VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0166-6
Math. Slovaca **63** (2013), No. 6, 1209–1226

AFFINE MAPS OF STATE SPACES AND STATE SPACES OF K_0 GROUPS

Zhu Xiaosheng

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. Let φ be a homomorphism from the partially ordered abelian group (S, v) to the partially ordered abelian group (G, u) with $\varphi(v) = u$, where v and u are order units of S and G respectively. Then φ induces an affine map φ^* from the state space $\mathrm{St}(G, u)$ to the state space $\mathrm{St}(S, v)$. Firstly, in this paper, we give some suitable conditions under which φ^* is injective, surjective or bijective. Let R be a semilocal ring with the Jacobson radical $J(R)$ and let $\pi\colon R \to R/J(R)$ be a canonical map. We discuss the affine map $(K_0\pi)^*$. Secondly, for a semiprime right Goldie ring R with the maximal right quotient ring Q, we consider the relations between $\mathrm{St}(R)$ and $\mathrm{St}(Q)$. Some results from [ALFARO, R.: *State spaces, finite algebras, and skew group rings*, J. Algebra **139** (1991), 134–154] and [GOODEARL, K. R.—WARFIELD, R. B., Jr.: *State spaces of K_0 of noetherian rings*, J. Algebra **71** (1981), 322–378] are extended.

Introduction

A preordered abelian group is a pair $(G, \leq)$ consisting of an abelian group G together with a specified translation-invariant pre-order $\leq$ on G. Denote the positive cone $\{x \in G \mid x \geq 0\}$ of G by G^+. An order-unit in a preordered abelian group G is an element $u \in G^+$ such that for any $x \in G$, there is a positive integer n with $x \leq nu$. Let G be a preordered abelian group with an order-unit u. A state on (G, u) is an order preserving homomorphism f from G to $\mathbb{R}$ with $f(u) = 1$. The state space of (G, u) is the set $\mathrm{St}(G, u)$ of all states on (G, u).

2010 Mathematics Subject Classification: Primary 16A18, 16A54, 16E20; Secondary 18F30, 19A49.
Keywords: affine map, state space, K_0 group, ordered abelian group, semilocal ring.
This work was partially supported by the National Natural Science Foundation of China (No. 10971090, No. 11071097) and the Natural Science Foundation of Jiangsu Province of China (BK 2007133).

For a preordered abelian group (G, u), if $G^+ - G^+ = G$ and $G^+ \cap (-G^+) = \{0\}$, then (G, u) is called an ordered group or a partially order group. Throughout this paper, all order groups considered always means partially order groups with order units. Ordered groups arise naturally in many contexts. One of the most fruitful ways to study ordered groups is via the states on the group; in fact, the order structure in some cases can be recovered from the state space. One major application of the theory of ordered groups is in K-theory. If A is a (simple) unital C^*-algebra, the structure of the ordered group $K_0(A)$ is described by the states on $K_0(A)$ (see [7: § 6]). In [6], B. Blackadar and M. Rørdam proved that every state on a subsemigroup of a scaled pre-ordered semigroup can be extended to the whole semigroup and every state on $K_0(A)$ for a unital C^*-algebra A comes from a quasitrace.

In this paper, all rings considered are associative with identity 1 unless otherwise specified. All modules will be unitary. For a ring R, let $K_0(R)$ be the Grothendieck group of the category of finitely generated projective right R-modules (as defined in [5], [12], [14], [15]). Let $FP(R)$ denote the class of finitely generated projective right R-modules. Set

$$FP(R)/\sim = \{[A] \mid A \in FP(R)\},$$

where $\sim$ is the equivalence relation of stable isomorphism on $FP(R)$ and $[A]$ denotes the equivalence class of A relative to $\sim$ (i.e., the stable isomorphism class of A). We define

$$K_0(R)^+ = FP(R)/\sim$$

and use $K_0(R)^+$ to define a relation $\leq$ on $K_0(R)$, so that for $x, y \in K_0(R)$, we have $x \leq y$ if and only if $y - x \in K_0(R)^+$. Thus $\leq$ is a preorder, called the natural preorder on $K_0(R)$. Clearly, $[R]$ is an order-unit of $K_0(R)$ (see [10]). The state space $\mathrm{St}(R)$ of the ring R is defined to be $\mathrm{St}(K_0(R), [R])$. The relations between the state space of the fixed ring and the state space of the skew group ring were considered by R. Alfaro in [2]. For commutative rings, hereditary noetherian prime rings and simple orders over Dedekind domains, the structure of their state spaces has been given by K. Goodearl and R. B. Warfield in [11]. For state spaces of other rings (such as a ring in relation to the ideals of localization on noncommutative rings and of ramificaion in generically Galois actions, $S * G$ for a commutative noetherian domain S and a finite group G acting faithfully as automorphisms of the ring S), some structure theorems were obtained by R. Alfaro in [1]. In this work, we mainly consider the natural affine map φ^* from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$ induced by the homomorphism φ from (S, v) to (G, u) with $\varphi(v) = u$ and the structure of state spaces of K_0 groups.

In outline the paper is as follows. Let

$$\varphi \colon (S, v) \to (G, u)$$

be a homomorphism between two ordered groups with $\varphi(v) = u$. Then φ induces an affine map

$$\varphi^*\colon \mathrm{St}(G, u) \to \mathrm{St}(S, v).$$

In the Section 1, we give some suitable conditions under which φ^* is injective (Proposition 1.2, Theorem 1.3, Corollary 1.8) and some conditions for which φ^* is bijective (Theorem 1.4, Corollary 1.7). As an application of the above results, some results in [1] (such as [1: Theorem 4.6] and [1: Corollary 4.7]) are extended. For a semilocal ring R, we prove that the following statements are equivalent:

(1) $(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$ is an affine embedding.
(2) $(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$ is an affine homeomorphism.
(3) $K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_0(R/J(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$, where π is a canonical map from R to $R/J(R)$ and $K_0\pi$ is a natural map from $K_0(R)$ to $K_0(R/J(R))$ induced by π.

Let R be a semiprime right Goldie ring and Q be the maximal right quotient ring of R with the inclusion map $\varphi\colon R \hookrightarrow Q$. In the Section 2, we consider the relations between the cokernel (kernel) of $K_0(\varphi)$ and the state spaces of R and Q. Some results in [11] are generalized.

1. Affine maps of state spaces induced by homomorphisms of ordered abelian groups

Goodearl and Warfield in [11] proved that if R is any nonzero indecomposable commutative ring, then $\mathrm{St}(R)$ consists of a single point (see [11: Proposition 3.8]). In fact, if R is a nonzero indecomposable commutative ring, let P be a prime ideal of R, and let F_P be the quotient field of R/P. Then the map

$$\varphi_P\colon R \to R/P \hookrightarrow F_P$$

induces an order preserving group homomorphism

$$K_0\varphi_P\colon K_0(R) \to K_0(F_P)$$

and

$$\ker K_0\varphi_P \subseteq \bigcap_{f\in \mathrm{St}(R)} \ker f.$$

Furthermore, $(K_0\varphi_P)^*$ is a bijective affine map from $\mathrm{St}(F_P)$ to $\mathrm{St}(R)$. Alfaro in [1] showed the following results:

(1) if R is a quasilocal ring, then $\mathrm{St}(R)$ is a point (see [1: Proposition 4.3]) and
(2) if R is a semiperfect ring, then

$$(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$$

is affinely homeomorphic (see [1: Proposition 4.5]).

In fact, if R is a quasilocal ring, then

$$K_0\pi\colon K_0(R) \to K_0(R/J(R))$$

induces an affinely homeomorphic

$$(K_0\pi)^*\colon \operatorname{St}(R/J(R)) \to \operatorname{St}(R),$$

where $\pi\colon R \to R/J(R)$ is a canonical map. These results motivate us to consider the following problems: if

$$\varphi\colon (S,v) \to (G,u)$$

is a homomorphism between two ordered groups with $\varphi(v) = u$, under what suitable conditions, the affine map

$$\varphi^*\colon \operatorname{St}(G,u) \to \operatorname{St}(S,v)$$

is either injective or bijective? In this section, we shall consider these problems.

Let G be an abelian group. Then $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-vector space. Define

$$\operatorname{rank}(G) = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Note that $\operatorname{rank}(G)$ is equal to the cardinality of a maximal $\mathbb{Z}$-independent subset of G since if $\{v_i\}$ is a maximal $\mathbb{Z}$-independent subset of G then

$$F = \bigoplus\nolimits_{\mathbb{Z}} v_i$$

is a subgroup of G with G/F torsion. So $F \otimes_{\mathbb{Z}} \mathbb{Q} \cong G \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus it is a $\mathbb{Q}$-vector space with dimension equal to the cardinality of $\{v_i\}$ (see [4]).

Let (G,u) be an ordered abelian group. If $\operatorname{rank}(G) = n < \infty$, then there exist

$$v_1, v_2, \ldots, v_n \in G$$

such that

$$\{v_1, v_2, \ldots, v_n\}$$

is a maximal $\mathbb{Z}$-independent subset of G and for any $x \in G$, there are $d \in \mathbb{N}$ and $d_1, d_2, \ldots, d_n \in \mathbb{Z}$ such that

$$dx = d_1v_1 + d_2v_2 + \cdots + d_nv_n.$$

In this case, we call

$$\{v_1, v_2, \ldots, v_n\}$$

a $\mathbb{Z}$-basis for G.

$$(d_1/d, d_2/d, \ldots, d_n/d)$$

is called the n-tuple of coordinates of x relative to the $\mathbb{Z}$-basis

$$\{v_1, v_2, \ldots, v_n\}.$$

Let (S,v) and (G,u) be ordered abelian groups with

$$\operatorname{rank}(G) = n < \infty,$$

and let $\varphi\colon S \to G$ be a homomorphism with $\varphi(v) = u$. If

$$\{v_1, v_2, \ldots, v_n\}$$

is a $\mathbb{Z}$-basis of G, for any $y \in S$, then there are integers

$$m_1, m_2, \ldots, m_n$$

and a positive integer m such that

$$m\varphi(y) = m_1 v_1 + m_2 v_2 + \cdots + m_n v_n.$$

In this case,

$$(m_1/m, m_2/m, \ldots, m_n/m)$$

is called the n-tuple of coordinates of $\varphi(y)$ under the $\mathbb{Z}$-basis

$$\{v_1, v_2, \ldots, v_n\}.$$

Let

$$V_\varphi = \big\{(m_1/m, \ldots, m_n/m) \mid m\varphi(y) = m_1 v_1 + \cdots + m_n v_n \text{ and } y \in S\big\}.$$

Then V_φ is a set of n-dimensional vectors over $\mathbb{Q}$. The number of vectors in a maximal independent subset of V_φ is called the rank of V_φ or the rank of φ, denote it by $\operatorname{rank}(V_\varphi)$ or $\operatorname{rank}\varphi$. Clearly, the rank of V_φ (or the rank of φ) is independent of the choice of a $\mathbb{Z}$-basis in G. If φ is surjective, then $\operatorname{rank}(V_\varphi) = n$. But, in general, the converse is not true.

If (S, v) and (G, u) are preordered abelian groups, we call

$$\varphi\colon (S, v) \to (G, u)$$

a homomorphism of preordered groups with order unit if φ preserves order, that is $x \le y$ in S implies that $\varphi(x) \le \varphi(y)$ in G and $\varphi(v) = u$. We say that φ is an embedding of preordered abelian groups with order unit if, in addition, φ is injective and for any $x, y \in S$, $\varphi(x) \le \varphi(y)$ implies $x \le y$. The operation taking (G, u) to $\operatorname{St}(G, u)$ is clearly a contravariant functor from the category of preordered abelian groups with order unit to the category of compact convex sets and affine continuous maps (see [11]).

Lemma 1.1. *Let $\varphi\colon (S, v) \to (G, u)$ be a homomorphism between two ordered groups with $\varphi(v) = u$ and*

$$\operatorname{rank}(G) = n < \infty.$$

Then $\operatorname{rank}(G) = \operatorname{rank}(V_\varphi) = n$ *if and only if the cokernel of φ is a torsion group.*

P r o o f.

($\Longrightarrow$): Since $\operatorname{rank}(G) = \operatorname{rank}(V_\varphi) = n$, we have that $\operatorname{rank}(G) = \operatorname{rank}(\varphi(S))$. So $G/\varphi(S)$ is a torsion group.

($\Longleftarrow$): Clearly. $\square$

Proposition 1.2. *Let φ be a homomorphism from an ordered abelian group (S, v) to an ordered abelian group (G, u) with $\varphi(v) = u$. If*

$$\operatorname{rank}(G) = \operatorname{rank}(V_\varphi) = n < \infty,$$

then φ^* is an injective affine map from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$.

Proof. By Lemma 1.1, $G/\operatorname{Im}(\varphi)$ is a torsion group. For any $x \in G$, there is $m \in \mathbb{N}$ such that $mx \in \operatorname{Im}(\varphi)$. Then there is $y \in S$ such that $mx = \varphi(y)$. If there are $f, g \in \mathrm{St}(G, u)$ such that $\varphi^*(f) = \varphi^*(g)$, then

$$\begin{aligned} mf(x) &= f(mx) = f(\varphi(y)) = \varphi^*(f)(y) \\ &= \varphi^*(g)(y) = g(\varphi(y)) = g(mx) = mg(x). \end{aligned}$$

So $f(x) = g(x)$, that is $f = g$. Thus φ^* is an injective affine map from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$. □

Theorem 1.3. *Let φ be a homomorphism from the ordered abelian group (S, v) to the ordered abelian group (G, u). Suppose there is a finite positive $\mathbb{Z}$-basis $\{v_1, v_2, \ldots, v_n\}$ (i.e., each $v_i \in G^+$) in G satisfying the following conditions:*

1. *For any $y \in G^+$, the ith coordinate of y relative to the positive $\mathbb{Z}$-basis $\{v_1, v_2, \ldots, v_n\}$ is non-negative, $1 \le i \le n$.*
2. *There is $f \in \mathrm{St}(G, u)$ such that $f(v_i) > 0$ for all $1 \le i \le n$.*

Then φ^* is an injective affine map from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$ if and only if the cokernel of φ is a torsion group.

Proof.

($\Longrightarrow$): It is sufficient to prove that $\operatorname{rank}(V_\varphi) = n$ by Lemma 1.1. Suppose $\operatorname{rank}(V_\varphi) = m$. We shall show that $m = n$. Let $\{\alpha_1, \ldots, \alpha_m\}$ be a maximal independent set of V_φ such that α_1 is the n-tuple of coordinates of $\varphi(v) = u$. Let

$$A = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

Consider the system of m homogeneous linear equations in n unknows

$$AX = 0. \tag{1.1}$$

If $m < n$, we can find a nonzero solution $(a_1, \ldots, a_n)$ of (1.1) such that $|a_i| < f(v_i)$ for all $1 \le i \le n$. Define

$$g\colon G \to \mathbb{R} \qquad \text{by} \quad g(v_i) = f(v_i) - a_i, \quad 1 \le i \le n$$

and for any $y \in G$ with $my = t_1v_1 + \cdots + t_nv_n$, we have

$$g(y) = (1/m) \sum_{i=1}^{n} t_i g(v_i)$$

for $m \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{Z}$. It is not difficult to prove that g is an order preserving homomorphism from G to the additive group of $\mathbb{R}$. Let $\alpha_1 = (b_{11}, \ldots, b_{1n})$. Because α_1 is the n-tuple of coordinates of $\varphi(v) = u$, we have

$$1 = f(u) = b_{11}f(v_1) + \cdots + b_{1n}f(v_n) = b_{11}g(v_1) + \cdots + b_{1n}g(v_n) = g(u).$$

So g is a state on (G, u). Evidently, $f \neq g$ as $(a_1, \ldots, a_n)$ is nonzero solution of (1.1). Next we shall show that $\varphi^*(f) = \varphi^*(g)$. For any $x \in S$, $\varphi(x) \in G$, there exist $k_1, \ldots, k_n \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that

$$k\varphi(x) = k_1v_1 + \cdots + k_nv_n.$$

Then

$$\alpha = (k_1/k, \ldots, k_n/k) \in V_\varphi.$$

Since $\{\alpha_1, \ldots, \alpha_m\}$ is a maximal independent set of V_φ, there are

$$d_1, \ldots, d_m \in \mathbb{Q}$$

such that

$$\alpha = d_1\alpha_1 + \cdots + d_m\alpha_m.$$

Hence

$$\varphi^*(f)(x) = f\varphi(x) = \alpha \begin{bmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{bmatrix} = \alpha \begin{bmatrix} g(v_1) \\ \vdots \\ g(v_n) \end{bmatrix} = g\varphi(x) = \varphi^*(g)(x).$$

So $\varphi^*(f) = \varphi^*(g)$. This contradicts the assumption that φ^* is injective. Thus $m = n$.

($\Longleftarrow$): By Proposition 1.2 and Lemma 1.1. □

Theorem 1.4. *Let φ be a homomorphism from an ordered abelian group (S, v) to an ordered abelian group (G, u) and* $\operatorname{rank}(G) = n < \infty$. *Assume*

(1) $\ker\varphi \subseteq \bigcap_{\overline{f} \in \mathrm{St}(S,v)} \ker \overline{f}$*;*

(2) *There are a $\mathbb{Z}$-basis $\{v_1, \ldots, v_n\}$ in G and a maximal independent set $\{\alpha_1, \ldots, \alpha_n\}$ in V_φ (relative to the $\mathbb{Z}$-basis $\{v_1, \ldots, v_n\}$) such that for any $\overline{f} \in \mathrm{St}(S, v)$ and any n-tuple $(t_1, \ldots, t_n)$ $(t_1, \ldots, t_n \in \mathbb{Z})$ with*

$$\sum_{i=1}^{n} t_iv_i \in G^+,$$

we have

$$(t_1,\dots,t_n)\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}^{-1}\begin{bmatrix}\overline{f}(u_1)\\ \vdots\\ \overline{f}(u_n)\end{bmatrix}\geq 0,$$

where each $u_i\in S$ and the n-tuple of coordinates of each $\varphi(u_i)$ is α_i, for all $1\leq i\leq n$.

Then φ^* is a bijective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$.

Proof. Let $\{v_1,\dots,v_n\}$ be a $\mathbb{Z}$-basis of G and $\{\alpha_1,\dots,\alpha_n\}$ be a maximal independent set of V_φ satisfying condition (2). Then there are $u_1,\dots,u_n\in S$ such that

$$m_i\varphi(u_i)=a_{i1}v_1+\cdots+a_{in}v_n,\qquad \text{for}\quad m_i\in\mathbb{N},\quad a_{ij}\in\mathbb{Z}$$

and

$$\alpha_i=(1/m_i)(a_{i1},\dots,a_{in}),\qquad 1\leq i\leq n.$$

Let

$$M=\begin{bmatrix}m_1&&\\&\ddots&\\&&m_n\end{bmatrix},\quad B=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix},\qquad A=\begin{bmatrix}a_{11}&\dots&a_{1n}\\ \dots&\dots&\dots\\ a_{n1}&\dots&a_{nn}\end{bmatrix}.$$

Then $M,A\in\mathbb{M}_{n\times n}(\mathbb{Z})$, $B=M^{-1}A\in\mathbb{M}_{n\times n}(\mathbb{Q})$ and $|A|\neq 0$. For any $\overline{f}\in\mathrm{St}(S,v)$, let

$$A_i=\begin{bmatrix}a_{11}&\dots&a_{1i-1}&m_1\overline{f}(u_1)&a_{1i+1}&\dots&a_{1n}\\ \dots&\dots&\dots&\dots&\dots&\dots&\dots\\ a_{n1}&\dots&a_{ni-1}&m_n\overline{f}(u_n)&a_{ni+1}&\dots&a_{nn}\end{bmatrix},$$

then $A_i\in\mathbb{M}_{n\times n}(\mathbb{R})$, $1\leq i\leq n$. Next we shall prove that there exists $f\in\mathrm{St}(G,u)$ such that $\varphi^*(f)=\overline{f}$. Define a map

$$f\colon G\to\mathbb{R}\qquad\text{by}\quad f(v_i)=|A_i|/|A|,\quad 1\leq i\leq n,\quad\text{and}$$

$$f(y)=(1/k)\sum_{i=1}^{n}k_1f(v_i),$$

for any $y\in G$ with

$$ky=k_1v_1+\cdots+k_nv_n.$$

Clearly, f is a homomorphism from abelian group G to the additive group of $\mathbb{R}$. For each $w\in G^+$, there are $t\in\mathbb{N}$, $t_1,\dots,t_n\in\mathbb{Z}$ such that

$$tw=t_1v_1+\cdots+t_nv_n.$$

Then

$$f(w) = (1/t)(t_1, \dots, t_n) \begin{bmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{bmatrix} = (1/t)(t_1, \dots, t_n) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^{-1} \begin{bmatrix} \overline{f}(u_1) \\ \vdots \\ \overline{f}(u_n) \end{bmatrix} \geq 0$$

by condition (2). So f is order preserving. Since $u = \varphi(v) \in G$, there are an $r \in \mathbb{N}$ and $r_1, \dots, r_n \in \mathbb{Z}$ such that

$$ru = r\varphi(v) = r_1 v_1 + \cdots + r_n v_n.$$

Then

$$f(u) = (1/r)(r_1, \dots, r_n) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^{-1} \begin{bmatrix} \overline{f}(u_1) \\ \vdots \\ \overline{f}(u_n) \end{bmatrix} = \overline{f}(v) = 1$$

by condition (1). Thus f is a state on G. Now we shall show that $\varphi^*(f) = \overline{f}$. In fact, for each $x \in S$, there exist $r \in \mathbb{N}$, $r_1, \dots, r_n \in \mathbb{Z}$ such that

$$r\varphi(x) = r_1 v_1 + \cdots + r_n v_n.$$

Then

$$\alpha = (1/r)(r_1, \dots, r_n) \in V_\varphi$$

and so there are $d_1, \dots, d_n \in \mathbb{Q}$ such that $\alpha = d_1\alpha_1 + \cdots + d_n\alpha_n$ as $\{\alpha_1, \dots, \alpha_n\}$ is a maximal independent set of V_φ. Take $p \in \mathbb{N}$ such that

$$(d'_1, \dots, d'_n) = p(d_1, \dots, d_n)M^{-1}$$

in which each $d'_i \in \mathbb{Z}$, $1 \leq i \leq n$, then

$$p\alpha = (d'_1, \dots, d'_n)A.$$

So

$$\varphi(prx) = pr\varphi(x) = \varphi(r(d'_1 m_1 u_1 + \cdots + d'_n m_n u_n)).$$

That is

$$prx - r(d'_1 m_1 u_1 + \cdots + d'_n m_n u_n) \in \ker\varphi \subseteq \bigcap_{\overline{f} \in \mathrm{St}(S,v)} \ker \overline{f}$$

by the condition (1). So

$$pr\overline{f}(x) = \overline{f}(prx) = r(d'_1 m_1 \overline{f}(u_1) + \cdots + d'_n m_n \overline{f}(u_n)) = prf\varphi(x) = pr\varphi^*(f)(x).$$

Thus $\overline{f}(x) = \varphi^*(f)(x)$. Therefore $\overline{f} = \varphi(f)$ and so φ^* is a surjective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$.

On the other hand,

$$\mathrm{rank}(G) = \mathrm{rank}(V_\varphi) = n$$

by condition (2). Proposition 1.2 shows that φ^* is injective. Thus φ^* is a bijective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$. □

Lemma 1.5. ([7: Theorem 6.8.3] or [9: Theorem 3.2]) *Let (G, G^+, u) be an ordered abelian group and let H be a subgroup of G which contains u. If f is any state on $(H, H \cap G^+, u)$, then f extends to a state on (G, u).*

Theorem 1.6. *Let $\varphi\colon (S, v) \to (G, u)$ be a homomorphism of ordered abelian groups with $\varphi(v) = u$ such that $\operatorname{rank}(\varphi(S)) = n < \infty$. Then φ^* is a surjective affine map from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$ if and only if*

(1) $\ker\varphi \subseteq \bigcap_{\overline{f}\in \mathrm{St}(S,v)} \ker \overline{f}$,

(2) *There are a $\mathbb{Z}$-basis $\{v_1, \dots, v_n\}$ in $\varphi(S)$ and a maximal independent set $\{\alpha_1, \dots, \alpha_n\}$ in V_φ (relative to the $\mathbb{Z}$-basis $\{v_1, \dots, v_n\}$) such that for any $\overline{f} \in \mathrm{St}(S, v)$ and any n-tuple $(t_1, \dots, t_n)$ $(t_1, \dots, t_n \in \mathbb{Z})$ with*

$$\sum_{i=1}^{n} t_i v_i \in \varphi(S)^+,$$

we have

$$(t_1, \dots, t_n) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^{-1} \begin{bmatrix} \overline{f}(u_1) \\ \vdots \\ \overline{f}(u_n) \end{bmatrix} \geq 0,$$

where each $u_i \in S$ and the n-tuple of coordinates of each $\varphi(u_i)$ is α_i, for all $1 \leq i \leq n$.

Proof. Suppose that φ^* is surjective. Then for any $\overline{f} \in \mathrm{St}(S, v)$ there is an $f \in \mathrm{St}(G, u)$ such that $\varphi^*(f) = \overline{f}$. So for any $x \in \ker\varphi$,

$$\overline{f}(x) = \varphi^*(f)(x) = f\varphi(x) = 0,$$

that is

$$\ker\varphi \subseteq \bigcap_{\overline{f}\in \mathrm{St}(S,v)} \ker \overline{f}$$

and then the condition (1) holds.

Let $\{v_1, \dots, v_n\}$ be a $\mathbb{Z}$-basis of $\varphi(S)$, $\{\alpha_1, \dots, \alpha_n\}$ a maximal independent set of V_φ and let $\{u_1, \dots, u_n\} \subseteq S$ such that the n-tuple of the coordinate of each $\varphi(u_i)$ is α_i relative to the $\mathbb{Z}$-basis $\{v_1, \dots, v_n\}$ for all $1 \leq i \leq n$. For any n-tuple $(t_1, \dots, t_n)$ with $\sum\limits_{i=1}^{n} t_i v_i \in \varphi(S)^+$ and $t_i \in \mathbb{Z}$ for all $1 \leq i \leq n$, we have that

$$(t_1, \dots, t_n) \begin{bmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{bmatrix} \geq 0$$

since f is a state on G. Hence

$$(t_1,\ldots,t_n)\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}^{-1}\begin{bmatrix}\overline{f}(u_1)\\ \vdots\\ \overline{f}(u_n)\end{bmatrix}\geq 0.$$

Thus the condition (2) holds.

Conversely, by Theorem 1.4, $\varphi^*|_{\mathrm{St}(\varphi(S),u)}$ is a surjective affine map from $\mathrm{St}(\varphi(S),u)$ to $\mathrm{St}(S,v)$. Further, using Lemma 1.5, we have that φ^* is a surjective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$. □

Corollary 1.7. *Let (G,u) be an ordered abelian group with a positive $\mathbb{Z}$-basis $\{v_1,v_2,\ldots,v_n\}$ (i.e. each $v_i\in G^+$) satisfying the following conditions:*

(a) *For any $y\in G^+$, the ith coordinate of y relative to the positive $\mathbb{Z}$-basis $\{v_1,v_2,\ldots,v_n\}$ is nonnegative, $1\leq i\leq n$;*

(b) *There is $f\in\mathrm{St}(G,u)$ such that $f(v_i)>0$ for all $1\leq i\leq n$.*

If φ is a homomorphism from an ordered abelian group (S,v) to the ordered abelian group (G,u) with $\varphi(v)=u$, then φ^* is a bijective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$ if and only if the following conditions are satisfied:

(1) $\ker\varphi\subseteq\bigcap_{\overline{f}\in\mathrm{St}(S,v)}\ker\overline{f}$

(2) *There are a $\mathbb{Z}$-basis $\{v_1,\ldots,v_n\}$ in G and a maximal independent set $\{\alpha_1,\ldots,\alpha_n\}$ in V_φ (relative to the $\mathbb{Z}$-basis $\{v_1,\ldots,v_n\}$) such that for any $\overline{f}\in\mathrm{St}(S,v)$ and any n-tuple $(t_1,\ldots,t_n)$ $(t_1,\ldots,t_n\in\mathbb{Z})$ with*

$$\sum_{i=1}^{n}t_iv_i\in G^+,$$

we have

$$(t_1,\ldots,t_n)\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}^{-1}\begin{bmatrix}\overline{f}(u_1)\\ \vdots\\ \overline{f}(u_n)\end{bmatrix}\geq 0,$$

where each $u_i\in S$ and the n-tuple of coordinates of each $\varphi(u_i)$ is α_i, for all $1\leq i\leq n$.

Proof. By Propositions 1.2, Theorems 1.3, 1.4 and 1.6. □

Corollary 1.8. *Let φ be a homomorphism from an ordered abelian group (S,v) to an ordered abelian group (G,u) with $\varphi(v)=u$. For a finite rank subgroup G_0 of G containing u, let $S_0=\varphi^{-1}(G_0)$. Give S_0 and G_0 the relative orderings with respective to S and G. If for any such (S_0,v) and (G_0,u), $\mathrm{rank}(G_0)=\mathrm{rank}(V_{\varphi|_{S_0}})$, then φ^* is an injective affine map from $\mathrm{St}(G,u)$ to $\mathrm{St}(S,v)$.*

P r o o f. Let $f, g \in \mathrm{St}(G, u)$ and suppose that $\varphi^*(f) = \varphi^*(g)$. Let G_0 be any finite rank subgroup of G containing u with the relative ordering and $S_0 = \varphi^{-1}(G_0)$. Then $f|_{G_0}, g|_{G_0} \in \mathrm{St}(G_0, u)$ and

$$\varphi^*|_{\mathrm{St}(G_0,u)}(f|_{G_0}) = \varphi^*|_{\mathrm{St}(G_0,u)}(g|_{G_0}) \in \mathrm{St}(S_0).$$

By Proposition 1.2, $f|_{G_0} = g|_{G_0}$. So $f = g$. Thus φ^* is an injective affine map from $\mathrm{St}(G, u)$ to $\mathrm{St}(S, v)$. □

Next we shall give the some applications of the above results. We shall extend [1: Theorem 4.6 and Corollary 4.7].

Recall that a ring R is said to be a semilocal ring if $R/J(R)$ is a left artinian ring, or, equivalently, if $R/J(R)$ is a semisimple ring. Let R be a ring and I an ideal of R. The canonical map $\pi\colon R \to R/I$ induces a natural map

$$K_0\pi\colon K_0(R) \to K_0(R/I).$$

If $I \subseteq J(R)$, then $K_0\pi$ is an injection (see [3], [5], [12], [14], [15]).

Theorem 1.9. *Let R be a semilocal ring and let $\pi\colon R \to R/J(R)$ denote the canonical map. Then the following statements are equivalent:*

(1) $(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$ *is an affine embedding.*

(2) $(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$ *is an affine homeomorphism.*

(3) $K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_0(R/J(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

P r o o f.

(1) $\Longleftrightarrow$ (2):

($\Longrightarrow$): By [8: Lemma 2.2],

$$K_0\pi(K_0(R)^+) = (K_0\pi(K_0(R)))^+$$

and so $\big(K_0(R), K_0(R)^+, [R]\big)$ and $\big(K_0\pi(K_0(R)), (K_0\pi(K_0(R)))^+, [R/J(R)]\big)$ are isomorphic as ordered groups with order unit. Note that

$$(K_0\pi(K_0(R)))^+ = (K_0\pi(K_0(R))) \cap (K_0(R/J(R)))^+.$$

By Lemma 1.5, each state on

$$(K_0\pi(K_0(R)), (K_0\pi(K_0(R)))^+, [R/J(R)])$$

can be extended to a state on

$$(K_0(R/J(R)), K_0(R/J(R))^+, [R/J(R)]).$$

Thus $(K_0\pi)^*\colon \mathrm{St}(R/J(R)) \to \mathrm{St}(R)$ is surjective. So (2) holds.

($\Longleftarrow$): Clearly.

(1) $\Longleftrightarrow$ (3): Since R is semilocal,

$$R/J(R) = R_1 \oplus \cdots \oplus R_k$$

with each R_i simple artinian of length n_i. Then

$$(K_0(R/J(R)), [R/J(R)]) \xrightarrow{\psi} (\mathbb{Z}^k, (n_1, \ldots, n_k))$$

where $\mathbb{Z}^k$ has the component-wise order. It is clear that $\bigl(\mathbb{Z}^k, (n_1, \ldots, n_k)\bigr)$ fulfills the conditions of Theorem 1.3. So

$$(\psi K_0\pi)^*\colon \operatorname{St}(\mathbb{Z}^k, (n_1, \ldots, n_k)) \xrightarrow{\psi^*} \operatorname{St}(R/J(R)) \xrightarrow{(K_0\pi)^*} \operatorname{St}(R)$$

is an affine embedding if and only if the cokernel of $\psi K_0\pi$ is a torsion group. Note that ψ is an isomorphism, then

$$(K_0\pi)^*\colon \operatorname{St}(R/J(R)) \to \operatorname{St}(R)$$

is an affine embedding if and only if the cokernel of $K_0\pi$ is a torsion group. But the coker$(K_0\pi)$ is a torsion group if and only if

$$K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_0(R/J(R)) \otimes_{\mathbb{Z}} \mathbb{Q}. \qquad \square$$

Let R be a semilocal ring and $\pi\colon R \to R/J(R)$ a canonical map. By the Wedderburn-Artin Theorem,

$$R/J(R) = B_1 \oplus \cdots \oplus B_n$$

where $B_i \cong \mathbb{M}_{d_i}(D_i)$, each D_i is a division ring and $d_i \in \mathbb{N}$. For each $[Q] \in K_0(R/J(R))^+$, we denote the image of $[Q]$ in $K_0(R/J(R))/\operatorname{Im}(K_0\pi)$ by $\overline{[Q]}$ and the order of $\overline{[Q]}$ by $o(\overline{[Q]})$. Let

$$T_i = \bigl\{[Q \otimes_{R/J(R)} B_i] \mid [Q] \in K_0(R/J(R))^+\bigr\}$$

and $W_i = \bigl\{[P] \mid [P] \in K_0(R)^+ \text{ and } rK_0\pi([P]) = m[B_i] \text{ for some } r, m \in \mathbb{Z}^+\bigr\}$.

Corollary 1.10. *Let R be a semilocal ring and $\pi\colon R \to R/J(R)$ the canonical map. Assume that*

$$R/J(R) = B_1 \oplus \cdots \oplus B_n$$

where $B_i \cong \mathbb{M}_{d_i}(D_i)$, each D_i is a division ring and $d_i \in \mathbb{N}$. If $(K_0\pi)^*$ is an injective affine map from $\operatorname{St}(R/J(R))$ to $\operatorname{St}(R)$ and for any

$$f \in \operatorname{St}(R/J(R)), f(T_i) = ((K_0\pi)^*(f))(W_i)$$

when $o(\overline{[B_i]}) < \infty$, then R is a semiperfect ring.

Proof. Since R is a semilocal ring,

$$K_0\pi\colon K_0(R) \to K_0(R/J(R))$$

is injective (see [3], [5], [12], [14], [15]). So it is sufficient to prove that $\operatorname{coker}(K_0\pi) = 0$.

By the proof of Theorem 1.9, $o(\overline{[B_i]}) < \infty$ for all $1 \le i \le n$. Then

$$f(T_i) = ((K_0\pi)^*(f))(W_i) \tag{$*$}$$

for any $f \in \mathrm{St}(R/J(R))$ and all $1 \le i \le n$. For each $z \in K_0(R/J(R))$, there are

$$b \in \mathbb{N}, \quad b_1, \ldots, b_n \in \mathbb{Z}$$

such that

$$bz = b_1[B_1] + \cdots + b_n[B_n].$$

Define

$$f_j\colon K_0(R/J(R)) \to \mathbb{R} \qquad \text{by} \quad f_j([B_i]) = \begin{cases} 1, & j = i \\ 0, & j \ne i \end{cases} \qquad \text{and} \quad f_j(z) = b_j/b.$$

Clearly, $f_j \in \mathrm{St}(R/J(R))$ $(j = 1, \ldots, n)$. For each $[Q] \in K_0(R/J(R))^+$ and each f_j, there exists $[P_{ji}] \in W_i$ such that

$$f_j([Q \otimes_{R/J(R)} B_i]) = ((K_0\pi)^*(f_j))([P_{ji}])$$

for all $1 \le i \le n$ by $(*)$. Evidently,

$$f_j K_0\pi([P_{li}]) = 0, \qquad \text{when} \quad i \ne j$$

for all $1 \le i, j, l \le n$. Let

$$P = P_{11} \oplus P_{22} \oplus \cdots \oplus P_{nn}.$$

Then

$$(f_j K_0\pi)([P]) = (f_j K_0\pi)([P_{jj}]).$$

So

$$\begin{aligned} f_j([Q]) &= \sum_{i=1}^{n} f_j([Q \otimes_{R/J(R)} B_i]) = \sum_{i=1}^{n} ((K_0\pi)^*(f_j))([P_{ji}]) \\ &= \sum_{i=1}^{n} (f_j K_0\pi)([P_{ji}]) = (f_j K_0\pi)([P_{jj}]) = f_j(K_0\pi([P])) \end{aligned}$$

for all $j = 1, \ldots, n$. So

$$w = [Q] - K_0\pi([P]) \in \bigcap_{j=1}^{n} \ker f_j.$$

Since $w \in K_0(R/J(R))$, there are the $a \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$aw = a_1[B_1] + \cdots + a_n[B_n].$$

Then

$$0 = f_j(w) = a_j/a, \qquad (j = 1, \ldots, n)$$

and so $aw = 0$. But $K_0(R/J(R)) \cong \mathbb{Z}^n$, i.e. $\mathrm{Tor}(K_0(R/J(R))) = 0$; hence $w = 0$. That is $[Q] = K_0\pi([P])$. Thus $\mathrm{coker}(K_0\pi) = 0$ and so

$$K_0(R) \cong K_0(R/J(R)).$$

Therefore R is a semiperfect ring. $\square$

Remark. The condition "for any

$$f \in \mathrm{St}(R/J(R)), \qquad f(T_i) = ((K_0\pi)^*(f))(W_i)$$

for all $1 \leq i \leq n$" in Corollary 1.10 is not redundant. For example, consider the ordered group with order-unit $(\mathbb{Z}, m)$ where m is a positive integer greater than one. Now consider the subgroup $m\mathbb{Z}$ of $\mathbb{Z}$. We have an embedding of ordered groups with order-unit $(m\mathbb{Z}, m) \subset (\mathbb{Z}, m)$. By [8: Theorem 6.3], there is a semilocal ring R such that the embedding $K_0(R) \hookrightarrow K_0(R/J(R))$ coincides with the above embedding. Since that map is not an isomorphism, the ring R is not semiperfect. On the other hand, $\mathrm{St}(R/J(R))$ is affinely homeomorphic to $\mathrm{St}(R)$.

Corollary 1.11. (cf. [1: Theorem 4.6 and Corollary 4.7]) *Let R be a semilocal ring. Assume that central idempotents in $R/J(R)$ can be lifted to idempotents (not necessarily central) in R modulo the Jacobson radical $J(R)$. Then* $\mathrm{St}(R/J(R))$ *is affinely homeomorphic to* $\mathrm{St}(R)$. *Hence* $\mathrm{St}(R)$ *is affinely homeomorphic to a finite dimensional simplex.*

Proof. By Theorem 1.9 and Proposition 1.2. □

2. Localizations and state spaces

A ring R is called a semiprime ring if R has no nonzero nilpotent right ideal. A module M is said to have finite uniform dimension if it contains no infinite direct sum of nonzero submodules. A ring R is called a right Goldie ring if R has finite right uniform dimension and R satisfies the a.c.c. on right annihilators. By Goldie's Theorem (see [13: Theorem 2.3.6]), R has a right quotient ring $Q(R)$ which is semisimple artinian.

Proposition 2.1. *Let R be a semiprime right Goldie ring, and let Q be the maximal right quotient ring of R with the inclusion map $\varphi\colon R \to Q$. Then the following statements are equivalent:*

(1) $\mathrm{rank}(\mathrm{Im}((K_0\varphi)^*\colon \mathrm{St}(Q) \to \mathrm{St}(R))) = \mathrm{rank}(K_0(Q))$.

(2) *The natural map $K_0\varphi\colon K_0(R) \to K_0(Q)$ has a torsion cokernel.*

(3) $\mathrm{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$. (*That is* $\mathrm{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q})$ *can be embedded in* $\mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$).

Proof. Since Q is a maximal quotient ring of the semiprime right Goldie ring, Q is semisimple. Then $Q = Q_1 \oplus \cdots \oplus Q_n$ in which each $Q_i \cong \mathbb{M}_{d_i}(k_i)$, where d_i is a positive integer and k_i is a division ring, for all $1 \leq i \leq n$ and so

$$K_0(Q) \cong K_0(Q_1) \oplus \cdots \oplus K_0(Q_n).$$

Hence $\operatorname{rank}(K_0(Q)) = n$. Consider the map

$$f_i\colon K_0(Q) \to \mathbb{R} \qquad \text{by} \quad f_i([M]) = \text{s.f.rank}(M \otimes_Q Q_i)_{Q_i},$$

for each $[M] \in K_0(Q)^+$ and

$$f_i([M] - m[Q]) = \text{s.f.rank}(M \otimes_Q Q_i)_{Q_i} - m,$$

for each $[M] - m[Q] \in K_0(Q)$, $1 \le i \le n$, where $\text{s.f.rank}(M \otimes_Q Q_i)_{Q_i}$ denotes the stably free rank of $M \otimes_Q Q_i$ as a power stably free Q_i-module. Clearly, $f_i \in \operatorname{St}(Q)$. In fact, $f_1, \ldots, f_n$ are affinely independent points of $\operatorname{St}(Q)$ and each point of $\operatorname{St}(Q)$ can be represented by $f_1, \ldots, f_n$. So

$$\operatorname{rank}(\operatorname{Im}((K_0\varphi)^*\colon \operatorname{St}(Q) \to \operatorname{St}(R))) = \operatorname{rank}(K_0(Q)) = n$$

if and only if

$$f_1 K_0\varphi,\ f_2 K_0\varphi,\ \ldots,\ f_n K_0\varphi$$

are affinely independent points of $\operatorname{St}(K_0(R))$. It is not difficult to see that $\{f_1, \ldots, f_n\}$ is a basis for the free abelian group $\operatorname{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q})$. Considering the exact sequence of $\mathbb{Z}$-modules

$$K_0(R) \xrightarrow{K_0\varphi} K_0(Q) \xrightarrow{\pi} K_0(Q)/\operatorname{Im}(K_0\varphi) \to 0,$$

we obtain the following exact sequence

$$\begin{aligned} &0 \to \operatorname{Hom}_{\mathbb{Z}}(K_0(Q)/\operatorname{Im}(K_0\varphi), \mathbb{Q}) \\ &\xrightarrow{\operatorname{Hom}(\pi,\mathbb{Q})} \operatorname{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q}) \xrightarrow{\operatorname{Hom}(K_0\varphi,\mathbb{Q})} \operatorname{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q}). \end{aligned}$$

So $K_0(Q)/\operatorname{Im}(K_0\varphi)$ is torsion if and only if $\operatorname{Hom}_{\mathbb{Z}}(K_0(Q)/\operatorname{Im}(K_0\varphi), \mathbb{Q}) = 0$ if and only if $\operatorname{Hom}(K_0\varphi, \mathbb{Q})$ is injective if and only if $f_1K_0\varphi, \ldots, f_nK_0\varphi$ are affinely independent points of $\operatorname{St}(K_0(R))$. The proof is complete. □

Let R be a commutative ring, let P be a prime ideal of R, and let B be a finitely generated projective R-module. Then B/BP is a finitely generated projective module over the domain R/P, and we use r_P to denote the rank of B/BP. Clearly, $r_P(B)$ is equal to the dimension of the vector space $B \otimes_R F_P$, where F_P is the quotient field of R/P, and $r_P(B)$ is also equal to the rank of the free module B_P over R_P (see [5], [11], [14], [15]). Evidently, $r_P(-)$ is additive on direct sums, we see that $r_P(-)$ induces an additive map $s_P\colon K_0(R) \to \mathbb{Z}$ such that

$$s_P([A] - [B]) = r_P([A]) - r_P([B])$$

for all finitely generated projective R-modules A and B. Also $s_P([R]) = 1$. Thus $s_P \in \operatorname{St}(R)$.

Let R be a right noetherian ring, let P be a prime ideal of R and let M be a finitely generated projective right R-module. The authors of [1] defined

$$r_P(M) = \operatorname{rank}(M/MP)/\operatorname{rank}(R/P),$$

where the rank of a module M is the Goldie (uniform) dimension of M, that is the largest non-negative integer n (or ∞) such that M contains a direct sum of n non-zero submodules. If Q is maximal right quotient ring of R/P, then

$$r_P(M) = \operatorname{length}(M \otimes_R Q)/\operatorname{length}(Q).$$

Since r_P is additive on finite direct sums and $r_P(R) = 1$, one can see that r_P induces a state $s_P \in \mathrm{St}(R)$ such that

$$s_P([M] - [W]) = r_P(M) - r_P(W)$$

for all finitely generated projective right R-modules M and W (see [1], [2], [11]).

Corollary 2.2. (cf. [11: Proposition 7.1]) *Let R be a semiprime right Goldie ring, and let Q be the maximal right quotient ring of R with the inclusion map $\varphi\colon R \to Q$. Then $\{s_P \mid P$ ranges over the minimal prime ideals of $R\}$ is a basis for the free group $\mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$ if and only if the natural map $K_0\varphi : K_0(R) \to K_0(Q)$ has a torsion kernel and a torsion cokernel.*

Proof. Since Q is a maximal quotient ring of the semiprime right Goldie ring, Q is semisimple. Then $Q = Q_1 \oplus \cdots \oplus Q_n$ in which each $Q_i \cong \mathbb{M}_{d_i}(k_i)$, where d_i is a positive integer and k_i is a division ring, for all $1 \le i \le n$. For each i, the ideal $\widehat{Q}_i = \sum\{Q_j \mid j \ne i\}$ is a minimal prime ideal of Q; and this gives all the minimal ideals of Q. Let $P_i = \widehat{Q}_i \cap R$. By [13: Proposition 2.2], $P_1, \dots, P_n$ are the minimal prime ideals. Evaluating the states $s_{\widehat{Q}_i}$ in $\mathrm{St}(Q)$, we find that $s_{\widehat{Q}_i} = f_i$, where each f_i is the same as in the proof of Proposition 2.1 and $s_{P_i} = s_{\widehat{Q}_i} K_0\varphi$ for each i. So $s_{P_i} \in \mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$. By the proof of the Proposition 2.1, $\{s_{P_1}, \dots, s_{P_n}\}$ is a basis for $\mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$ if and only if

$$\mathrm{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q}) \overset{\mathrm{Hom}(K_0\varphi, \mathbb{Q})}{\cong} \mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q}). \tag{2.1}$$

($\Longrightarrow$): Since $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, from the exact sequence of $\mathbb{Z}$-modules

$$0 \to \ker K_0\varphi \to K_0(R) \xrightarrow{K_0\varphi} K_0(Q) \to \operatorname{coker} K_0\varphi \to 0$$

we have the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathbb{Z}}(\operatorname{coker} K_0\varphi, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Q}) \\ &\xrightarrow{\mathrm{Hom}(K_0\varphi, \mathbb{Q})} \mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\ker K_0\varphi, \mathbb{Q}) \to 0. \end{aligned} \tag{2.2}$$

By the assumption of the Proposition,

$$\{s_{P_1}, \dots, s_{P_n}\}$$

is a basis for $\mathrm{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$, then (2.1) holds. So

$$\mathrm{Hom}_{\mathbb{Z}}(\ker \varphi, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}}(\operatorname{coker} K_0\varphi, \mathbb{Q}) = 0$$

by (2.1) and (2.2). Thus $\ker K_0\varphi$ and $\operatorname{coker} K_0\varphi$ are torsion.

($\Longleftarrow$): If $\ker K_0\varphi$ and $\operatorname{coker} K_0\varphi$ are torsion, then

$$\operatorname{Hom}_{\mathbb{Z}}(\ker K_0\varphi, \mathbb{Q}) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} K_0\varphi, \mathbb{Q}) = 0.$$

(2.2) gives (2.1). So

$$\left\{s_{_P} \mid P \text{ ranges over the minimal prime ideals of } R\right\}$$

is a basis for the free group $\operatorname{Hom}_{\mathbb{Z}}(K_0(R), \mathbb{Q})$. □

Acknowledgement. The author would like to give many thanks to the referee for many useful comments and suggestions.

REFERENCES

[1] ALFARO, R.: *State spaces, finite algebras, and skew group rings*, J. Algebra **139** (1991), 134–154.

[2] ALFARO, R.: *States of skew groups rings and fixed rings*, Comm. Algebra **18** (1990), 3381–3394.

[3] ARA, P.: *Extensions of exchange rings*, J. Algebra **197** (1997), 409–423.

[4] ARNOLD, D. M.: *Finite Rank Torsion Free Abelian Groups and Rings*. Lecture Notes in Math. 931, Springer-Verlag, New York-Berlin, 1982.

[5] BASS, H.: *Algebraic K-Theory*, Benjamin, New York, 1968.

[6] BLACKADAR, B.—RØRDAM, M.: *Extending states on preordered semigroups and the exstence of quasitraces on C^*-algebras*, J. Algebra **152** (1992), 240–247.

[7] BLACKADAR, B.: *K-Theory for Operator Algebras*. Math. Sci. Res. Inst. Publ. 5, Springer-Verlag, New York, 1986.

[8] FACCHINI, A.—HERBRA, D.: *K_0 of a semilocal ring*, J. Algebra **225** (2000), 47–69.

[9] GOODEARL, K. R.—HANDELMAN, D.: *Rank functions and K_0 of regular rings*. J. Pure Appl. Algebra **7** (1976), 195–216.

[10] GOODEARL, K. R.: *Partially ordered Grothendieck groups*. In: Algebra and Its Applications. Lecture Notes in Pure and Appl. Math. 91, Dekker, New York, 1984, pp. 71–90.

[11] GOODEARL, K. R.—WARFIELD, Jr., R. B.: *State spaces of K_0 of noetherian rings*. J. Algebra **71** (1981), 322–378.

[12] MAGURN, B. A.: *An algebraic introduction to K-theory*. Encyclopedia Math. Appl. 87, Cambridge University Press, Cambridge, 2002.

[13] McCONNELL, J. C.—ROBSON, J. C.: *Noncommutative Noetherian Rings*, John Wiley and Sons, New York, 1987.

[14] ROSENBERG, J.: *Algebraic K-Theory and Its Applications*. Grad. Texts in Math. 147, Springer-Verlag, New York, 1994.

[15] SILVESTER, J. R.: *Introduction to Algebraic K-Theory*, Chapman & Hall, London-New York, 1981.

Received 3. 6. 2011
Accepted 17. 2. 2012

Department of Mathematics
Nanjing University
Nanjing 210093
P.R. CHINA
E-mail: zhuxs@nju.edu.cn