VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0168-4
Math. Slovaca **63** (2013), No. 6, 1233–1246

ENTIRE FUNCTIONS SHARING SETS OF SMALL FUNCTIONS WITH THEIR DIFFERENCE OPERATORS OR SHIFTS

BaoQin Chen* — ZongXuan Chen**

(*Communicated by Ján Borsík*)

ABSTRACT. We show some interesting results concerning entire functions sharing two sets of small functions CM with their difference operators or shifts.

1. Introduction and main results

Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane, unless specifically stated otherwise. We use the standard notations in the Nevanlinna theory of meromorphic functions (see e.g., [10, 12, 18, 19]). For a meromorphic function $f(z)$, we denote by $S(f)$ the set of all meromorphic functions $a(z)$ such that $T(r,a) = o(T(r,f))$ for all r outside of a set with finite logarithmic measure. Functions in the set $S(f)$ are called small functions compared to $f(z)$. And if $a(z) \in S(f)$, we write $T(r,a) = S(r,f)$ (see [8]). Moreover, we also use the notation $\hat{S}(f) = S(f) \cup \{\infty\}$.

For a set $S \subset \hat{S}(f)$, we define that

$$E_f(S) = \bigcup_{a \in S} \big\{ z \mid f(z) - a(z) = 0, \ \text{counting multiplicities} \big\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \big\{ z \mid f(z) - a(z) = 0, \ \text{ignoring multiplicities} \big\}.$$

2010 Mathematics Subject Classification: Primary 30D35; Secondary 39B32.
Keywords: uniqueness theory, shift, difference, shared values.
This work was supported by the National Natural Science Foundation of China (No: 11171119 and 11301091) and Guangdong Natural Science Foundation (No: S2013040014347).

We say that two meromorphic functions f and g share a set S CM, resp. IM, provided that $E_f(S) = E_g(S)$, resp. $\overline{E}_f(S) = \overline{E}_g(S)$.

The classical results in the uniqueness theory of meromorphic functions are the five values and four values theorems due to Nevanlinna [16], see also [10, 18]. When considering sharing sets, it is well known that there exists a set S containing seven elements such that if f and g are two non-constant entire functions and $E_f(S) = E_g(S)$, then $f = g$ (see [18: Theorem 10.56]).

We firstly recall the following result concerning an entire function f sharing a set with its derivative f'.

Theorem A. ([14]) *Let f be a non-constant entire function and a_1, a_2 be two distinct complex numbers. If f and f' share the set $\{a_1, a_2\}$ CM, then f takes one of the following conclusions:*

(i) $f = f'$;

(ii) $f + f' = a_1 + a_2$;

(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$, *with* $a_1 + a_2 = 0$, *where* c, c_1, c_2 *are non-zero constants which satisfy* $c^2 \neq 1$ *and* $c_1 c_2 = \frac{1}{4} a_1^2 \left(1 - \frac{1}{c^2}\right)$.

For the case when two entire functions share common sets, we recall the following result.

Theorem B. ([6]) *Let $S_1 = \{1, -1\}$, $S_2 = \{0\}$. If f and g are non-constant entire functions of finite order such that f and g share the sets S_1 and S_2 CM, then $f = g$ or $f \cdot g = 1$.*

Recently, a number of papers (including [1,3,4,7–9,11,13,15,17]) have focused on value distribution in difference analogues of meromorphic functions. In a recent paper [15], considering Theorems A and B, Liu investigated the cases when $f(z)$ shares sets with its shift $f(z+c)$ or difference operator $\Delta_c f := f(z+c) - f(z)$, where c is a non-zero constant, and proved the following Theorems C–E.

Theorem C. ([15]) *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$, and let $a(z) \in S(f)$ be a non-vanishing periodic entire function with period c. If $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, then $f(z)$ must take one of the following conclusions:*

(i) $f(z) \equiv f(z+c)$;

(ii) $f(z) + f(z+c) \equiv 0$;

(iii) $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, *where* $\frac{h_1(z+c)}{h_1(z)} = -e^{\gamma}$, $\frac{h_2(z+c)}{h_2(z)} = e^{\gamma}$, $h_1(z) h_2(z) = a(z)^2 (1 - e^{-2\gamma})$ *and* γ *is a polynomial.*

Remark 1. From the proof of Theorem C (see [15]), we see that the condition that $a(z)$ is non-vanishing can be replaced by a much weaker condition that $a(z) \not\equiv 0$.

Theorem D. ([15]) *Under the assumptions of* Theorem C*, if $f(z)$ and $f(z+c)$ share the sets $\big\{a(z), -a(z)\big\}$, $\{0\}$ CM, then $f(z) = \pm f(z+c)$ for all $z \in \mathbb{C}$.*

Remark 2. Theorem D is a corollary of Theorem C and its assumption yields that $f(z)$ and $f(z+c)$ share the value 0 CM. An interesting question is whether the conclusion still holds if we replace the set $\{0\}$ with the set $\{b(z)\}$, where $b(z) \in S(f) \setminus \{a(z), -a(z)\}$. Considering this question, we prove the following result.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$, and let $a(z)(\not\equiv 0)$, $b(z) \in S(f)$ be two distinct periodic entire functions with period c. If $f(z)$ and $f(z+c)$ share the sets $\{a(z), -a(z)\}$ and $\{b(z)\}$ CM, then $f(z) = \pm f(z+c)$ for all $z \in \mathbb{C}$. Moreover, if $b(z) \not\equiv 0$, then $f(z) \equiv f(z+c)$.*

Remark 3. Suppose $f(z)$ and $f(z+c)$ share the sets $\{a_1(z), a_2(z)\}$ and $\{b_1(z)\}$ CM in Theorem 1.1, where $a_1(z)$, $a_2(z)$, $b_1(z) \in S(f)$ are three distinct periodic entire functions with period c. This situation can be dealt with by taking $g(z) = f(z) - \frac{a_1(z)+a_2(z)}{2}$. Obviously, $g(z)$ and $g(z+c)$ share the sets $\Big\{\frac{a_1(z)-a_2(z)}{2}, \frac{a_2(z)-a_1(z)}{2}\Big\}$ and $\Big\{b_1(z) - \frac{a_1(z)+a_2(z)}{2}\Big\}$ CM. By Theorem 1.1, we have $f(z) \equiv f(z+c)$, if $b_1(z) - \frac{a_1(z)+a_2(z)}{2} \not\equiv 0$; we have $f(z) = f(z+c)$ or $f(z+c) + f(z) = a_1(z) + a_2(z)$ for all $z \in \mathbb{C}$, if $b_1(z) \equiv \frac{a_1(z)+a_2(z)}{2}$.

Another interesting question is what happens if $f(z+c)$ is replaced by $P(z, f(z))$ in Theorem D, where $P(z, f(z))$ is a linear difference polynomial in f. Corresponding to this question, we have the following result.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$, and let*

$$P(z, f(z)) = b_k(z)f(z+kc) + \cdots + b_1(z)f(z+c) + b_0(z)f(z), \tag{1.1}$$

where $b_k(z) \not\equiv 0$, $b_0(z), \ldots, b_k(z) \in S(f)$ and k is a nonnegative integer. Suppose that $a(z) \in S(f)$ is a periodic entire function with period c such that $a(z) \not\equiv 0$. If $f(z)$ and $P(z, f(z))$ share the sets $\{a(z), -a(z)\}$ and $\{0\}$ CM, then $P(z, f(z)) = \pm f(z)$ for all $z \in \mathbb{C}$.

If the coefficients of $P(z, f(z))$ in Theorem 1.2 are all polynomials, we prove the following result.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$, and let*

$$P(z, f(z)) = b_k(z)f(z+kc) + \cdots + b_1(z)f(z+c) + b_0(z)f(z),$$

where $b_k(z) \not\equiv 0$, $b_0(z), \ldots, b_k(z)$ are polynomials, and k is a nonnegative integer. Suppose that $a_1(z), \ldots, a_n(z) \in S(f)$ are distinct periodic entire functions with period c such that $a_i(z) \not\equiv 0$, $i = 1, 2, \ldots, n$, where n is a positive integer.

If $f(z)$ and $P(z,f(z))$ share the sets $\{a_1(z),\dots,a_n(z)\}$ and $\{0\}$ CM, then $P(z,f(z)) = tf(z)$ for all $z \in \mathbb{C}$, where $t \in \mathbb{C}\setminus\{0\}$.

Remark 4. For two sets S_1, S_2 such that $S_1 \subset S_2$, the condition $E_f(S_2) = E_g(S_2)$ does not mean that $E_f(S_1) = E_g(S_1)$. Thus Theorem 1.3 is not a corollary of Theorem 1.2 and their proofs are different.

Theorem E. ([15]) *Let $f(z)$ be a transcendental entire function of finite order, and let a be a non-zero finite constant. If $f(z)$ and $\Delta_c f$ share the set $\{a,-a\}$ CM, then $f(z+c) \equiv 2f(z)$.*

Remark 5. As mentioned in [15], it is quite natural to ask what happens if the set $\{a,-a\}$ is replaced by the set $\{a(z),b(z)\}$, where $a(z), b(z) \in S(f)$ are two distinct periodic entire functions with period c such that $a(z), b(z) \not\equiv 0$. Considering Theorem 1.1 and Theorem E, we obtain the following Theorem 1.4.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}\setminus\{0\}$, and let $a(z)(\not\equiv 0)$, $b(z) \in S(f)$ be two periodic entire functions with period c such that $a(z)$ and $b(z)$ are linearly dependent over the complex field, but $b(z) \not\equiv \pm a(z)$. If $f(z)$ and $\Delta_c f$ share the sets $\{a(z),-a(z)\}$ and $\{b(z)\}$ CM, and if the inequality*

$$N\left(r, \frac{1}{f(z)-b(z)}\right) \geq \lambda T(r,f), \tag{1.2}$$

holds for $\lambda \in (2/3, 1]$, then

$$\frac{\Delta_c f - b(z)}{f(z)-b(z)} = t,$$

where $t \in \mathbb{C}\setminus\{0\}$.

The following result is a corollary of Theorem 1.2.

Theorem 1.5. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}\setminus\{0\}$, and let $a(z) \in S(f)$ be a periodic entire function with period c such that $a(z) \not\equiv 0$. If $f(z)$ and $\Delta_c f$ share the sets $\{a(z),-a(z)\}$ and $\{0\}$ CM, then $f(z+c) \equiv 2f(z)$.*

2. Proof of Theorem 1.1

Halburd–Korhonen [7] and Chiang–Feng [4] investigated the value distribution theory of difference expressions, including the difference analogue of the logarithmic derivative lemma, independently. We recall the following result.

Lemma 2.1. ([7: Corollary 2.2]) *Let $f(z)$ be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^{\delta}}\right),$$

for all r outside of a possible exceptional set with finite logarithmic measure.

By [9: Lemma 2.1], we have $T(r+|c|, f(z)) = (1+o(1))T(r,f)$ for all r outside of a set with finite logarithmic measure, when $f(z)$ is of finite order.

The Lemma 2.2 below can be proved by a similar reasoning as in the proof of [2: Lemma 3(b)]. We omit those details.

Lemma 2.2. *Let $g(z)$ be a transcendental meromorphic function and let $E \subset (0,\infty)$ be a set of finite logarithmic measure. Then we have*

$$\varlimsup_{\substack{r\to\infty \\ r\in(0,\infty)\setminus E}} \frac{\log T(r,g)}{\log r} = \varlimsup_{\substack{r\to\infty \\ r\in(0,\infty)}} \frac{\log T(r,g)}{\log r} = \rho(g).$$

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function, and let $a(z)$ be a meromorphic function such that $a(z) \in S(f)$. Then we have $\rho(a) \le \rho(f)$.*

P r o o f. This follows immediately from Lemma 2.2. □

P r o o f o f T h e o r e m 1.1. By Theorem D, we see that our conclusion holds if $b(z) \equiv 0$. Next we suppose that $b(z) \not\equiv 0$.

If $a(z)$ is not a constant, then $a(z)$ is transcendental by the fact that $a(z)$ is a periodic entire function. As $f(z)$ is of finite order and $a(z) \in S(f)$, by Lemma 2.3, we see that $a(z)$ is also of finite order.

Since sums, differences, products and quotients of functions of finite order are again of finite order, we see that if $f(z)$ is a transcendental entire function of finite order, then

$$\frac{(f(z+c)-a(z))(f(z+c)+a(z))}{(f(z)-a(z))(f(z)+a(z))}$$

is of finite order.

Moreover, since $f(z)$ and $f(z+c)$ share the sets $\{a(z), -a(z)\}$ CM, it follows that

$$\frac{(f(z+c)-a(z))(f(z+c)+a(z))}{(f(z)-a(z))(f(z)+a(z))}$$

is an entire function of finite order without zeros. By Hadamard's factorization theorem, an entire function of finite order without zeros is of the form $\mathrm{e}^{p(z)}$, where $p(z)$ is a polynomial. That is

$$(f(z+c)-a(z))(f(z+c)+a(z)) = (f(z)-a(z))(f(z)+a(z))\mathrm{e}^{p(z)}. \tag{2.1}$$

Similarly, since $f(z)$ and $f(z+c)$ share the set $\{b(z)\}$ CM, we have

$$f(z+c)-b(z)=(f(z)-b(z))\mathrm{e}^{q(z)}, \tag{2.2}$$

where $q(z)$ is a polynomial.

Note that $a(z), b(z)\in S(f)$ are periodic entire functions with period c. By Lemma 2.1 and (2.2), we have

$$T\big(r,\mathrm{e}^{q(z)}\big)=m\big(r,\mathrm{e}^{q(z)}\big)=m\left(r,\frac{f(z+c)-b(z)}{f(z)-b(z)}\right)=o\left(\frac{T(r,f-b)}{r^{\delta}}\right),$$

outside of a possible exceptional set with finite logarithmic measure.

That is

$$T\big(r,\mathrm{e}^{q(z)}\big)=S(r,f). \tag{2.3}$$

Similarly, from (2.1) and Lemma 2.1, we get

$$\begin{aligned}T\big(r,\mathrm{e}^{p(z)}\big)&=m\big(r,\mathrm{e}^{p(z)}\big)\\&=m\left(r,\frac{(f(z+c)-a(z))(f(z+c)+a(z))}{(f(z)-a(z))(f(z)+a(z))}\right)\\&\le m\left(r,\frac{f(z+c)-a(z)}{f(z)-a(z)}\right)+m\left(r,\frac{f(z+c)+a(z)}{f(z)+a(z)}\right)\\&=S(r,f).\end{aligned}\tag{2.4}$$

If $\mathrm{e}^{q(z)}\equiv 1$, it follows from (2.2) that $f(z)\equiv f(z+c)$.

If $\mathrm{e}^{q(z)}\not\equiv 1$, substituting (2.2) into (2.1), we obtain that

$$f(z)P(z,f)=Q(z,f), \tag{2.5}$$

where

$$P(z,f)=\big(\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\big)f(z), \tag{2.6}$$

$$Q(z,f)=2b(z)\mathrm{e}^{q(z)}\big(\mathrm{e}^{q(z)}-1\big)f(z)-b(z)^2\big(\mathrm{e}^{q(z)}-1\big)^2-a(z)^2\big(\mathrm{e}^{p(z)}-1\big). \tag{2.7}$$

Note that $\mathrm{e}^{q(z)}\not\equiv 1$ and $b(z)\not\equiv 0$. By (2.5)–(2.7), we observe that $\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\not\equiv 0$. Indeed, if $\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\equiv 0$, we have $Q(z,f)\equiv 0$. It implies that $T(r,f)=S(r,f)$ by (2.3),(2.4) and (2.7), which is impossible.

Thus, by (2.5)–(2.7) and the Clunie Lemma [5: Lemma 2], we see that

$$T\big(r,\big(\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\big)f(z)\big)=m\big(r,\big(\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\big)f(z)\big)=m(r,P(z,f))=S(r,f).$$

Combining this with (2.3) and (2.4) gives that

$$T(r,f)\le T\big(r,\big(\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\big)f(z)\big)+T\big(r,1/\big(\mathrm{e}^{2q(z)}-\mathrm{e}^{p(z)}\big)\big)=S(r,f),$$

a contradiction. □

3. Proof of Theorem 1.2

As in the proof of Theorem 1.1 it follows that

$$(P(z,f(z))-a(z))(P(z,f(z))+a(z))=(f(z)-a(z))(f(z)+a(z))\mathrm{e}^{p(z)}, \quad (3.1)$$

$$P(z,f(z))=f(z)\mathrm{e}^{q(z)}, \quad (3.2)$$

where $p(z)$ and $q(z)$ are polynomials.

If $\mathrm{e}^{2q(z)}\equiv 1$, it follows from (3.2) that $P(z,f(z))\equiv \pm f(z)$.

If $\mathrm{e}^{2q(z)}\not\equiv 1$, from (3.2) and Lemma 2.1, we get

$$\begin{aligned} T\big(r,\mathrm{e}^{q(z)}\big) &= m\big(r,\mathrm{e}^{q(z)}\big)=m\left(r,\frac{P(z,f(z))}{f(z)}\right)\\ &\le m\left(r,\frac{f(z+kc)}{f(z)}\right)+\cdots+m\left(r,\frac{f(z+c)}{f(z)}\right)+m(r,b_k(z)) \quad (3.3)\\ &\qquad \cdots+m(r,b_0(z))+O(1)\\ &= S(r,f), \end{aligned}$$

where the exceptional set associated with $S(r,f)$ has at most finite logarithmic measure.

Note that $f(z)$ and $P(z,f(z))$ share the set $\{a(z),-a(z)\}$ CM. Let z_0 be a common zero of $(P(z,f(z))-a(z))(P(z,f(z))+a(z))$ and $(f(z)-a(z))\cdot$ $\cdot(f(z)+a(z))$ such that $a(z_0)\ne 0$. Then

$$P(z_0,f(z_0))=\pm f(z_0)=\pm a(z_0). \quad (3.4)$$

From (3.2) and (3.4), we have

$$\mathrm{e}^{2q(z_0)}=\left(\frac{P(z_0,f(z_0))}{f(z_0)}\right)^2=1.$$

Hence all zeros of $(f(z)-a(z))(f(z)+a(z))$ are zeros of $\mathrm{e}^{2q(z)}-1$ as long as they are not zeros of $a(z)$. Thus, we deduce that

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f(z)^2-a(z)^2}\right) &\le N\left(r,\frac{1}{\mathrm{e}^{2q(z)}-1}\right)+N\left(r,\frac{1}{a(z)}\right)\\ &\le 2T\big(r,\mathrm{e}^{q(z)}\big)+S(r,f)=S(r,f), \end{aligned}$$

which implies

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{f(z)-a(z)}\right)+\overline{N}\left(r,\frac{1}{f(z)+a(z)}\right)\\ &\le \overline{N}\left(r,\frac{1}{f(z)^2-a(z)^2}\right)+\overline{N}\left(r,\frac{1}{a(z)}\right)=S(r,f). \end{aligned} \quad (3.5)$$

If both $(P(z,f(z))-a(z))(P(z,f(z))+a(z))$ and $(f(z)-a(z))(f(z)+a(z))$ have no zeros, then (3.5) also holds.

Set $g(z)=\frac{f(z)+a(z)}{f(z)-a(z)}$. Then $f(z)=a(z)+\frac{2a(z)}{g(z)-1}$. So, we have

$$\begin{aligned} T(r,f) &\le T(r,a)+T\left(r,\frac{2a}{g-1}\right)+\log 2 \\ &\le 3T(r,a)+T(r,g-1)+O(1)=T(r,g)+S(r,f), \end{aligned} \tag{3.6}$$

and

$$T(r,g)\le 2T(r,f)+2T(r,a)+O(1)=2T(r,f)+S(r,f). \tag{3.7}$$

By (3.6) and (3.7), we see that $S(r,g)=S(r,f)$. Then, by (3.5), it follows from the second main theorem [12: Corollary 2.5.4] that

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+S(r,g) \\ &\le \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f+a}\right)+\overline{N}\left(r,\frac{1}{2a}\right)+S(r,f) \\ &= S(r,f). \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we have $T(r,f)\le S(r,f)$, which is a contradiction.

4. Proof of Theorem 1.3

Lemma 4.1. ([7: Corollary 3.4] or [13: Theorem 2.4]) *Let $w(z)$ be a non-constant finite order meromorphic solution of*

$$P(z,w)=0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r,a)=S(r,w)$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w),$$

where the exceptional set associated with $S(r,w)$ has at most finite logarithmic measure.

Proof of Theorem 1.3. As in the proof of Theorem 1.1 it follows that

$$\begin{aligned} &(P(z,f(z))-a_1(z))\cdots(P(z,f(z))-a_n(z)) \\ &=(f(z)-a_1(z))\cdots(f(z)-a_n(z))\mathrm{e}^{p(z)}, \end{aligned} \tag{4.1}$$

where $p(z)$ is a polynomial. Now (3.2) and (3.3) should hold.

If $q(z)\equiv q\in\mathbb{C}$, then from (3.2), we get $P(z,f(z))=tf(z)$, $t=\mathrm{e}^q\in\mathbb{C}\setminus\{0\}$.

If $q(z)$ is a nonconstant polynomial, for any given meromorphic function $g(z)$, we denote

$$Q(z, g(z)) := P(z, g(z)) - g(z)e^{q(z)}. \tag{4.2}$$

By (3.2) and (4.2), we have $Q(z, f(z)) \equiv 0$.

Since $a_1(z), \dots, a_n(z) \in S(f)$ are distinct periodic entire functions with period c such that $a_i(z) \not\equiv 0$, for $i = 1, 2, \dots, n$, we have

$$\begin{aligned} Q(z, a_i(z)) &= P(z, a_i(z)) - a_i(z)e^{q(z)} \\ &= b_k(z)a_i(z+kc) + \cdots + b_1(z)a_i(z+c) + b_0(z)a_i(z) - a_i(z)e^{q(z)} \\ &= (b_k(z) + \cdots + b_1(z) + b_0(z) - e^{q(z)})a_i(z). \end{aligned}$$

By the assumption that $b_0(z), \dots, b_k(z)$ are polynomials, $a_i(z) \not\equiv 0$ $(i = 1, 2, \dots, n)$, and $q(z)$ is a nonconstant polynomial, we see that

$$Q(z, a_i(z)) \not\equiv 0.$$

By Lemma 4.1, for $i = 1, 2, \dots, n$, we get

$$m\left(r, \frac{1}{f(z) - a_i(z)}\right) = S(r, f), \tag{4.3}$$

where the exceptional set associated with $S(r, f)$ has at most finite logarithmic measure.

Then by (4.3) and Lemma 2.1, for $i = 1, 2, \dots, n$, we see that

$$\begin{aligned} & m\left(r, \frac{P(z, f(z)) - a_i(z)}{f(z) - a_i(z)}\right) \\ \leq & m\left(r, b_k(z)\frac{f(z+kc) - a_i(z)}{f(z) - a_i(z)}\right) + \cdots + m\left(r, b_1(z)\frac{f(z+c) - a_i(z)}{f(z) - a_i(z)}\right) \\ & \qquad + m(r, b_0(z)) + m\left(r, \frac{(b_k(z) + \cdots + b_1(z) + b_0(z) - 1)a_i(z)}{f(z) - a_i(z)}\right) \\ =& S(r, f), \end{aligned} \tag{4.4}$$

where the exceptional set associated with $S(r, f)$ has at most finite logarithmic measure.

Therefore, by (4.1) and (4.4), we obtain

$$\begin{aligned} T\big(r, e^{p(z)}\big) &= m\big(r, e^{p(z)}\big) \\ &= m\left(r, \frac{(P(z, f(z)) - a_1(z)) \cdots (P(z, f(z)) - a_n(z))}{(f(z) - a_1(z)) \cdots (f(z) - a_n(z))}\right) \\ &\leq \sum_{i=1}^{n} m\left(r, \frac{P(z, f(z)) - a_i(z)}{f(z) - a_i(z)}\right) = S(r, f). \end{aligned} \tag{4.5}$$

Substituting (3.2) into (4.1) yields

$$
\begin{aligned}
&(\mathrm{e}^{nq(z)} - \mathrm{e}^{p(z)}) f(z) \cdot f(z)^{n-1} \\
&= \sum_{i=1}^{n} a_i(z) \cdot (\mathrm{e}^{(n-1)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-1} \\
&\quad - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_i(z) a_j(z) \cdot (\mathrm{e}^{(n-2)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-2} + \cdots \\
&\quad \cdots + (-1)^{n-1} a_1(z) \cdots a_n(z)(1 - \mathrm{e}^{p(z)}).
\end{aligned} \tag{4.6}
$$

Suppose that $\mathrm{e}^{nq(z)} - \mathrm{e}^{p(z)} \not\equiv 0$. Thus, by (4.6) and the Clunie Lemma [5], we see that

$$
T\big(r, \big(\mathrm{e}^{nq(z)} - \mathrm{e}^{p(z)}\big) f(z)\big) = m\big(r, \big(\mathrm{e}^{nq(z)} - \mathrm{e}^{p(z)}\big) f(z)\big) = S(r, f),
$$

which implies that $T(r, f) = S(r, f)$ by (3.3) and (4.5), a contradiction.

Therefore, we have $\mathrm{e}^{nq(z)} - \mathrm{e}^{p(z)} \equiv 0$. Since $q(z)$ is a nonconstant polynomial, we get $\mathrm{e}^{sq(z)} - \mathrm{e}^{p(z)} \not\equiv 0$, for $0 \leq s \leq n-1$. Now we consider the coefficient of the term $(\mathrm{e}^{(n-1)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-1}$. If $a_1(z) + \cdots + a_n(z) \not\equiv 0$, we rewrite (4.6) as follows

$$
\begin{aligned}
&\sum_{i=1}^{n} a_i(z) \cdot (\mathrm{e}^{(n-1)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-1} \\
&= \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_i(z) a_j(z) \cdot (\mathrm{e}^{(n-2)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-2} \\
&\quad - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sum_{l=1, l\neq i,j}^{n} a_i(z) a_j(z) a_l(z) \cdot (\mathrm{e}^{(n-3)q(z)} - \mathrm{e}^{p(z)}) f(z)^{n-3} + \cdots \\
&\quad \cdots + (-1)^{n} a_1(z) \cdots a_n(z)(1 - \mathrm{e}^{p(z)}).
\end{aligned}
$$

By the Clunie Lemma [5], we can similarly get the contradiction that $T(r, f) = S(r, f)$ again. Therefore, $a_1(z) + \cdots + a_n(z) \equiv 0$. By induction, we can prove that the coefficient of each term $(\mathrm{e}^{sq(z)} - \mathrm{e}^{p(z)}) f(z)^s$ $(s = 1, \ldots, n-1)$ is identically vanishing and hence we have

$$
(-1)^n a_1(z) \cdots a_n(z)(1 - \mathrm{e}^{p(z)}) \equiv 0,
$$

which is impossible. Thus Theorem 1.3 is proved. □

5. Proof of Theorem 1.4

Lemma 5.1. ([8: Lemma 2.3]) *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z) \in S(f)$ with period c,*

$$m\left(r, \frac{\Delta_c^n f}{f(z)-a(z)}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ has at most finite logarithmic measure.

Proof of Theorem 1.4. As in the proof of Theorem 1.1 it follows that

$$(\Delta_c f - a(z))(\Delta_c f + a(z)) = (f(z)-a(z))(f(z)+a(z))\mathrm{e}^{p(z)}, \tag{5.1}$$

$$\Delta_c f - b(z) = (f(z)-b(z))\mathrm{e}^{q(z)}, \tag{5.2}$$

where $p(z)$ and $q(z)$ are polynomials.

If $q(z) \equiv q \in \mathbb{C}$, then for $t = \mathrm{e}^q \in \mathbb{C} \setminus \{0\}$, it follows from (5.2) that

$$\frac{\Delta_c f - b(z)}{f(z)-b(z)} = t.$$

If $q(z)$ is a nonconstant polynomial, by (1.2), we get

$$\begin{aligned} m\left(r, \frac{1}{f(z)-b(z)}\right) &\le T(r,f) - N\left(r, \frac{1}{f(z)-b(z)}\right) + S(r,f) \\ &\le (1-\lambda)T(r,f) + S(r,f), \end{aligned} \tag{5.3}$$

where $\lambda \in (2/3, 1]$. By (5.2) and (5.3) and Lemma 5.1, we have

$$\begin{aligned} T\big(r, \mathrm{e}^{q(z)}\big) = m\big(r, \mathrm{e}^{q(z)}\big) &= m\left(r, \frac{\Delta_c f - b(z)}{f(z)-b(z)}\right) \\ &\le m\left(r, \frac{\Delta_c f}{f(z)-b(z)}\right) + m\left(r, \frac{1}{f(z)-b(z)}\right) + m(r, b(z)) + O(1) \\ &\le (1-\lambda)T(r,f) + S(r,f), \end{aligned} \tag{5.4}$$

where the exceptional set associated with $S(r,f)$ has at most finite logarithmic measure.

Note that $f(z)$ and $\Delta_c f$ share the set $\{a(z), -a(z)\}$ CM. Let z_0 be a common zero of $(\Delta_c f - a(z))(\Delta_c f + a(z))$ and $(f(z)-a(z))(f(z)+a(z))$ such that $a(z_0) \neq 0$ and $b(z_0) \pm a(z_0) \neq 0$. Then

$$\Delta_c f(z_0) = \pm f(z_0) = \pm a(z_0). \tag{5.5}$$

As $a(z_0) \neq 0$ and $b(z_0) \pm a(z_0) \neq 0$, by (5.2), we deduce that

$$\frac{\Delta_c f(z_0) - b(z_0)}{f(z_0) - b(z_0)} = \mathrm{e}^{q(z_0)}. \tag{5.6}$$

Since $a(z)$ and $b(z)$ are linearly dependent over the complex field and $b(z) \not\equiv \pm a(z)$, there exists a $\alpha \in \mathbb{C} \setminus \{-1, 1\}$ such that

$$b(z) = \alpha a(z).$$

Set $\beta = \frac{2}{\alpha - 1} + 1$. Thus, $\beta \neq 0$, and we have

$$a(z) + b(z) = \beta(b(z) - a(z)).$$

Consider four cases for (5.5) with (5.6):

(i) if $f(z_0) = a(z_0)$, $\Delta_c f(z_0) = a(z_0)$, then $\mathrm{e}^{q(z_0)} = 1$;

(ii) if $f(z_0) = a(z_0)$, $\Delta_c f(z_0) = -a(z_0)$, then $\mathrm{e}^{q(z_0)} = \beta$;

(iii) if $f(z_0) = -a(z_0)$, $\Delta_c f(z_0) = a(z_0)$, then $\mathrm{e}^{q(z_0)} = \frac{1}{\beta}$;

(iv) if $f(z_0) = -a(z_0)$, $\Delta_c f(z_0) = -a(z_0)$, then $\mathrm{e}^{q(z_0)} = 1$.

Then we can deduce that

$$(\mathrm{e}^{q(z_0)} - 1)(\mathrm{e}^{q(z_0)} - \beta)\left(\mathrm{e}^{q(z_0)} - \frac{1}{\beta}\right) = 0.$$

Hence all zeros of $(f(z) - a(z))(f(z) + a(z))$ are zeros of $\mathrm{e}^{q(z)} - 1$, $\mathrm{e}^{q(z)} - \beta$ or $\mathrm{e}^{q(z)} - \frac{1}{\beta}$ as long as they are not zeros of $a(z)$ or $b(z) \pm a(z)$.

Thus, we see that

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{f(z)^2 - a(z)^2}\right) &\leq N\left(r, \frac{1}{\mathrm{e}^{q(z)} - 1}\right) + N\left(r, \frac{1}{\mathrm{e}^{q(z)} - \beta}\right) \\
&\quad + N\left(r, \frac{1}{\mathrm{e}^{q(z)} - \frac{1}{\beta}}\right) + N\left(r, \frac{1}{a(z)}\right) + N\left(r, \frac{1}{b(z) \pm a(z)}\right) \\
&\leq 3T\left(r, \mathrm{e}^{q(z)}\right) + S(r, f) \\
&\leq 3(1 - \lambda)T(r, f) + S(r, f),
\end{aligned}$$

which implies that

$$\overline{N}\left(r, \frac{1}{f(z) - a(z)}\right) + \overline{N}\left(r, \frac{1}{f(z) + a(z)}\right) \leq 3(1 - \lambda)T(r, f) + S(r, f). \tag{5.7}$$

If both $(\Delta_c f - a(z))(\Delta_c f + a(z))$ and $(f(z) - a(z))(f(z) + a(z))$ have no zeros, then (5.7) also holds.

Set $g(z) = \frac{f(z)+a(z)}{f(z)-a(z)}$. Thus, we can get (3.6) and $S(r,g) = S(r,f)$ as in the proof of Theorem 1.2. Then we get from (5.7) and the second main theorem [12: Corollary 2.5.4] that

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + S(r,g) \\ &\le \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f+a}\right) + \overline{N}\left(r, \frac{1}{2a}\right) + S(r,f) \\ &\le 3(1-\lambda)T(r,f) + S(r,f). \end{aligned} \tag{5.8}$$

From (3.6) and (5.8), we have $(3\lambda - 2)T(r,f) \le S(r,f)$. This is impossible for the number $\lambda \in (2/3, 1]$. The proof of Theorem 1.4 is thus completed. $\square$

REFERENCES

[1] BERGWEILER, W.—LANGLEY, J. K.: *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 133–147.

[2] CHEN, Z. X.: *On the complex oscillation theory of $f^{(k)} + Af = F$*, Proc. Edinb. Math. Soc. (2) **36** (1993), 447–461.

[3] CHEN, Z. X.—SHON, K. H.: *On zeros and fixed points of differences of meromorphic functions*, J. Math. Anal. Appl. **344** (2008), 373–383.

[4] CHIANG, Y. M.—FENG, S. J.: *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (2008), 105–129.

[5] CLUNIE, J.: *On integral and meromorphic functions*, J. London Math. Soc. (2) **37** (1962), 17–27.

[6] GROSS, F.—OSGOOD, C. F.: *Entire functions with common preimages.* In: Factorization Theory of Meromorphic Functions. Lecture Notes in Pure and Appl. Math. 78, Marcel Dekker, New York, 1982, pp. 19–24.

[7] HALBURD, R. G.—KORHONEN, R. J.: *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), 477–487.

[8] HALBURD, R. G.—KORHONEN, R. J.: *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 463–478.

[9] HALBURD, R. G.—KORHONEN, R. J.: *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. London Math. Soc. (3) **94** (2007), 443–474.

[10] HAYMAN, W. K.: *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[11] HEITTOKANGAS, J.—KORHONEN, R. J.—LAINE, I.—RIEPPO, J.—ZHANG, J. L.: *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. **355** (2009), 352–363.

[12] LAINE, I.: *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[13] LAINE, I.—YANG, C. C.: *Clunie theorems for difference and q-difference polynomials*, J. London Math. Soc. (2) **76** (2007), 556–566.

[14] LI, P.—YANG, C. C.: *Value sharing of an entire function and its derivatives*, J. Math. Soc. Japan **51** (1999), 781–799.

[15] LIU, K.: *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359** (2009), 384–393.
[16] NEVANLINNA, R.: *Le Théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.
[17] QI, X. G.—YANG, L. Z.—LIU, K.: *Uniqueness and periodicity of meromorphic functions concerning the difference operator*, Comput. Math. Appl. **60** (2010), 1739–1746.
[18] YANG, C. C.—YI, H. X.: *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003.
[19] YANG, L.: *Value Distribution Theory*, Springer-Verlag, Berlin-Heidelberg, 1993.

Received 31. 5. 2011
Accepted 18. 10. 2011

* *College of Science*
Guangdong Ocean University
Zhanjiang, 524088
P. R. China
E-mail: chenbaoqin_chbq@126.com

** *Corresponding author:*
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631
P. R. CHINA
E-mail: chzx@vip.sina.com