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q-SUBHARMONICITY AND q-CONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. In this paper we study q-subharmonic and q-plurisubharmonic functions in $\mathbb{C}^n$. Next as an application, we give the notion of q-convex domains in $\mathbb{C}^n$ which is an extension of weakly q-convex domains introduced and investigated in [10]. In the end of the paper we show that the q-convexity is the local property and give some examples about q-convex domains.

1. Introduction

The notion about q-plurisubharmonic functions in the case of functions of C^2-class has been introduced and investigated first by Andreotti and Grauert in [2]. After that, Hunt and Murray gave a natural extension of this notion to the class of upper semi-continuous functions (see [13]). Next, in [10] Ho has introduced the class of q-subharmonic functions and weakly q-convex domains in $\mathbb{C}^n$ and proved that the equation $\overline{\partial} u = g$ has solutions u for every $\overline{\partial}$-closed form g of bidegree $(0, r)$ ($r \leqslant q$) on these domains. Recently, H. Ahn and N. Q. Dieu have proved a version of Donnelly-Fefferman theorem for the $\overline{\partial}$-equation on q-convex domains (see [1]).

In this note we continue to study the two classes of q-subharmonic and q-plurisubharmonic functions and domains in $\mathbb{C}^n$ defined by these functions. In Section 2 we recall the definitions of q-subharmonic and q-plurisubharmonic

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functions. We establish some basic properties of the two classes of these functions and give some relations between them. In Section 3 we introduce the notion of q-convex domains in $\mathbb{C}^n$. It is an extension of the weakly q-convex domains introduced and investigated by L.-H. Ho in [10]. The main result in this section is to prove that the q-convexity is the local property (see Theorem 3.3). We remark that the implication from local q-convexity to q-convexity is more difficult than the classical proofs for pseudoconvexity and hyperconvexity. More precisely, in the pseudoconvex case, the idea is to use plurisubharmonicity of $-\log d_\Omega(z)$. However, we show that, for general, q-convex domains, $-\log d_\Omega$ may not be q-subharmonic. On the other hand, for hyperconvexity, we use the upper boundedness of local functions in the patching processing (see [12]). This fact is again not available in our context. Therefore, it is necessary to introduce new techniques in the proof for the q-convex case. An another remarkable result is to establish the q-convexity of Hartogs domains which we present in the end of the paper.

2. q-subharmonic and q-plurisubharmonic functions in $\mathbb{C}^n$

First we recall the following definition of q-subharmonic functions which has been introduced by H. Ahn and N. Q. Dieu in [1] (also see [10]).

Definition 2.1. Let Ω be an open set in $\mathbb{C}^n$ and let $1 \leqslant q \leqslant n$. A semi-continuous function u defined in Ω is called a q-subharmonic function if for every q-dimension space L in $\mathbb{C}^n$, $u|_L$ is a subharmonic function on $L \cap \Omega$. This means that for every compact subset $K \Subset L \cap \Omega$ and every continuous harmonic function h on K such that $u \leqslant h$ on ∂K then $u \leqslant h$ on K.

The set of all q-subharmonic functions in Ω is denoted by $SH_q(\Omega)$.

Compared with subharmonic and plurisubharmonic functions in potential and pluripotential theory it is easy to see that 1-subharmonic functions are plurisubharmonic and n-subharmonic functions are subharmonic.

Next we recall the definition of q-plurisubharmonic functions given by Hunt and Murray in [13] (also see [6]).

Definition 2.2. Let Ω be an open set in $\mathbb{C}^n$ and $u\colon \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. Let q be an integer, $0 \leqslant q \leqslant n-1$. u is said to be a q-plurisubharmonic function on Ω if for every complex linear subspace

L of dimension $q+1$ intersecting Ω, for every closed ball $\bar{\mathbb{B}}$ in L and for every smooth plurisuperharmonic function g defined in a neighborhood of $\bar{\mathbb{B}}$ in L satisfying $u \leqslant g$ on $\partial\mathbb{B}$ it follows that $u \leqslant g$ on $\bar{\mathbb{B}}$. Here a function g is said to be plurisuperharmonic if $-g$ is a plurisubharmonic function. The set of all q-plurisubharmonic functions in Ω is denoted by $PSH_q(\Omega)$.

The following basic properties of q-subharmonic functions can be proved in the same way as for subharmonic functions.

Proposition 2.3. *Let Ω be an open set in $\mathbb{C}^n$ and $1 \leqslant q \leqslant n$. Then the following hold:*

1) *$SH_q(\Omega)$ is a convex cone.*
2) *If $\{u_\alpha\}$, $\alpha \in A$ is a family of q-subharmonic functions and $u = \sup\limits_{\alpha\in A} u_\alpha < +\infty$, u is upper semi-continuous then u is a q-subharmonic function.*
3) *If $\{u_j\}_{j=1}^{\infty}$ is a decreasing sequence of q-subharmonic functions then so is $u = \lim\limits_{j\to+\infty} u_j$.*
4) *If u is a q-subharmonic function in Ω then $u_\varepsilon := u * \varrho_\varepsilon$ is smooth q-subharmonic in Ω_ε, where $\Omega_\varepsilon = \{z \in \Omega : \ d(z, \partial\Omega) > \varepsilon\}$ and $\varrho_\varepsilon = \varrho(z/\varepsilon)/|\varepsilon|^{2n}$, ϱ is a non-negative smooth radial function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int\limits_{\mathbb{C}^n} \varrho \, \mathrm{d}V_n = 1$. Moreover, $u * \varrho_\varepsilon$ is decreasing to u when $\varepsilon \downarrow 0$.*
5) *If χ is a convex increasing function in $\mathbb{R}$ and u is q-subharmonic in Ω, then so is $\chi \circ u$.*

Proof. In fact, the proof of this proposition are from properties of subharmonic functions. However, for convenience to readers we provide a bit more details. The proof of 4) and 5) is exact as in [1: Proposition 1.2]. Now, we give the proof of 1). Assume that $u, v \in SH_q(\Omega)$, $\alpha, \beta \geqslant 0$. For every q-dimension space L in $\mathbb{C}^n$, since $u|_L, v|_L$ are subharmonic functions on $L\cap\Omega$ and $SH(L\cap\Omega)$ is a convex cone so it follows that $(\alpha u + \beta v)|_L = \alpha u|_L + \beta v|_L$ is a subharmonic function on $L\cap\Omega$. Hence, $\alpha u + \beta v \in SH_q(\Omega)$ and 1) follows. Similarly, it is easy to see that 2) and 3) hold because these properties are true for subharmonic functions. □

Now we give the following.

Theorem 2.4. *Let u be a upper-semicontinuous function on $\Omega \subset \mathbb{C}^n$ and $u \in L^1(\Omega, \mathrm{loc})$, where $L^1(\Omega, \mathrm{loc})$ denotes the set of locally integrable functions on Ω. Then the following statements are equivalent.*

1) *u is a q-subharmonic function in Ω.*

2) *For every subharmonic function g on $\Omega \cap L$, where L is a q-dimension subspace of $\mathbb{C}^n$ and for every ball $\mathbb{B}$ in $\mathbb{C}^n$ such that $u + g \leqslant 0$ on $\partial\mathbb{B} \cap L$ it follows that $u + g \leqslant 0$ on $\mathbb{B} \cap L$.*

3) $\mathrm{i}\partial\overline{\partial}u \wedge \omega^{q-1} \geqslant 0$ *in the sense of currents, where* $\omega := \mathrm{i}\partial\overline{\partial}|z|^2$.

4) *For each $(0,q)$-form $f = \sum\limits_{|J|=q}{}' f_J \mathrm{d}\overline{z}_J$ with constant coefficients and every non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ the inequality holds*

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int\limits_{\Omega} u f_{jK} \overline{f_{kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}} \geqslant 0.$$

P r o o f.

1) $\Longrightarrow$ 2) We assume that g is as in the statement of the theorem. From Definition 2.1 it follows that $u + g$ is a subharmonic function on $\Omega \cap L$. Hence, the maximum principle for subharmonic functions gives 2). By the hypothesis 2) and Definition 2.1 we infer that 2) $\Longrightarrow$ 1).

Now, we prove 1) $\Longleftrightarrow$ 3). First, we assume that $u \in C^2(\Omega)$. Let $u \in SH_q(\Omega)$ and $z_0 \in \Omega$. By [4, chapter IX] we can choose a system of coordinates $(z_1, \dots, z_n)$ of $\mathbb{C}^n$ such that the Hessian $\left(\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z_0)\right)$ is diagonal. Assume that $H \subset \mathbb{C}^n$ is a q-dimension subspace of $\mathbb{C}^n$ with $z_0 \in H$. Then by the hypothesis $u|_{\Omega \cap H}$ is subharmonic on $\Omega \cap H$ then $\sum\limits_{k \in K} \dfrac{\partial^2 u}{\partial z_k \partial \overline{z}_k}(z_0) \geqslant 0$ for all $K = \{k_1, k_2, \dots, k_q\} \subset \{1, 2, \dots, n\}$. It follows that $\mathrm{i}\partial\overline{\partial}u(z_0) \wedge \omega^{q-1} \geqslant 0$. Hence, $\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} \geqslant 0$ for all $z \in \Omega$. Conversely, assume that $\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} \geqslant 0$ for all $z \in \Omega$. Let L be a q-dimension space of $\mathbb{C}^n$. Since $\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} \geqslant 0$, $z \in \Omega$ it follows that $u \in SH(\Omega \cap L)$. Hence, $u \in SH_q(\Omega)$. Thus (1) $\Longleftrightarrow$ (3) holds in the case $u \in C^2(\Omega)$. Let u be as in the statement of the theorem. By putting $u_\varepsilon = u * \varrho_\varepsilon$ and applying the above results to u_ε we obtain the desired conclusion.

Finally, we prove 1) $\Longleftrightarrow$ 4). By Ho (see [10]), it is easy to see that this fact is true if $u \in \mathcal{C}^2(\Omega)$. In the case u is arbitrary we note that the assertion is true for u_ε. Let $\varepsilon \searrow 0$ we obtain the assertion for u and the proof of the theorem is complete. □

Next we obtain the following interest result which is an extension of a result of Ho (see [10: Theorem 2.4]).

Proposition 2.5. *Let Ω be an open set in $\mathbb{C}^n$. Assume that ρ is a smooth negative function in Ω such that*

$$\sup_{z\in\Omega}\left|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(z)\right|\leqslant 1$$

for every $1\leqslant j,k\leqslant n$. Then there exists a constant $C_1>0$ only depending on q, n such that $-\log(-\rho)+C_1|z|^2\in SH_q(\Omega)$ if and only if there exists a constant $C_2>0$ only depending on q, n such that

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}-C_2\rho\delta_{j,k}\right)u_{jK}\overline{u_{kK}}\geqslant 0 \tag{2.1}$$

for every $(0,q)$-form $u=\sum_{|J|=q}{}' u_J d\overline{z}_J$ satisfying $\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}=0$, $|K|=q-1$, where δ_{jk} denotes the Kronecker symbol.

P r o o f. The proof is almost the same as the ones given by Ho (see [10: Theorem 2.4]). For convenience to readers, we sketch the proof of the proposition. Assume that $-\log(-\rho)+C_1|z|^2\in SH_q(\Omega)$. By Theorem 2.4, we have

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n}\left(-\frac{1}{\rho}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}+\frac{1}{\rho^2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}+C_1\delta_{j,k}\right)u_{jK}\overline{u_{kK}}\geqslant 0$$

for all $(0,q)$-form u. Hence, if we let $C_2=C_1$ then (2.1) follows.

Conversely, assume that (2.1) is satisfied. Let u be an arbitrary $(0,q)$-form. First we prove that

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\geqslant qC_2\rho|u|^2-C_3\sum_{|K|=q-1}{}'\left|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\right||u|, \tag{2.2}$$

where C_3 is a positive constant only depending on q, n. Indeed, let v be a $(0,q)$-form with coefficients defined by

$$v_{h_1\dots h_q}(z)=\begin{cases}\displaystyle\sum_{j_1,\dots,j_q=1}^{n}\left(\delta_{j_1h_1}-\frac{\frac{\partial\rho}{\partial z_{j_1}}\frac{\partial\rho}{\partial\overline{z}_{h_1}}}{\sum_{k=1}^{n}|\frac{\partial\rho}{\partial z_k}|^2}\right)\dots \\ \qquad\dots\left(\delta_{j_qh_q}-\frac{\frac{\partial\rho}{\partial z_{j_q}}\frac{\partial\rho}{\partial\overline{z}_{h_q}}}{\sum_{k=1}^{n}|\frac{\partial\rho}{\partial z_k}|^2}\right)u_{j_1\dots j_q}(z) & \text{if } |\partial\rho(z)|\neq 0\\ u_{h_1\dots h_q}(z) & \text{if } |\partial\rho(z)|=0.\end{cases}$$

Put $w = u - v$. It is easy to see that

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} v_{jK}(z) = 0 \qquad \text{and} \qquad |w| \leqslant C_4 \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big|,$$

where $C_4 > 0$ is a constant only depending on q, n. Hence, we get

$$\begin{aligned}
&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u_{kK}} \\
&= \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} v_{jK} \overline{v_{kK}} + 2\Re \Bigg(\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} v_{jK} \overline{w_{kK}} \Bigg) \\
&\qquad + \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_{jK} \overline{w_{kK}} \\
&\geqslant qC_2 \rho |u|^2 - 2 \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} (|v_{jK}| \cdot |w_{kK}| + |w_{jK}| \cdot |w_{kK}|) \\
&\geqslant qC_2 \rho |u|^2 - C_3 \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big| |u|,
\end{aligned}$$

where $C_3 > 0$ is a constant only depending only q, n. Hence (2.2) is proved.

Now, using (2.2), we have

$$\begin{aligned}
&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 (-\log(-\rho))}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u_{kK}} \\
&= -\frac{1}{\rho} \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u_{kK}} + \frac{1}{\rho^2} \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big|^2 \\
&\geqslant -\frac{1}{\rho} \Big(qC_2 \rho |u|^2 - C_3 \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big| \cdot |u| \Big) + \frac{1}{\rho^2} \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big|^2 \\
&= -qC_2 |u|^2 + \sum_{|K|=q-1}{}' C_3 |u| . \frac{1}{\rho} \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big| + \frac{1}{\rho^2} \sum_{|K|=q-1}{}' \Big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \Big|^2.
\end{aligned}$$

Applying the inequality $-|xy| \geqslant -|\frac{x}{2}|^2 - |y|^2$ to the second term with $x = C_3|u|$, $y = \frac{1}{\rho} \big| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} u_{jK} \big|$, we get

$$\begin{aligned}\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} &\frac{\partial^2(-\log(-\rho))}{\partial z_j \partial \overline{z}_k} u_{jK}\overline{u_{kK}} \\ &\geqslant -qC_2|u|^2 - \sum_{|K|=q-1}{}' \left(\frac{C_3}{2}\right)^2 |u|^2 \\ &= -\left(qC_2 + \frac{n!}{4(q-1)!(n-q+1)!}(C_3)^2\right)|u|^2.\end{aligned}$$

Put $C_1 = C_2 + \frac{n!}{4q(q-1)!(n-q+1)!}(C_3)^2$, we have $-\log(-\rho) + C_1|z|^2 \in SH_q(\Omega)$ and the desired conclusion follows. □

Remark 2.6. Let Ω be an open set in $\mathbb{C}^n$ and $u \in \mathrm{L}^1(\Omega, \mathrm{loc})$.

1) if $u \in C^2(\Omega)$ such that $\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) = 0$, for all $j \neq k$ and $z \in \Omega$. Then $u \in SH_q(\Omega)$ if and only if $\sum\limits_{j,k \in J} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \geqslant 0$, for all $|J| = q$ and for all $z \in \Omega$. Indeed, it is easy to see that

$$\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} = 2^q \sum_{|J|=q}{}' \sum_{j,k \in J} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\, \mathrm{d}V_J$$

where $\mathrm{d}V_J = \bigwedge\limits_{j \in J} \frac{i}{2} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$. Thus the desired conclusion follows from Theorem 2.4.

2) If u is a q-subharmonic function in Ω then u is also a r-subharmonic function in Ω for every $q \leqslant r \leqslant n$. Indeed, since u is q-subharmonic then $\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} \geqslant 0$ for all $z \in \Omega$. Hence

$$\mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{r-1} = \mathrm{i}\partial\overline{\partial}u(z) \wedge \omega^{q-1} \wedge \omega^{r-q} \geqslant 0$$

for every $r \geqslant q$. Thus u is r-subharmonic.

3) By 2) of Theorem 2.4 then every q-subharmonic function in Ω is $(q-1)$-plurisubharmonic.

4) In [14] Slodkowski proved that if $\varphi \in PSH_q(\Omega)$ and $\psi \in PSH_q(\Omega)$ then $\varphi + \psi \in PSH_{p+q}(\Omega)$. On the other hand, $SH_{p+1}(\Omega) + SH_{q+1}(\Omega) \subset SH_{\max(p+1,q+1)}(\Omega)$. Thus there exists $\varphi \in PSH_q(\Omega) \backslash SH_{q+1}(\Omega)$. Indeed, in the converse case, we have

$$PSH_q(\Omega) + PSH_q(\Omega) \subset PSH_q(\Omega).$$

However, if we take $\varphi(z_1, z_2) = |z_1|^2 - 2|z_2|^2$, $\psi(z_1, z_2) = -2|z_1|^2 + |z_2|^2$ then φ, $\psi \in PSH_1(\mathbb{C}^2)$. But $\varphi + \psi \notin PSH_1(\mathbb{C}^2)$ and we get a contradiction.

Corollary 2.7. *Let u be in $SH_q(\Omega) \cap L^1(\Omega, \mathrm{loc})$ and let $D \subset \mathbb{C}$ be an open set. Then $u \circ \pi$ is a $(q+1)$-subharmonic function on $\Omega \times D$, where $\pi \colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ denotes the canonical projection. Moreover, for every $n > 1$ and for all $1 < q \leqslant n$, there exists a q-subharmonic function on Ω such that $\varphi \circ \pi$ is not q-subharmonic on $\Omega \times \mathbb{C}$.*

P r o o f. Let $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$. Then

$$\mathrm{i}\partial\overline{\partial}|z|^2 = \mathrm{i}\partial\overline{\partial}|z'|^2 + \mathrm{i}\partial\overline{\partial}|z_{n+1}|^2,$$

and

$$(\mathrm{i}\partial\overline{\partial}|z|^2)^q = (\mathrm{i}\partial\overline{\partial}|z'|^2)^q + q(\mathrm{i}\partial\overline{\partial}|z'|^2)^{q-1} \wedge \mathrm{i}\partial\overline{\partial}|z_{n+1}|^2.$$

On the other hand, since $u \in SH_q(\Omega)$ then $\mathrm{i}\partial\overline{\partial}u \wedge (\mathrm{i}\partial\overline{\partial}|z'|^2)^{q-1} \geqslant 0$. Hence, it follows that

$$\begin{aligned} &\mathrm{i}\partial\overline{\partial}(u \circ \pi) \wedge (\mathrm{i}\partial\overline{\partial}|z|^2)^q \\ &= \mathrm{i}\partial\overline{\partial}u \wedge (\mathrm{i}\partial\overline{\partial}|z'|^2)^q + qi\partial\overline{\partial}u \wedge (\mathrm{i}\partial\overline{\partial}|z'|^2)^{q-1} \wedge \mathrm{i}\partial\overline{\partial}|z_{n+1}|^2 \geqslant 0. \end{aligned}$$

This shows that $u \circ \pi \in SH_{q+1}(\Omega \times D)$. Next, we prove the second conclusion of the corollary. Consider the function $\varphi(z) = |z|^2 - q|z_1|^2$. Remark 2.6 implies that φ is a q-subharmonic function in Ω. However, $\varphi_1 = \varphi \circ \pi$ is not q-subharmonic in $\Omega \times \mathbb{C}$. Indeed, we have

$$\varphi_1(z, z_{n+1}) = |z|^2 - q|z_1|^2.$$

Hence

$$\frac{\partial^2 \varphi_1}{\partial z_1 \partial \overline{z}_1} + \sum_{j=n-q+2}^{n+1} \frac{\partial^2 \varphi_1}{\partial z_j \partial \overline{z}_j} = -1 < 0.$$

From remark 2.6 the desired conclusion follows. □

Theorem 2.8. *Let $\Omega \subset \mathbb{C}^n$ be an open set and $\pi \colon \mathbb{C}^{n+m} \longrightarrow \mathbb{C}^n$ denote the canonical projection. Then $u \circ \pi$ is q-plurisubharmonic on $\Omega \times \mathbb{C}^m$ for every q-plurisubharmonic function u on Ω.*

We need the following.

Lemma 2.9. *Let $\Omega \subset \mathbb{C}^n$ be an open set and u an upper semi-continuous function on Ω. Then $\widetilde{u} = u \circ \pi$ is n-plurisubharmonic on $\Omega \times \mathbb{C}$, where $\pi \colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ is the canonical projection.*

P r o o f. Let $\widetilde{\mathbb{B}}$ be an open ball in $\Omega\times\mathbb{C}$ with the radius $r>0$ and $\widetilde{g}$ a C^∞-function on $\widetilde{\mathbb{B}}$ which is plurisuperharmonic on $\Omega\times\mathbb{C}$ such that $\widetilde{u}\leqslant\widetilde{g}$ on $\partial\widetilde{\mathbb{B}}$. For each $(z^0,z^0_{n+1})\in\mathbb{B}$ which we may assume that $(z^0,z^0_{n+1})=0$, put $g(z)=\widetilde{g}(z,0)$ and $g_{n+1}(z_{n+1})=\widetilde{g}(0,z_{n+1})$. Then g_{n+1} is a C^∞-function. Since $-g_{n+1}$ is plurisubharmonic it follows that

$$\begin{aligned}(\widetilde{u}-\widetilde{g})(0)&=(u-g)(0)=\widetilde{u}(0,r)-\widetilde{g}(0)\\&\leqslant\widetilde{u}(0,r)+\sup_{|z_{n+1}|=r}-g_{n+1}(z_{n+1})\\&\leqslant\sup_{|z_{n+1}|=r}(\widetilde{u}(0,z_{n+1})-\widetilde{g}(0,z_{n+1}))\leqslant\sup_{\partial\widetilde{\mathbb{B}}}(\widetilde{u}-\widetilde{g}).\end{aligned}$$

Hence by Definition 2.2 it follows that $\widetilde{u}$ is n-plurisubharmonic. The lemma is proved. □

Lemma 2.10. *Let $T\colon\mathbb{C}^n\to\mathbb{C}^n$ be an isomorphism and let $V\subset\mathbb{C}^n$ be an open set. Then $u\circ T\in PSH_q(T^{-1}(V))$ for every $u\in PSH_q(V)$.*

P r o o f. Let $\bar{\mathbb{B}}$ be a closed ball in V and φ a smooth plurisuperharmonic function in $\mathbb{C}^n$ such that $u\circ T\leqslant\varphi$ on $\partial\mathbb{B}$. Then $u\leqslant\varphi\circ T^{-1}$ on $\partial(T(\mathbb{B}))$. Since $-\varphi\circ T^{-1}$ is plurisubharmonic on $\mathbb{C}^n$ then $u+(-\varphi\circ T^{-1})$ is q-plurisubharmonic. The maximum principle for q-plurisubharmonic functions in [7] implies that $u-\varphi\circ T^{-1}\leqslant 0$ on $T(\mathbb{B})$. Hence $u\circ T\leqslant\varphi$ on $\bar{\mathbb{B}}$. □

P r o o f o f T h e o r e m 2.8. Without loss of generality we may assume that $m=1$. First we note that by [13] a function u is q-plurisubharmonic on $\Omega\subset\mathbb{C}^n$ if and only if u is q-plurisubharmonic on $\Omega\cap L$ for every complex linear subspace L of dimension $q+1$. Now let L be a $(q+1)$-dimension linear subspace of $\mathbb{C}^{n+1}$, it remains to show that $\widetilde{u}=u\circ\pi$ is q-plurisubharmonic on $(\Omega\times\mathbb{C})\cap L$. Put $\pi_0=\pi|_L$. Consider the two cases.

Case 1. Assume that $\ker\pi_0=0$. Then $\pi_0\colon L\to\pi_0(L)$ is an isomorphism. It is easy to see that $\pi_0(\Omega\times\mathbb{C}\cap L)=\Omega\cap\pi_0(L)$ and u is q-plurisubharmonic on $\Omega\cap\pi_0(L)$. By Lemma 2.10 we infer that $\widetilde{u}=u\circ\pi$ is q-plurisubharmonic on $(\Omega\times\mathbb{C})\cap L$.

Case 2. If $\ker\pi_0\neq0$. Put $L'=\pi_0(L)$. Then $L=L'\times\mathbb{C}$. Indeed, assume that $e_1,e_2,\ldots,e_{q+1}$ is a basis of L. Write $e_j=(\mathrm{e}^j,z^j_{n+1})$. Then $\mathrm{e}^1,\mathrm{e}^2,\ldots,\mathrm{e}^{q+1}\subset L'$. Notice that $\dim L'\leqslant q$. Hence $\mathrm{e}^1,\mathrm{e}^2,\ldots,\mathrm{e}^{q+1}$ is linearly dependent. Without loss of generality we may assume that $\mathrm{e}^1=\sum\limits_{j=2}^{q+1}\lambda_j\mathrm{e}^j$. Thus, the element

$e_1 - \sum_{j=2}^{q+1} \lambda_j e_j = (0, z_{n+1}) \in L$, where $z_{n+1} \neq 0$. From this result it is easy to see that $L = L' \times \mathbb{C}$. By Lemma 2.9 it follows that $\widetilde{u} = u \circ \pi$ is q-plurisubharmonic on $\Omega \times \mathbb{C} \cap L$. Hence, theorem 2.8 is completely proved. □

Now we extend the notion of q-subharmonic functions with respect to a positive $(1,1)$-form Θ. Let Θ be a positive $(1,1)$-form with C^∞ coefficients and let $1 \leqslant q \leqslant n$. We give the following definition.

Definition 2.11. Let Ω be an open set in $\mathbb{C}^n$ and $u\colon \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. Let Θ be a positive $(1,1)$-form with C^∞ coefficients and $1 \leqslant q \leqslant n$. We say that u is a Θ^{q-1}-subharmonic function on Ω if for every ball $\mathbb{B} \Subset \Omega$ there exists a decreasing sequence $\{u_j\} \subset C^2(\mathbb{B})$ such that $\mathrm{i}\partial\overline{\partial} u_j \wedge \Theta^{q-1} \geqslant 0$ in $\mathbb{B}$ and $\{u_j\}$ pointwise converges to u on $\mathbb{B}$. The set of all Θ^{q-1}-subharmonic functions in Ω is denoted by $SH_{\Theta^{q-1}}(\Omega)$.

Remark 2.12. Assume that $u, u_1, u_2, \ldots \in C^2(\Omega)$ is such that $\mathrm{i}\partial\overline{\partial} u_j \wedge \Theta^{q-1} \geqslant 0$, $j = 1, 2, \ldots$ and $\{u_j\}$ is decreasing and pointwise converges to u. Then $\mathrm{i}\partial\overline{\partial} u \wedge \Theta^{q-1} \geqslant 0$ and, hence, a function $u \in C^2(\Omega)$ is Θ^{q-1}-subharmonic if and only if $\mathrm{i}\partial\overline{\partial} u \wedge \Theta^{q-1} \geqslant 0$. Indeed, the sufficiency is obvious. In order to prove the necessary condition we assume that $\mathrm{i}\partial\overline{\partial} u \wedge \Theta^{q-1}$ is not positive. Then there exists a positive $(n-q, n-q)$-form η with compact support in Ω such that

$$\int_\Omega \mathrm{i}\partial\overline{\partial} u \wedge \Theta^{q-1} \wedge \eta < 0$$

On the other hand, since $\{u_j\}$ is decreasing and pointwise converges to u we have

$$\lim_{j\to+\infty} \int_\Omega (u_j - u)\mathrm{i}\partial\overline{\partial}(\Theta^{q-1} \wedge \eta) = 0$$

Hence we have

$$0 \leqslant \lim_{j\to+\infty} \int_\Omega u_j \mathrm{i}\partial\overline{\partial}(\Theta^{q-1} \wedge \eta) = \int_\Omega \mathrm{i}\partial\overline{\partial} u \wedge \Theta^{q-1} \wedge \eta < 0$$

and we get a contradiction.

Now, we give some basic properties of Θ^{q-1}-subharmonic functions.

Proposition 2.13. *Let Θ be a positive $(1,1)$-form with C^∞ coefficients and let $1 \leqslant q \leqslant n$. Then the following hold:*

1) *$SH_{\Theta^{q-1}}(\Omega)$ is a convex cone.*

2) *If u is Θ^{q-1}-subharmonic in Ω then u is Θ^{r-1}-subharmonic for every $q \leqslant r \leqslant n$.*

3) *If χ is a smooth convex increasing function in $\mathbb{R}$ and u is Θ^{q-1}-subharmonic in Ω then so is $\chi \circ u$.*

4) *If $\{v_j\}_{j=1}^{\infty}$ is a decreasing sequence of Θ^{q-1}-subharmonic functions then so is $v = \lim\limits_{j\to+\infty} v_j$.*

Proof.

1) Assume that $u, v \in SH_{\Theta^{q-1}}(\Omega)$ and $\alpha, \beta \geqslant 0$ and $\mathbb{B} \Subset \Omega$ is a ball. By Definition 2.11 there exist two decreasing sequences $\{u_j\}, \{v_j\} \subset C^2(\mathbb{B})$ which are pointwise convergent to u and v on $\mathbb{B}$, respectively, such that $\mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} \geqslant 0$ and $\mathrm{i}\partial\overline{\partial}v_j \wedge \Theta^{q-1} \geqslant 0$ in $\mathbb{B}$. It follows that $\{\alpha u_j + \beta v_j\} \subset C^2(\mathbb{B})$ pointwise converges to $\alpha u + \beta v$ on $\mathbb{B}$ and $\mathrm{i}\partial\overline{\partial}(\alpha u_j + \beta v_j) \wedge \Theta^{q-1} = \alpha \mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} + \beta \mathrm{i}\partial\overline{\partial}v_j \wedge \Theta^{q-1} \geqslant 0$ in $\mathbb{B}$. Hence, the desired conclusion follows.

2) Assume that $u \in SH_{\Theta^{q-1}}(\Omega)$ and r is an integer with $q \leqslant r \leqslant n$. For every ball $\mathbb{B} \Subset \Omega$ by definition 2.11 there exists a decreasing sequence $\{u_j\} \subset C^2(\mathbb{B})$ which is pointwise convergent to u on $\mathbb{B}$ such that $\mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} \geqslant 0$ in $\mathbb{B}$. Since

$$\mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{r-1} = \mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} \wedge \Theta^{r-q} \geqslant 0,$$

it follows that $u \in SH_{\Theta^{r-1}}(\Omega)$.

3) Assume that $u \in SH_{\Theta^{q-1}}(\Omega)$. Let $\mathbb{B} \Subset \Omega$ be a ball. Definition 2.11 implies that there exists a decreasing sequence $\{u_j\} \subset C^2(\mathbb{B})$ which pointwise converges to u on $\mathbb{B}$ such that $\mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} \geqslant 0$ in $\mathbb{B}$. Since χ is a smooth convex increasing function in $\mathbb{R}$ so the sequence $\{\chi \circ u_j\} \subset C^2(\mathbb{B})$ decreasing pointwise to $\chi \circ u$ on $\mathbb{B}$ and

$$\mathrm{i}\partial\overline{\partial}\chi \circ u_j \wedge \Theta^{q-1} = \chi' \circ u_j \mathrm{i}\partial\overline{\partial}u_j \wedge \Theta^{q-1} + \chi'' \circ u_j \mathrm{i}\partial u_j \wedge \overline{\partial}u_j \wedge \Theta^{q-1} \geqslant 0.$$

Hence, $\chi \circ u \in SH_{\Theta^{q-1}}(\Omega)$.

We prove 4). First we prove the following assertion. Let $\{u_j\}$ and $\{v_j\}$ be two decreasing sequences of continuous functions which pointwise converge to u and v in Ω respectively. Assume that $u < v$ on Ω and $\Omega' \Subset \Omega$. Then there exists a subsequence $\{u_{k_j}\}$ of the sequence $\{u_j\}$ such that $u_{k_j} < v_j$ in Ω' for every j. Indeed, we begin with v_1. Let $z \in \Omega$. Since $\lim\limits_{j\to\infty}(u_j - v_1)(z) = (u - v_1)(z) \leqslant (u - v)(z) < 0$ so there exist $r_z > 0$ and $M_z > 0$ such that $u_j < v_1$ in $B(z, r_z)$ for every $j \geqslant M_z$. Moreover, because $\Omega' \Subset \Omega$ so there exist $z^1, \ldots, z^m$ such that $\Omega' \Subset \bigcup\limits_{l=1}^{m} B(z^l, r_{z^l})$. Put $k_1 = \max\{M_{z^l} : l = 1, 2, \ldots, m\}$. It is clear that

$u_{k_1} < v_1$ in Ω'. Similarly, for v_2 we can choose $k_2 > k_1$ such that $u_{k_2} < v_2$ in Ω'. Continuing this process we get the desired subsequence $\{k_j\}$.

Next we give the proof of 4). We may assume that $v_{j+1} < v_j$ in Ω for all $j \geqslant 1$. Let $\mathbb{B} \Subset \Omega$ be a ball. Choose a ball $\mathbb{B}_1$ such that $\mathbb{B} \Subset \mathbb{B}_1 \Subset \Omega$. Since v_1 is Θ^{q-1}-subharmonic there exists a sequence $\{v_1^k\} \subset C^2(B_1)$, $v_1^k \searrow v_1$ on $\mathbb{B}_1$ as $k \to \infty$. Since v_2 is Θ^{q-1}-subharmonic there exists a sequence $\{v_2^k\} \subset C^2(B_1)$, $v_2^k \searrow v_2$ on $\mathbb{B}_1$ as $k \to \infty$.

Using the above result we can choose a subsequence $\{v_2^{l_k^2}\}$ of the sequence $\{v_2^k\}$ that $v_2^{l_k^2} < v_1^k$ for all k. Now again since v_3 is Θ^{q-1}-subharmonic there exists a sequence $\{v_3^k\} \subset C^2(B_1)$, $v_3^k \searrow v_3$ on $\mathbb{B}_1$ as $k \to \infty$.

Repeating the above result we can choose a subsequence $\{v_3^{l_k^3}\}$ of the sequence $\{v_3^k\}$ such that $v_3^{l_k^3} < v_2^{l_k^2}$ for all k. Continuing the process by induction we choose subsequences $\{v_{m+1}^{l_k^{m+1}}\}$ of the sequences $\{v_{m+1}^k\}$ such that $v_{m+1}^{l_k^{m+1}} < v_m^{l_k^m}$ for all k, m and $v_{m+1}^{l_k^{m+1}} \searrow v_{m+1}$ as $k \to \infty$.

Put $u_j = v_j^{l_j^j} \in C^2(\mathbb{B}_1)$, $j \geqslant 1$. It is easy to see that the sequence $\{u_j\}$ is decreasing in $\mathbb{B}$. For every $z \in \mathbb{B}$ we have $v(z) \leqslant v_j(z) \leqslant v_j^{l_j^j}(z) = u_j(z)$, for all $j \geqslant 1$. At the same time, we have $v(z) \leqslant \lim\limits_{j\to+\infty} u_j(z) = \lim\limits_{j\to+\infty} v_j^{l_j^j}(z) \leqslant \lim\limits_{j\to+\infty} v_k^{l_j^k}(z) = v_k(z)$ for every k. Therefore we get $\lim\limits_{j\to+\infty} u_j(z) = v(z)$. The proof is complete. □

Now, we consider in particular $\Theta = \mathrm{i}\partial\overline{\partial}\psi$, where $\psi \in C^\infty(\Omega)$ is a strictly plurisubharmonic function. We have relations between the $(\mathrm{i}\partial\overline{\partial}\psi)^{q-1}$-subharmonic and $(q-1)$-plurisubharmonic functions as follows.

Proposition 2.14. *Let $\psi \in C^\infty(\Omega)$ be a plurisubharmonic function and $u \in C^2(\Omega)$. Then the following conditions are equivalent.*

1) *u is $(\mathrm{i}\partial\overline{\partial}\psi)^{q-1}$-subharmonic in Ω if and only if $u|_{L\cap\Omega}$ is $(\mathrm{i}\partial\overline{\partial}\psi|_{L\cap\Omega})^{q-1}$-subharmonic in $L \cap \Omega$ for every q-dimension subspace L of $\mathbb{C}^n$.*
2) *if u is a $(\mathrm{i}\partial\overline{\partial}\psi)^{q-1}$-subharmonic function then u is a $(q-1)$-plurisubharmonic function. Hence, every $(\mathrm{i}\partial\overline{\partial}\psi)^{q-1}$-subharmonic function satisfies the local maximum principle.*

Proof.

1) *Necessary.* Let L be a q-dimension space of $\mathbb{C}^n$. By a unitary change of coordinates we may assume that L has the basis $\{e_1, e_2, \ldots, e_q\}$, where $e_j =$

$(\underbrace{0,\dots,0,1}_{j},0,\dots,0)$ for $1 \leqslant j \leqslant q$. Since u is $(\mathrm{i}\partial\overline{\partial}\psi)^{q-1}$-subharmonic in Ω, we have $\mathrm{i}\partial\overline{\partial}u \wedge (\mathrm{i}\partial\overline{\partial}\psi)^{q-1} \geqslant 0$. Hence, $\mathrm{i}\partial\overline{\partial}u \wedge (\mathrm{i}\partial\overline{\partial}\psi)^{q-1} \wedge \mathrm{i}\mathrm{d}z_{q+1} \wedge \mathrm{d}\overline{z}_{q+1} \wedge \cdots \cdots \wedge \mathrm{i}\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n \geqslant 0$. It follows that $\mathrm{i}\partial\overline{\partial}u|_{L\cap\Omega} \wedge (\mathrm{i}\partial\overline{\partial}\psi|_{L\cap\Omega})^{q-1} \geqslant 0$.

Sufficient. Let $x_0 \in \Omega$. Choose an orthogonal basis $\{f_1,\dots,f_n\}$ of $\mathbb{C}^n$ with coordinates $w=(w_1,\dots,w_n)$ such that $\mathrm{i}\partial\overline{\partial}\psi(x_0) = \mathrm{i}\sum_{j=1}^{n} \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$ and $\mathrm{i}\partial\overline{\partial}u(x_0) = \mathrm{i}\sum_{j=1}^{n} \lambda_j \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$. Hence, we have

$$\mathrm{i}\partial\overline{\partial}u(x_0) \wedge (\mathrm{i}\partial\overline{\partial}\psi(x_0))^{q-1} = (q-1)! \sum_{|J|=q}{}' \lambda_J \,\mathrm{d}V_J$$

where $\lambda_J = \sum_{k\in J} \lambda_k$ and $\mathrm{d}V_J = \bigwedge_{k\in J} (\mathrm{i}\mathrm{d}w_k \wedge \mathrm{d}\overline{w}_k)$.

Now applying the hypothesis to the subspace $L = \operatorname{span}\{f_k : k \in J\}$ containing x_0, we have

$$\mathrm{i}\partial\overline{\partial}u(x_0)|_L \wedge (\mathrm{i}\partial\overline{\partial}\psi(x_0)|_L)^{q-1} = (q-1)!\lambda_J \,\mathrm{d}V_J \geqslant 0.$$

Hence, $\lambda_J \geqslant 0$.

On the other hand, the positivity is invariant with respect to complex isomorphisms of $\mathbb{C}^n$, we get $\mathrm{i}\partial\overline{\partial}u(x_0) \wedge (\mathrm{i}\partial\overline{\partial}\psi(x_0))^{q-1} \geqslant 0$. This completes the proof of the sufficiency.

2) We may assume that $u \in C^2$. If u is not a $(q-1)$-plurisubharmonic function then there exists $x_0 \in \Omega$ such that the matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(x_0)\right)$ has at least q negative eigenvalues. Take a q-dimension subspace L of $\mathbb{C}^n$ containing x_0 such that $\left(\frac{\partial^2 u|_L}{\partial z_j \partial \overline{z}_k}(x_0)\right)$ has at least q negative eigenvalues. Hence, we have

$$\mathrm{i}\partial\overline{\partial}u|_{L\cap\Omega}(x_0) \wedge \left(\mathrm{i}\partial\overline{\partial}\psi|_{L\cap\Omega}(x_0)\right)^{q-1} < 0$$

which contradicts 1). The proof is complete. □

3. Applications to q-convexity

In this section, we will introduce the notion about q-convex domains and establish some results concerning to these domains. We begin with the following.

Definition 3.1. A domain Ω in $\mathbb{C}^n$ is said to be q-convex if there exists a continuous q-subharmonic exhaustion function in Ω.

By repeating arguments as in the proof of [11: Theorem 2.6.11] we see that if Ω is q-convex then the exhaustion function in the above definition can be chosen to be smooth.

Remark 3.2.

1) The above definition is similar as [1: Definition 1.3]. The difference between the two these definitions is as follows. In the above definition we require that the q-subharmonic exhaustion function is continuous but in [1: Definition 1.3] not to have. The reason is late (see Theorem 3.3 below) we use the continuity of the q-subharmonic exhaustion function to prove the equivalence of the global q-subharmonicity and the local q-subharmonicity of a domain in $\mathbb{C}^n$.

2) By a result in [10] (see [10: Theorem 2.4]) we notice that every weakly q-convex domain in the sense of Ho (see [10: Definition 2.1]) is q-convex.

3) By [7], every n-dimensional connected non compact complex manifold has a strongly subharmonic exhaustion function with respect to any hermitian metric ω. Thus, every open set in $\mathbb{C}^n$ is n-convex.

4) A pseudoconvex domain in $\mathbb{C}^n$ is exactly 1-convex. However if we take $\Omega = \mathbb{B}(0,1) \setminus \{0\} \subset \mathbb{C}^n$. Then, as in 2), Ω is a n-convex domain. We notice that the function $-\log d(\bullet, \partial\Omega) = -\log|z|$ in a neighborhood of 0, and hence, it is not subharmonic in Ω. Thus, there exist q-convex domains in $\mathbb{C}^n$ but the function $-\log d(\bullet, \Omega)$ is not q-subharmonic on Ω. This is also the difference between the q-convex and the pseudoconvex domains.

Now we are position to prove the equivalence between the local q-convexity and the global q-convexity. Namely, we have the following.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then Ω is q-convex if and only if for each $a \in \partial\Omega$ there is a neighborhood W of a such that $W \cap \Omega$ is q-convex.*

The proof of the necessity is obvious. In order to prove the sufficiency we need the following auxiliary lemmas.

Lemma 3.4. *Assume that $\rho > 1$ is a smooth exhaustion function in $\Omega \subset \mathbb{C}^n$. Then there exists a smooth increasing convex function χ such that $\chi \circ \rho \geqslant \rho$ and*

$$\sup_{z\in\Omega} \left| \frac{\partial^2(-\mathrm{e}^{-\chi\circ\rho})}{\partial z_j \partial \overline{z}_k}(z) \right| \leqslant 1, \qquad \text{for all} \quad j,k. \tag{3.1}$$

P r o o f. Let λ be a smooth positive function in $(0,+\infty)$ be chosen such that $\lambda(t) \geqslant \sup\{\lambda_{j,k}(t) : \ j,k = 1,\dots,n\}$, where $\lambda_{j,k}$ is defined by

$$\lambda_{j,k}(t) = \begin{cases} \sup\limits_{z\in\{\rho\leqslant t\}} \Big\{ \big|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(z)\big|, \big|\frac{\partial\rho}{\partial z_j}(z)\frac{\partial\rho}{\partial\overline{z}_k}(z)\big|, t, 1\Big\} & \text{if } \Omega\cap\{\rho\leqslant t\}\neq\varnothing \\ 1 & \text{if } \Omega\cap\{\rho\leqslant t\}=\varnothing. \end{cases}$$

By [5: Lemma VIII-5.7] there exists a smooth positive convex function χ such that χ, χ', $\chi'' \geqslant \lambda$ and $(1+\chi'+\chi'')^2\mathrm{e}^{-\chi} \leqslant \frac{1}{\lambda}$. It is clear that χ is a smooth increasing convex function. We prove that χ satisfies (3.1). Indeed, for every $j,k\in\{1,2,\dots,n\}$ we have

$$\frac{\partial^2(-\mathrm{e}^{-\chi\circ\rho})}{\partial z_j\partial\overline{z}_k} = \mathrm{e}^{-\chi\circ\rho}\left(\chi'\circ\rho\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k} + \chi''\circ\rho\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k} - (\chi'\circ\rho)^2\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}\right).$$

Moreover, since

$$\begin{aligned}\lambda\circ\rho(z) &\geqslant \sup_{w\in\{\rho\leqslant\rho(z)\}}\left\{\left|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(w)\right|, \left|\frac{\partial\rho}{\partial z_j}(w)\frac{\partial\rho}{\partial\overline{z}_k}(w)\right|, \rho(z), 1\right\} \\ &\geqslant \sup\left\{\left|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(z)\right|, \left|\frac{\partial\rho}{\partial z_j}(z)\frac{\partial\rho}{\partial\overline{z}_k}(z)\right|, \rho(z), 1\right\}\end{aligned}$$

for every $z\in\Omega$, and χ, χ', $\chi''\geqslant 1$, we get

$$\begin{aligned}\left|\frac{\partial^2(-\mathrm{e}^{-\chi\circ\rho})}{\partial z_j\partial\overline{z}_k}\right| &\leqslant \mathrm{e}^{-\chi\circ\rho}\left(\chi'\circ\rho\left|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\right| + \chi''\circ\rho\left|\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}\right| + (\chi'\circ\rho)^2\left|\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}\right|\right) \\ &\leqslant \mathrm{e}^{-\chi\circ\rho}\lambda\circ\rho\left(\chi'\circ\rho+\chi''\circ\rho+(\chi'\circ\rho)^2\right) \\ &\leqslant \mathrm{e}^{-\chi\circ\rho}\lambda\circ\rho\left(1+\chi'\circ\rho+\chi''\circ\rho\right)^2 \leqslant 1.\end{aligned}$$

Thus, (3.1) is proved, and the proof of the lemma is complete. □

Now we give the proof of the sufficiency of Theorem 3.3.

P r o o f o f t h e s u f f i c i e n c y o f T h e o r e m 3.3. We split the proof into three steps.

Step 1. We prove that if ρ is a smooth negative function in an open set Ω of $\mathbb{C}^n$ such that

$$\sup_{z\in\Omega}\left|\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(z)\right| \leqslant 1, \qquad \text{for all} \quad 1\leqslant j, \quad k\leqslant n$$

and

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k} - C_2\rho\delta_{j,k}\right)u_{jK}\overline{u_{kK}} \geqslant 0$$

for every $(0,q)$-form $u = \sum_{|J|=q}{}' u_J \mathrm{d}\overline{z}_J$ satisfying $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} u_{jK} = 0$, where $|K| = q-1$ and $C_2 > 0$ is a constant only depending on q, n then for every the continuous increasing convex function $\tau\colon (-\infty; \sup_\Omega \rho) \to \mathbb{R}^+$, for every $\varepsilon > 0$ and for every $\varphi \in \mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega) \cap PSH^-(\Omega)$ such that $\sup_\Omega |\partial\varphi| < \infty$, there exists a constant $C_1 > 0$ only depending on q, n and $\sup_\Omega \left|\frac{\partial\varphi}{\varphi}\right|$ such that

$$-\log\left(1 - \frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\right) + C_1|z|^2 \in SH_q(\Omega).$$

We can assume that τ is smooth (since if τ is continuous then we can approximate τ by a sequence of smooth increasing convex functions). Let $u = \sum_{|J|=q}{}' u_J \mathrm{d}\overline{z_J}$ be a $(0,q)$-form. Since $\varphi \in PSH^-(\Omega)$ so we have

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k} u_{jK}\overline{u_{kK}} \geqslant 0.$$

Hence, by a computation we have

$$\begin{aligned}
&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\left(-\log\left(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\right)\right)}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}} \\
&\quad= \frac{\tau(-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi} \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\left(\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\right)}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}} \\
&\qquad+ \left(\frac{\tau(-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\right)^2 \sum_{|K|=q-1}{}' \left|\sum_{j=1}^{n} \frac{\partial\left(\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\right)}{\partial z_j} u_{jK}\right|^2 \\
&\quad\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi} \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}} \\
&\qquad+ \left(\frac{1}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\right)^2 \sum_{|K|=q-1}{}' \left|\sum_{j=1}^{n} \frac{\partial(\tau\circ(\rho-\varepsilon)+\varphi)}{\partial z_j} u_{jK}\right|^2.
\end{aligned}$$

Next using the inequality $(x+y)^2 \geqslant \frac{x^2}{2} - y^2$, we get

$$
\begin{aligned}
&\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2\big(-\log(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)})\big)}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\\
&\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\\
&\quad+\Big(\frac{1}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\Big)^2.\\
&\quad\cdot\sideset{}{'}\sum_{|K|=q-1}\Big(\frac{1}{2}\Big|\sum_{j=1}^{n}\frac{\partial\tau\circ(\rho-\varepsilon)}{\partial z_j}u_{jK}\Big|^2-\Big|\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}u_{jK}\Big|^2\Big).
\end{aligned}
$$

Moreover, since τ is increasing and $\varphi<0$ so

$$
\frac{1}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}<\frac{1}{-\varphi}.
$$

From

$$
\Big|\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}u_{jK}\Big|^2\leqslant\sum_{j=1}^{n}\Big|\frac{\partial\varphi}{\partial z_j}\Big|^2.\sum_{j=1}^{n}|u_{jK}|^2\leqslant|\partial\varphi|^2.|u|^2,
$$

it follows that

$$
\begin{aligned}
&\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2\big(-\log\big(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\big)\big)}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\\
&\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\\
&\quad+\Big(\frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\Big)^2\sideset{}{'}\sum_{|K|=q-1}\frac{1}{2}\Big|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\Big|^2-C_3|u|^2,
\end{aligned}
$$

where C_3 is a positive constant only depending q, n and $\sup\limits_{\Omega}|\frac{\partial\varphi}{\varphi}|$. On the other hand, by (2.2) we have

$$
\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u_{kK}}\geqslant qC_2\rho|u|^2-C_4\sideset{}{'}\sum_{|K|=q-1}\Big|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\Big||u|,
$$

where C_4 are a positive constant also only depending on q, n. Therefore,

$$\begin{aligned}&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\big(-\log\big(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\big)\big)}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}}\\ &\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\bigg(qC_2\rho|u|^2 - C_4\sum_{|K|=q-1}{}' \Big|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\Big||u|\bigg)\\ &\quad+\bigg(\frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\bigg)^2 \sum_{|K|=q-1}{}' \frac{1}{2}\Big|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\Big|^2 - C_3|u|^2.\end{aligned}$$

Applying the inequality $-|xy| \geqslant -|x|^2 - \frac{1}{4}|y|^2$ to

$$x = C_4|u|,\ \ y = \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)-\varphi}\Big|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}u_{jK}\Big|$$

we get

$$\begin{aligned}&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\big(-\log\big(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\big)\big)}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}}\\ &\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{\tau(-\varepsilon)-\tau\circ(\rho-\varepsilon)}qC_2\rho|u|^2 - \bigg(\sum_{|K|=q-1}{}' (C_4)^2 + C_3\bigg)|u|^2.\end{aligned}$$

Moreover, by Taylor expansion we have $\tau(-\varepsilon) - \tau\circ(\rho-\varepsilon) = -\rho\tau'(t)$ with $\rho(z)-\varepsilon < t < -\varepsilon$, and since $0 < \tau'(\rho-\varepsilon) \leqslant \tau'(t)$ so we get

$$\begin{aligned}&\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2(-\log(1-\frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}))}{\partial z_j\partial\overline{z}_k} u_{jK}\overline{u_{kK}}\\ &\qquad\geqslant \frac{\tau'\circ(\rho-\varepsilon)}{-\rho\tau'(t)}qC_2\rho|u|^2 - \bigg(\sum_{|K|=q-1}{}' (C_4)^2 + C_3\bigg)|u|^2\\ &\qquad\geqslant -qC_2|u|^2 - \bigg(\frac{n!}{(q-1)!(n-q+1)!}(C_4)^2 + C_3\bigg)|u|^2.\end{aligned}$$

Thus, if put $C_1 = C_2 + \frac{1}{q}\Big(\frac{n!}{(q-1)!(n-q+1)!}(C_4)^2 + C_3\Big)$ then

$$-\log\bigg(1 - \frac{\tau\circ(\rho-\varepsilon)+\varphi}{\tau(-\varepsilon)}\bigg) + C_1|z|^2 \in SH_q(\Omega).$$

Step 2. Choose open subsets $V_j \Subset U_j \Subset W_j$, $j = 1,2,\dots,m$ such that $\partial\Omega \Subset \bigcup_{j=1}^{m} V_j$, and $W_j \cap \Omega$ is q-convex, $j = 1,2,\dots,m$. By Lemma 3.4 we can chose a smooth q-subharmonic exhaustion function ψ_j in $W_j \cap \Omega$ such that

$$\sup_{W_j\cap\Omega}\left|\frac{\partial^2(-\mathrm{e}^{-\psi_j})}{\partial z_h \partial \overline{z}_l}\right| \leqslant 1, \qquad \text{for all} \quad h, l.$$

Put $\rho_j = -\mathrm{e}^{-\psi_j}$, $j = 1, 2, \ldots, m$. Using [4: Lemma 2] (also see the proof of [6: Proposition 3.2]) we find a continuous increasing function $\tau\colon (-\infty; 0) \to \mathbb{R}$ such that $\lim\limits_{t\to 0} \tau(t) = +\infty$ and $|\tau \circ \rho_j - \tau \circ \rho_k| \leqslant 1/2$ for every $j, k \in \{1, 2, \ldots, m\}$ with $U_j \cap U_k \cap \Omega \neq \varnothing$. Therefore for every $\varepsilon > 0$ sufficiently small we have

$$|\tau \circ (\rho_j(z) - \varepsilon) - \tau \circ (\rho_k(z) - \varepsilon)| \leqslant \frac{2}{3}, \qquad \text{for all} \quad z \in U_j \cap U_k \cap \Omega.$$

Choose $\chi_j \in \mathcal{C}_0^\infty(U_j)$ satisfying $0 \leqslant \chi_j \leqslant 1$ and $\chi_j \equiv 1$ on $\overline{V}_j$. Let $C > 0$ so large that $|z|^2 - C < 0$ on Ω and that $\chi_j(z) + C|z|^2$ is plurisubharmonic in Ω for every j. Put $\varphi_j = \chi_j(z) - 1 + C(|z|^2 - C) \in PSH^-(\Omega)$. By Proposition 2.5 it is easy to see that ρ_j and φ_j satisfy the hypothesis of Step 1, and hence, there exists a constant $C_j > 0$ only depending on q, n and $\sup\limits_{W_j\cap\Omega}\left|\frac{\partial\varphi_j}{\varphi_j}\right|$ such that $-\log\bigl(1 - \frac{\tau\circ(\rho_j-\varepsilon)+\varphi_j}{\tau(-\varepsilon)}\bigr) + C_j|z|^2 \in SH_q(W_j \cap \Omega)$. Let $D = \max\{C_j :\ j = 1, 2, \ldots, m\}$. Then we have

$$-\log\left(1 - \frac{\tau \circ (\rho_j - \varepsilon) + \varphi_j}{\tau(-\varepsilon)}\right) + D|z|^2 \in SH_q(W_j \cap \Omega), \qquad j = 1, 2, \ldots, m$$

for every ε sufficiently small.

Moreover, it is clear that

$$\tau(\rho_j - \varepsilon) + \varphi_j < \tau(\rho_k - \varepsilon) + \varphi_k, \tag{3.2}$$

on $\partial U_j \cap V_k \cap \Omega$. Hence,

$$-\log\left(1 - \frac{\tau \circ (\rho_j - \varepsilon) + \varphi_j}{\tau(-\varepsilon)}\right) < -\log\left(1 - \frac{\tau \circ (\rho_k - \varepsilon) + \varphi_k}{\tau(-\varepsilon)}\right)$$

on $\partial U_j \cap V_k \cap \Omega$. For each $z \in \Omega \cap \bigl(\bigcup\limits_j V_j\bigr)$, put $I_z = \bigl\{j \in \{1, 2, \ldots, m\} :\ z \in U_j\bigr\}$, and define

$$\phi^\varepsilon(z) := \max_{j\in I_z}\left\{-\log\left(1 - \frac{\tau \circ (\rho_j(z) - \varepsilon) + \varphi_j(z)}{\tau(-\varepsilon)}\right)\right\}.$$

Then, we deduce that $\phi^\varepsilon + D|z|^2 \in SH_q\bigl(\Omega \cap \bigl(\bigcup\limits_j V_j\bigr)\bigr)$, for every ε sufficiently small.

Step 3. Let $\{\varepsilon_k\} \searrow 0$, $k \to \infty$ such that $\phi^{\varepsilon_k} + D|z|^2 \in SH_q\big(\Omega \cap \big(\bigcup_j V_j\big)\big)$ for all k. Define $\phi = \sup_k \phi^{\varepsilon_k}$. Let K be a compact subset of Ω such that $\big(\Omega \backslash \big(\bigcup_j V_j\big)\big) \Subset K$ and $\partial K \Subset \bigcup_j V_j$. Let $M = \max_{\partial K}(\phi + D|z|^2)$ and

$$\varphi := \begin{cases} \max(\phi + D|z|^2, M) & \text{on } \Omega\backslash K \\ M & \text{on } K. \end{cases}$$

It is clear that φ is an exhaustion function for Ω. We prove that φ is continuous on Ω. Indeed, first we prove ϕ^ε is continuous on $\Omega \cap \big(\bigcup_j V_j\big)$ for every ε sufficiently small. Let $x_0 \in \Omega \cap \big(\bigcup_j V_j\big)$. Choose a neighborhood $D_{x_0} \subset \Omega \cap \big(\bigcup_j V_j\big)$ of x_0 such that $D_{x_0} \cap U_j = \varnothing$ if $x_0 \notin \overline{U}_j$, $j = 1, \dots, m$. Assume that $x_0 \in V_{j_0}$. For each j such that $x_0 \in \partial U_j$, by (3.2) we have

$$\tau \circ (\rho_j - \varepsilon) + \varphi_j < \tau \circ (\rho_{j_0} - \varepsilon) + \varphi_{j_0} \qquad \text{on} \quad D_{x_0},$$

if $D_{x_0} \subset V_{j_0}$ is chosen small enough. Hence we have

$$\phi^\varepsilon|_{D_{x_0}} := \max_{j \in I_{x_0}} \left\{ -\log \left(1 - \frac{\tau \circ (\rho_j(z) - \varepsilon) + \varphi_j(z)}{\tau(-\varepsilon)} \right) \right\}.$$

It is follows that $\phi^\varepsilon|_{D_{x_0}}$ is continuous, and hence, ϕ^ε is continuous on $\Omega \cap \big(\bigcup_j V_j\big)$.

Now, we prove φ is continuous. Let $\{\varphi < M + 1\} \Subset \Omega' \Subset \Omega$ and $\Omega \cap \big(\bigcup_j V_j\big) \Subset K' \Subset K$. It is easy to see that $\lim_{k\to\infty} \sup_{\Omega'\backslash K'} \phi^{\varepsilon_k} = 0$ and $\inf_{\Omega'\backslash K'} \phi > 0$ (since we can choose K', K sufficiently lager and near Ω). Hence there exists k_0 such that

$$\sup_{\Omega'\backslash K'} \phi^{\varepsilon_k} < \inf_{\Omega'\backslash K'} \phi, \qquad \text{for all} \quad k \geqslant k_0.$$

Thus, $\phi|_{\Omega'\backslash K'} = \sup\{\phi^{\varepsilon_k}_{\Omega'\backslash K'} : \ k = 1, 2, \dots, k_0\}$. Moreover since $\phi^{\varepsilon_k}_{\Omega'\backslash K'}$ is continuous, it follows that $\phi|_{\Omega'\backslash K'}$ is continuous. Thus φ is continuous on Ω', and hence, φ is a continuous q-subharmonic exhaustion function for Ω. The desired conclusion follows. □

As well-known that, if Ω is a pseudoconvex domain in $\mathbb{C}^n$ and $K \Subset \Omega$ then $\Omega \setminus K$ is not pseudoconvex. A raised question here is in the case of q-convex domains how is the above situation? The following proposition shows that if we take a small enough subset out a q-convex domain then q-convexity may be broken.

Proposition 3.5. *Let A be an analytic subset of $\Omega \subset \mathbb{C}^n$ with $\dim A = k$. Then $\Omega \setminus A$ is not $(n-k-1)$-convex.*

P r o o f. Assume that $\Omega \setminus A$ is $(n-k-1)$-convex and φ is a continuous $(n-k-1)$-subharmonic exhaustion function of $\Omega \setminus A$. Take arbitrary a point $\xi_0 \in R(A)$, the regular locus of A. We may assume that $\xi_0 = 0$. Consider the tangent plan $T_0 A$ and a subspace L of $\mathbb{C}^n$ containing $T_0 A$ with $\dim L = k+1$. Write $\mathbb{C}^n = L^\perp \oplus L$. We have $\dim(L^\perp \cap A) = 0$. Indeed, in the converse case we can find a complex line l in $L^\perp \cap T_0 A$ and we get a contradiction because $l \subset T_0 A \subset L$ and $L \cap L^\perp = \{0\}$. Thus, there exists a neighborhood $U' \times U''$ of $0 \in L^\perp \oplus L$ such that $(\partial U' \times U'') \cap A$ is empty. Given $\xi'' \in L \setminus \pi(A)$, where $\pi \colon L^\perp \oplus L \to L$ is the orthogonal projection. Then $(U' \times \{\xi''\}) \cap A$ is empty. Since $\varphi(\bullet, \xi'')$ is subharmonic on U', we have

$$\sup_{U' \times \{\xi''\}} \varphi(\bullet, \xi'') = \sup_{\partial U' \times \{\xi''\}} \varphi(\bullet, \xi'') \leqslant \sup_{\partial U' \times U''} \varphi < +\infty. \tag{3.3}$$

On the other hand, $L \setminus \pi(A)$ is dense in L because $\dim L > \dim A \geqslant \dim \pi(A)$ we can find a sequence $\{(\xi_k', \xi_k'')\} \subset U' \times (U'' \setminus \pi(A))$ converging to $0 = \xi_0 \in R(A)$. Thus, $\varphi(\xi_k', \xi_k'') \to +\infty$ which contradicts (3.3). The proof is complete. $\square$

Finally, we give a result about q-convexity of Hartogs domains.

Proposition 3.6. *Let $1 < p, q \leqslant n$ and let $\Omega \subset \mathbb{C}^n$ be a $(q-1)$-convex domain. Assume that φ is a continuous $(p-1)$-subharmonic function in $\mathbb{C}^n$. Then the Hartogs domain*

$$\Omega_\varphi = \big\{(z,\lambda) \in \Omega \times \mathbb{C} : \ |\lambda| < \mathrm{e}^{-\varphi(z)}\big\}$$

is $\max(p,q)$*-convex.*

P r o o f. By Corollary 2.7 $\varphi \circ \pi$ is a continuous p-subharmonic function on $\Omega \times \mathbb{C}$, where $\pi \colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ denotes the canonical projection. Hence, the function $\phi(z,\lambda) := \log|\lambda| + \varphi(z)$ is continuous p-subharmonic. Thus $\big\{(z,\lambda) \in \mathbb{C}^n \times \mathbb{C} : |\lambda| < \mathrm{e}^{-\varphi(z)}\big\}$ is a p-convex domain with the exhaustion function $\varrho(z,\lambda) = -\frac{1}{\log|\lambda| + \varphi(z)}$.

On the other hand, since $\Omega \subset \mathbb{C}^n$ is a $(q-1)$-convex domain so that there is a continuous $(q-1)$-subharmonic exhaustion function ψ of Ω. Hence again by Corollary 2.7 $\psi \circ \pi$ is a continuous q-subharmonic function on $\Omega \times \mathbb{C}$. Thus

$$\widetilde{\varrho}(z,\lambda) = \max\{\psi(z), 0\} + |\lambda|$$

is a continuous q-subharmonic exhaustion function of $\Omega \times \mathbb{C}$. Hence $\Omega \times \mathbb{C}$ is a q-convex domain. It is easy to check that $\Omega_\varphi = (\Omega \times \mathbb{C}) \cap \{(z,\lambda) \in \mathbb{C}^n \times \mathbb{C} : |\lambda| < \mathrm{e}^{-\varphi(z)}\}$ is $\max(p,q)$-convex. This completes the proof. □

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