty).$$

In the case when $g(t, u) = 0$ for $(t, u) \in [t_0, \infty) \times \mathbb{R}_+$, we deduce the following corollary from Lemma 2.2.

**Corollary 2.1.** *Assume that:*

1. *Conditions* H2.1–H2.3 *hold.*

2. *The function $V \in V_0$ is such that the inequality*

$$D^+V(t, \phi(t)) \leq 0$$

*is valid for any $t \in [t_0, \infty)$ and any function $\phi \in C\big[[t-\tau, t], \mathbb{R}^n\big]$ for which $V(t+s, \phi(t+s)) \leq V(t, \phi(t)),\ \ s \in [-\tau, 0]$.*

*Then*

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(0)), \qquad t \in [t_0, \infty).$$

## 3. Main results

**Theorem 3.1.** *Assume that:*

1. *Conditions* H2.1–H2.3 *hold.*
2. *There exists a function $V \in V_0$ such that*

$$a(|x|) \leq V(t, x), \qquad a \in K, \quad (t, x) \in [t_0, \infty) \times \Omega, \tag{3.1}$$

*and the inequality*

$$D^+V(t, \phi(t)) \leq 0$$

*is valid for any $t \in [t_0, \infty)$ and any function $\phi \in C\big[[t-\tau, t], \mathbb{R}^n\big]$ for which $V(t+s, \phi(t+s)) \leq V(t, \phi(t))$, $s \in [-\tau, 0]$.*

*Then the zero solution of system* (2.1) *is stable.*

Proof. Let $\varepsilon > 0$. It follows from the properties of the function $V$ that there exists a constant $\delta = \delta(t_0, \varepsilon) > 0$ such that if $x \in \Omega$: $|x| < \delta$, then $\sup\limits_{|x|<\delta} V(t_0, x) < a(\varepsilon)$.

Let $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$: $\|\varphi_0\| < \delta$. Then $|\varphi_0(0)| \leq \|\varphi_0\| < \delta$ and therefore

$$V(t_0, \varphi_0(0)) < a(\varepsilon). \tag{3.2}$$

Let $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (2.1), (2.2). Since all the conditions of Corollary 2.1 are met, then

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(0)), \quad t \in [t_0, \infty). \tag{3.3}$$

From (3.1), (3.2) and (3.3) there follow the inequalities

$$a(|x(t; t_0, \varphi_0)|) \leq V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(0)) < a(\varepsilon),$$

whence we obtain that $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t \geq t_0$. This implies that the zero solution of system (2.1) is stable. □

**Theorem 3.2.** *Let the conditions of* Theorem 3.1 *hold, and let a function $b \in K$ exist such that*

$$V(t,x) \le b(|x|), \qquad (t,x) \in (t_0,\infty) \times \Omega. \tag{3.4}$$

*Then the zero solution of system* (2.1) *is uniformly stable.*

Proof. Let $\varepsilon > 0$ be chosen. Choose $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

Let $\varphi_0 \in C\big[[-\tau,0],\Omega\big]\colon \|\varphi_0\| < \delta$ and $x(t) = x(t;t_0,\varphi_0)$ be the solution of problem (2.1), (2.2).

As in Theorem 3.1, we prove that

$$a(|x(t;t_0,\varphi_0)|) \le V(t,x(t;t_0,\varphi_0)) \le V(t_0,\varphi_0(0)), \qquad t \ge t_0.$$

From the above inequalities and (3.4), we get to the inequalities

$$a(|x(t;t_0,\varphi_0)|) \le V(t_0,\varphi_0(0)) \le b(|\varphi_0(0)|) \le b(\|\varphi_0\|) < b(\delta) < a(\varepsilon),$$

from which it follows that $|x(t;t_0,\varphi_0)| < \varepsilon$ for $t \ge t_0$. This proves the uniform stability of the zero solution of system (2.1). □

**Theorem 3.3.** *Assume that:*

1. *Conditions* H2.1–H2.3 *hold.*
2. *There exists a function $V \in V_0$ such that* (3.1) *hold,*

$$a(|x|) \le V(t,x) \le b(|x|), \qquad a,b \in K, \quad (t,x) \in [t_0,\infty) \times \Omega, \tag{3.5}$$

*and the inequality*

$$D^+V(t,\phi(t)) \le -c(|\phi(t)|), \qquad c \in K \tag{3.6}$$

*is valid for any $t \in [t_0,\infty)$ and any function $\phi \in C\big[[t-\tau,t],\mathbb{R}^n\big]$ for which $V(t+s,\phi(t+s)) \le V(t,\phi(t))$, $s \in [-\tau,0]$.*

*Then the zero solution of system* (2.1) *is uniformly asymptotically stable.*

Proof.

1. Let $\alpha = \text{const} > 0$ be such that $\{x \in \mathbb{R}^n : \ |x| \le \alpha\} \subset \Omega$.

For any $t \in [t_0,\infty)$ denote

$$V^{-1}_{t,\alpha} = \big\{x \in \Omega : \ V(t,x) \le a(\alpha)\big\}.$$

From (3.5), we deduce

$$V^{-1}_{t,\alpha} \subset \{x \in \mathbb{R}^n : \ |x| \le \alpha\} \subset \Omega.$$

From condition 2 of Theorem 3.3, it follows that for any $t_0 \in \mathbb{R}$ and any function $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$ such that $\varphi_0(0) \in V^{-1}_{t_0,\alpha}$ we have $x(t; t_0, \varphi_0) \in V^{-1}_{t,\alpha}$, $t \geq t_0$.

Let $\varepsilon > 0$ be chosen. Choose $\eta = \eta(\varepsilon)$ so that $b(\eta) < a(\varepsilon)$, and let $T > \frac{b(\alpha)}{c(\eta)}$.

If we assume that for each $t \in [t_0, t_0 + T]$ the inequality $|x(t; t_0, \varphi_0)| \geq \eta$ is valid, then from (3.6) we get

$$\begin{aligned} V(t, x(t; t_0, \varphi_0)) &\leq V(t_0, \varphi_0(0)) - \int\limits_{t_0}^{t} c(|x(s; t_0, \varphi_0)|)\,\mathrm{d}s \\ &\leq b(\alpha) - c(\eta)T < 0, \end{aligned}$$

which contradicts (3.6). The contradiction obtained shows that there exists $t^* \in [t_0, t_0 + T]$ such that $|x(t^*; t_0, \varphi_0)| < \eta$.

Then from (3.5) and (3.6) it follows that for $t \geq t^*$ (hence for any $t \geq t_0 + T$) the following inequalities hold

$$\begin{aligned} a(|x(t; t_0, \varphi_0)|) &\leq V(t; x(t; t_0, \varphi_0)) \leq V(t^*, x(t^*; t_0, \varphi_0)) \\ &\leq b(|x(t^*; t_0, \varphi_0)|) < b(\eta) < a(\varepsilon). \end{aligned}$$

Therefore, $|x(t; t_0, \varphi_0| < \varepsilon$ for $t \geq t_0 + T$.

2. Let $\lambda = \text{const} > 0$ be such that $b(\lambda) < a(\alpha)$. Then, if $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$ and $\|\varphi_0\| < \lambda$, (3.5) implies

$$V(t_0, \varphi_0(0)) \leq b(|\varphi_0(0)|) \leq b(\|\varphi_0\|) < b(\lambda) < a(\alpha),$$

which shows that $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$ is such that $\varphi_0(0) \in V^{-1}_{t_0,\alpha}$. From what we proved in item 1, it follows that the zero solution of system (2.1) is uniformly attractive and since Theorem 3.2 implies that it is uniformly stable, then the solution $x \equiv 0$ is uniformly asymptotically stable. □

**Corollary 3.1.** *If in* Theorem 3.3 *condition* (3.6) *is replaced by the condition*

$$D^+V(t, \phi(t)) \leq -cV(t, \phi(t)), \tag{3.7}$$

*where* $V(t+s, \phi(t+s)) \leq V(t, \phi(t))$, $s \in [-\tau, 0]$, $t \in [t_0, \infty)$, $c = \text{const} > 0$*, then the zero solution of system* (2.1) *is uniformly asymptotically stable.*

P r o o f. The proof of Corollary 3.1 is analogous to the proof of Theorem 3.3. It uses the fact that

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(0)) \exp[-c(t - t_0)]$$

for $t \geq t_0$, which is obtained from (3.7).

In fact, let $\alpha = \text{const} > 0$: $\{x \in \mathbb{R}^n : \ |x| \le \alpha\} \subset \Omega$. Choose $\lambda > 0$ so that $b(\lambda) < a(\alpha)$. Let $\varepsilon > 0$ and $T \ge \frac{1}{c} \ln \frac{a(\alpha)}{a(\varepsilon)}$. Then for $\varphi_0 \in C\big[[-\tau, 0], \Omega\big]$: $\|\varphi_0\| < \lambda$ and $t \ge t_0 + T$ the following inequalities hold

$$V(t, x(t; t_0, \varphi_0)) \le V(t_0, \varphi_0(0)) \exp[-c(t - t_0)] < a(\varepsilon),$$

whence, in view of (3.5), we deduce that the solution $x \equiv 0$ of system (2.1) is uniformly attractive. □

## 4. Examples

*Example* 4.1. Consider the following impulsive system

$$\dot{x}(t) = Ax(t) + B \max_{s \in [t-\tau, t]} x(s), \qquad t > 0, \tag{4.1}$$

where $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$, $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$.

Let $V(t, x) = |x|^2 = \langle x, x \rangle$, where $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ is the dot product of $x, y \in \mathbb{R}^n$.

Let $\varphi_0 \in C\big[[-\tau, 0], \mathbb{R}^n_+\big]$. Denote by $x(t) = x(t; 0, \varphi_0)$ the solution of system (4.1) satisfying the initial conditions

$$x(s) = \varphi_0(s) \ge 0, \qquad s \in [-\tau, 0); \quad x(0) > 0.$$

For $t \ge 0$, we have

$$D^+ V(t, x(t)) = 2\langle x(t), \dot{x}(t) \rangle = 2\Big\langle x(t), Ax(t) + B \max_{s \in [t-\tau, t]} x(s) \Big\rangle.$$

If $a_i \le 0$, $b_i \ge 0$, $b = \max_i b_i$ and $a_i \le -b$ for $i = 1, 2, \ldots, n$, then for $t \ge 0$, and for any $\phi \in C\big[[t - \tau, t], \mathbb{R}\big]$ such that $V(t + s, \phi(t + s)) < V(t, \phi(t))$, $s \in [-\tau, 0)$, we have

$$D^+ V(t, \phi(t)) \le 0, \qquad t \ge 0$$

and according to Theorem 3.2 the trivial solution of (4.1) is uniformly stable.

Next, let us investigate uniform asymptotic stability. If there is a constant $c > 0$ such that $a_i \le -(b + c)$ for $i = 1, 2, \ldots, n$, then for $V(t + s, \phi(t + s)) \le V(t, \phi(t))$, $s \in [-\tau, 0]$,

$$D^+ V(t, \phi(t)) \le -2c\langle \phi(t), \phi(t) \rangle = -2cV(t, \phi(t)), \qquad t \ge 0$$

and according to Corollary 3.1 the trivial solution of (4.1) is uniformly asymptotically stable.

*Example* 4.2. Consider a nonlinear scalar equation

$$\dot{x}(t) = f\Big(x(t), \mu \max_{s\in[t-\tau,t]} x(s)\Big), \qquad t > 0, \tag{4.2}$$

where $f(x, y)$ is a continuous function with $f(0,0) = 0$, $\frac{f(x,0)}{x} = -\sigma$ for some $\sigma > 0$ satisfying $\sigma \geq L|\mu|$ and $|f(x_1, y_1) - f(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$. Then conditions H2.1–H2.3 are valid.

Let $\varphi_0 \in C\big[[-\tau, 0], \mathbb{R}_+\big]$. Denote by $x(t) = x(t; 0, \varphi_0)$ the solution of system (4.2) satisfying the initial conditions

$$x(s) = \varphi_0(s) \geq 0, \qquad s \in [-\tau, 0); \quad x(0) > 0.$$

Choosing $V(t, x) = x^2$, we get for $t \geq 0$

$$\begin{aligned} D^+V(t, \phi(t)) &= 2\phi(t) f\Big(\phi(t), \mu \max_{s\in[t-\tau,t]} \phi(s)\Big) \\ &= 2\left[\frac{f(\phi(t), \mu \max\limits_{s\in[t-\tau,t]} \phi(s)) - f(\phi(t), 0)}{\phi(t)} + \frac{f(\phi(t), 0)}{\phi(t)}\right]\phi^2(t) \\ &= 2\left[\frac{L|\mu|| \max\limits_{s\in[t-\tau,t]} \phi(s)|}{|\phi(t)|} - \sigma\right]\phi^2(t) \leq 2(L|\mu| - \sigma)\phi^2(t) \leq 0 \end{aligned}$$

whenever $V(t + s, \phi(t + s)) \leq V(t, \phi(t))$, $s \in [-\tau, 0]$, $t \in \mathbb{R}_+$. It follows from Theorem 3.2 that the solution $x \equiv 0$ of (4.2) is uniformly stable.

*Example* 4.3. Consider the equation

$$\dot{N}(t) = rN(t)[1 - aN(t) - b \max_{s\in[t-\tau,t]} N(s)], \quad t > 0. \tag{4.3}$$

Equation (4.3) models the dynamics of a logistically growing population subjected to a density-dependent harvesting. There, $N(t)$ denotes the population density of a single species and the model parameters $r$, $a$ and $b$ are assumed to be positive.

Let $\varphi \in C\big[[-\tau, 0], \mathbb{R}_+\big]$ and $N(t) = N(t; 0, \varphi)$ be the solution of equation (4.3) satisfying the initial conditions

$$N(s) = \varphi(s) \geq 0, \qquad s \in [-\tau, 0); \quad N(0) > 0.$$

Gopalsamy [8] studied the equation (4.3) in the case $\max\limits_{s\in[t-\tau,t]} N(s) = N([t])$ and he showed that $N^* = \frac{1}{a+b}$ is asymptotically stable if $\alpha \geq 1$, where $\alpha = a/b$.

Particularly, it was shown that the equilibrium is stable for integer initial moments. The restriction is caused by the method of investigation: the reduction to difference equation. Our results are for all initial moments from $\mathbb{R}_+$, not only integers. Moreover, we consider uniform stability for the general case. Consequently, we may say that the approach of the paper allows to study stability of the class of equations in complete form.

For our needs, we translate the equilibrium point $N^*$ to origin by the transformation $x = b(N - N^*)$, which takes (4.3) into the following form

$$\dot{x}(t) = -r\left[x(t) + \frac{1}{1+\alpha}\right]\left[\alpha x(t) + \max_{s\in[t-\tau,t]} x(s)\right], \qquad t > 0. \tag{4.4}$$

Note that the function $f(x,y) = -r\left(x + \frac{1}{1+\alpha}\right)(\alpha x + y)$ is a continuous function and has continuous partial derivatives for $x, y \in \Omega \equiv S(\rho)$, $S(\rho) = \{x \in \mathbb{R} : |x| < \rho\}$, $\rho > 0$. If we evaluate the first partial derivatives of the function $f(x,y)$, we see that

$$\begin{aligned}
|\partial f/\partial x| &\le r\left(2\alpha\rho + \rho + \frac{\alpha}{1+\alpha}\right), \\
|\partial f/\partial y| &\le r\left(\rho + \frac{1}{1+\alpha}\right),
\end{aligned}$$

for $x, y \in S(\rho)$ If we choose $L = r(2\alpha\rho + 2\rho + 1)$ as a Lipschitz constant, one can see that the conditions H2.1–H2.3 are fulfilled for sufficiently small $r$.

Suppose that $\alpha \ge 1$ and $\rho < 1/(1+\alpha)$. Then for $V(x) = x^2$, $x \in S(\rho)$ and $t \ge 0$, we have

$$\begin{aligned}
D^+V(t,\phi(t)) &= -2\phi(t)\left[\phi(t) + \frac{1}{1+\alpha}\right]\left[\alpha\phi(t) + \max_{s\in[t-\tau,t]} \phi(s)\right] \\
&\le -2\left[\phi(t) + \frac{1}{1+\alpha}\right]\left[\alpha\phi^2(t) + |\phi(t)||\max_{s\in[t-\tau,t]} \phi(s)|\right] \\
&\le -2\left[\phi(t) + \frac{1}{1+\alpha}\right](\alpha - 1)\phi^2(t) \le 0,
\end{aligned}$$

whenever $V(t+s,\phi(t+s)) \le V(t,\phi(t))$, $s \in [-\tau, 0]$, $t \in \mathbb{R}_+$. Theorem 3.2 implies that the zero solution of (4.4) is uniformly stable. This in turn leads to uniform stability of the positive equilibrium $N^*$ of (4.3).

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