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GLOBAL SMOOTHNESS PRESERVATION WITH SECOND ORDER MODULUS OF SMOOTHNESS

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ABSTRACT. We establish the global smoothness preservation of a function f by the sequence of linear positive operators. Our estimate is in terms of the second order Ditzian-Totik modulus of smoothness. Application is given to the Bernstein operator.

1. Introduction

Over the recent years there has been considerable interest in the preservation of global smoothness properties by linear operators. This concept was introduced in [4], a recent book on the subject is [5]. Statements concerning global smoothness preservaton can be of various types. The first statements on this topic have been established as

$$\omega_s(B_n f, \varepsilon) \le M \cdot \omega_s(f, \varepsilon), \qquad \varepsilon \ge 0,$$

where ω_s is the classical sth order modulus of smoothness. That is, for $f \in C[0,1]$:

$$\omega_1(f,\varepsilon) := \sup\big\{|f(x) - f(y)| : \ |x-y| \le \varepsilon\big\}$$
$$\omega_s(f,\varepsilon) := \sup_{0<h\le\varepsilon} \max_{0\le x\le 1-sh} |\Delta_h^s f(x)|,$$

where $\Delta_h^s f(x)$ is the sth symmetric difference of the function f. For every function $f \in C[0,1]$ the Bernstein polynomial operator is given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1].$$

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The Lipschitz class of functions $\mathrm{Lip}_M(\alpha; [0,1])$ with exponent α for some $0 < \alpha \le 1$ and constant M consists of all continuous functions on $[0,1]$ which satisfy the inequality

$$|f(x) - f(y)| \le M|x-y|^\alpha, \qquad x, y \in [0,1],$$

with M an absolute positive constant independent of x, y and α. The equivalent to the last inequality is $\omega_1(f;\delta) \le M\delta^\alpha$ for all $\delta \in (0,1]$. A result on smoothness preservation by Bernstein operator on $C[0,1]$ was given in 1965 by Hajek in [17]:

Theorem A. *Let $f \in \mathrm{Lip}_M(1; [0,1])$. Then $B_n f \in \mathrm{Lip}_M(1; [0,1])$.*

This result was later generalized by Lindvall [18] and Brown et al. in [6], who showed that the same statement is also true if $\mathrm{Lip}_M(1; [0,1])$ is replaced by $\mathrm{Lip}_M(\alpha; [0,1])$, $0 < \alpha \le 1$. This means that, if global smoothness of a function $f \in C[0,1]$ is expressed by stating that it satisfies a certain Lipschitz condition, the same is true for its approximant $B_n f$. Theorem A was generalized in 1991 by Anastassiou, Cottin and Gonska in [4]:

Theorem B. *For the Bernstein operators B_n one has, for all $f \in C[0,1]$ and $\delta \ge 0$,*

$$\omega_1(B_n f, \delta) \le 1 \cdot \tilde{\omega}_1(f, \delta) \le 2 \cdot \omega_1(f, \delta). \tag{1.1}$$

Here $\tilde{\omega}_1(f,\cdot)$ denotes the least concave majorant of $\omega_1(f,\cdot)$. The constants 1 and 2 are best possible.

We recall that the least concave majorant of $\omega_1(f,t)$, $1 \ge t > 0$, is given by

$$\tilde{\omega}(f,t) = \sup\Big\{ \tfrac{(t-x)\omega(f;y) + (y-t)\omega(f;x)}{y-x} \ : \ 0 \le x \le t \le y \le 1 \Big\}.$$

Best constants for Bernstein-type oparators with the least concave majorant are obtained in de la Cal and Cárcamo [7]. In the multivariate setting, best constants in preservation inequalities for Bernstein-type operators can be found in de la Cal and Valle [10,11], de la Cal *et al.* [9] and de la Cal and Cárcamo [8]. On the other hand, D. X. Zhou obtained in [22] that the Bernstein operators do not preserve the Lipschitz classes of the second order $\mathrm{Lip}^*_M(\alpha, [0,1])$. Here $\mathrm{Lip}^*_M(\alpha, [0,1])$ denotes the set of all functions that satisfies the inequality $\omega_2(f,h) \le Mh^\alpha$ for all $h > 0$, with $M > 0$ and $0 < \alpha \le 2$. If we do not limit ourselves to functions satisfying a second order Lipschitz condition we can consider the estimate of the form

$$\omega_2(B_n f, h) \le c \cdot \omega_2(f,h), \qquad f \in C[0,1], \quad h > 0. \tag{1.2}$$

This type of estimates was studied first by Cottin and Gonska in [12], who obtained that there is such constant c, namely we can take $c = 4.5$. The following refinement and improvement of (1.2) is due to Păltănea (see [20])

$$2 \le \sup_{n \in N} \ \sup_{\substack{f \in C[0,1] \\ f \ne \text{linear}}} \ \sup_{h \in (0, \frac{1}{2}]} \frac{\omega_2(B_n f, h)}{\omega_2(f,h)} \le 3.$$

Preservation inequalities for the second modulus, that is, inequalities of the type

$$\omega_2(Lf;\delta) \le C(\delta)\omega_2(f;\delta), \qquad \delta \ge 0,$$

have been considered in Adell and de la Cal [1], Gonska and Kovacheva [16] and Adell and Pérez-Palomares [2]. Further the preservation of the Lipschitz class of second order for the Bernstein polynomials was stated and proved in Adell and de la Cal [1]. It is known that the usual second order modulus of continuity $\omega_2(f,\delta)$ is not appropriate to characterize the degree of approximation of $f \in C[0,1]$ by its Bernstein operator $B_n f$. This is possible using the Ditzian-Totik modulus of second order, given by

$$\omega_2^{\varphi}(f,\varepsilon) = \sup_{0<h\le\varepsilon} \sup_{x\pm h\varphi(x)\in[0,1]} \left|f\big(x-h\varphi(x)\big) - 2f(x) + f\big(x+h\varphi(x)\big)\right|, \quad (1.3)$$

where $\varphi^2(x) = x(1-x)$, $x \in [0,1]$. We recall the remarkable result of Knoop and Zhou from 1995 and independently of Totik, which is given in the book of DeVore and Lorentz ([13: Chap. 10]) as

$$\|B_n f - f\| \approx \omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right), \tag{1.4}$$

where $\approx$ means that there are two constants C_1, C_2 independent of f, n such that

$$C_1\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) \le \|B_n f - f\| \le C_2\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right).$$

So, the question is, how the global smoothness preservation of $B_n f$ can be expressed in terms of $\omega_2^{\varphi}(f,\cdot)$. To the end of the paper $\|\cdot\|$ denotes the usual supremum norm on the interval $[0,1]$. Given the operator $L\colon C[0,1] \to C[0,1]$ we recall that the norm of the operator L is defined as

$$\|L\| := \sup_{\substack{f\in C[0,1]\\ f\neq 0}} \frac{\|Lf\|}{\|f\|}.$$

In Section 2 we formulate our main results. In Section 3 we give the proofs.

2. Main results

Our main result states the following theorem:

Theorem 1. *If L is a linear positive operator preserving linear functions, $\varphi^2(x) = x(1-x)$, then the following estimate*

$$\omega_2^{\varphi}(Lf,h) \le \left[4\|L\| + 5 + \frac{28}{h^2}\left\|\frac{L((t-\cdot)^2,\cdot)}{\varphi^2(\cdot)}\right\|\right] \cdot \omega_2^{\varphi}(f,h) \tag{2.1}$$

holds true for all $h \in (0,1)$.

The second aim of this note is for the case when $\delta = \frac{1}{n}$, $n = 1, 2, \ldots$ to improve the estimate (1.1), more precisely to show that the constant 2 in (1.1) could be replaced by 1. We also generalize (1.1) in terms of the kth order modulus of continuity $\omega_k(f, \cdot)$. As usual by $(n)_k$ we denote

$$(n)_k := \begin{cases} n(n-1)(n-2)\cdots(n-k+1), & n \geq k, \\ 0 & n < k. \end{cases}$$

Our main results are:

Theorem 2. *For all $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$ if $f \in C[0,1]$ then the following*

$$\omega_k\left(B_n f, \frac{1}{n}\right) \leq \frac{(n)_k}{n^k} \cdot \omega_k\left(f, \frac{1}{n}\right) \tag{2.2}$$

holds true.

Theorem 3. *For all $f \in C^k[0,1]$, $k \geq 0$, $s \geq 1$ the following*

$$\omega_s\left(D^k B_n f, \frac{1}{n}\right) \leq \frac{(n)_{k+s}}{n^{k+s}} \cdot \omega_s\left(D^k f, \frac{1}{n}\right) \tag{2.3}$$

holds true.

Corollary 1. *For all $h \in (0,1)$ it holds that*

$$\omega_2^\varphi(B_n f, h) \leq 37 \cdot \omega_2^\varphi(f, h). \tag{2.4}$$

3. Proofs

Let us define the following function space

$$W^2_{\infty,\varphi} = \big\{ f \in AC[0,1] : \ f' \in AC_{loc}, \ \varphi^2 f'' \in L_\infty[0,1] \big\}, \tag{3.1}$$

where $AC[0,1]$ denotes the space of absolutely continuous functions, defined on the interval $[0,1]$, AC_{loc} denotes the class of functions, which are absolutely continuous on each closed subinterval $[a,b] \subset (0,1)$, and L_∞ is the class of bounded measurable functions on the interval $[0,1]$. The typical example of functions from this space is

$$u(x) = x\ln(x) + (1-x)\ln(1-x), \qquad x \in (0,1), \quad u(0) = u(1) = 0,$$

which was used as "universal tool" to obtain different results in approximation theory. It is easy to observe that

$$u''(x) = \frac{1}{x(1-x)}, \qquad x \in (0,1), \quad \varphi^2(x) u''(x) = 1.$$

The following result from [21] will be used later.

Lemma 1. *For $g \in W^2_{\infty,\varphi}$ we have*

$$\|Lg - g\| \le \|\varphi^2 g''\|_\infty \cdot \|(\varphi^2(\cdot))^{-1} L((t-\cdot)^2, \cdot)\|_\infty, \tag{3.2}$$

where L satisfies the conditions from Theorem 1.

As usual for $f \in C[0,1]$ we replace $\|\cdot\|_\infty$ by sup norm $\|\cdot\|_{C[0,1]} := \|\cdot\|$. In [15] Gavrea proved the following:

Lemma 2. *Let $h \in \left[\sin \frac{\pi}{2(m+1)}, \sin \frac{\pi}{2m}\right]$, $m \ge 1$. There exists function $g \in W^2_{\infty,\varphi}$, such that*

$$\|f - g\| \le \omega_2^\varphi \left(f, \sin \frac{\pi}{2(m+1)}\right) \le \omega_2^\varphi(f,h), \tag{3.3}$$

$$h^2 \cdot \|\varphi^2 g''\| \le c_m \cdot \omega_2^\varphi \left(f, \sin \frac{\pi}{2(m+1)}\right) \le c_m \cdot \omega_2^\varphi(f,h), \tag{3.4}$$

where

$$c_m = \frac{\sin^2\left(\frac{\pi}{2m}\right)}{\sin^2\left(\frac{\pi}{4(m+1)}\right)}, \qquad \lim_{m\to\infty} c_m = 4$$

and

$$c_m \le c_1 = \frac{1}{\sin^2 \frac{\pi}{8}} = \frac{2\sqrt{2}}{\sqrt{2}-1} \in (6,7). \tag{3.5}$$

From the definition of $\omega_2^\varphi(f,h)$, $h \in (0,1)$ it follows that

$$\omega_2^\varphi(f,h) \le 4\|f\|. \tag{3.6}$$

P r o o f o f T h e o r e m 1. For the function $g \in W^2_{\infty,\varphi}$ from Lemma 2 we write

$$\omega_2^\varphi(Lf,h) \le \omega_2^\varphi(Lf - Lg, h) + \omega_2^\varphi(Lg - g, h) + \omega_2^\varphi(g - f, h) + \omega_2^\varphi(f,h). \tag{3.7}$$

From Lemmas 1,2 and (3.6) we get

$$\begin{aligned}
\omega_2^\varphi(Lf,h) &\le 4\|L\| \cdot \|f-g\| + 4\|Lg - g\| + 4\|g - f\| + \omega_2^\varphi(f,h)\\
&\le 4\|L\| \cdot \omega_2^\varphi(f,h) + 4\|\varphi^2 g''\| \cdot \|(\varphi^2(\cdot))^{-1} L((t-\cdot)^2,\cdot)\|\\
&\quad + 4\omega_2^\varphi(f,h) + \omega_2^\varphi(f,h)\\
&\le \omega_2^\varphi(f,h) \cdot \left[4\|L\| + 5 + \frac{4c_1}{h^2} \cdot \left\|\left(\varphi^2(\cdot)\right)^{-1} L\big((t-\cdot)^2,\cdot\big)\right\|\right].
\end{aligned} \tag{3.8}$$

The estimates (3.5) and (3.8) complete the proof of Theorem 1. □

P r o o f o f C o r o l l a r y 1. It is known that for the Bernstein operator (see [19]) we have $\|B_n\| = 1$ and

$$B_n((t-x)^2, x) = \frac{x(1-x)}{n}.$$

Now from Theorem 1 we immediately get the proof of (2.4). □

Remark 1. It is clear that the constant 37 is not the best possible.

Proof of Theorem 2. We use the following representation of D^kB_nf, see [13: p. 121]:

$$D^kB_n(f,x) = n(n-1)\cdots(n-k+1)\cdot\sum_{v=0}^{n-k}\Delta^k f\left(\frac{\nu}{n}\right)\binom{n-k}{\nu}x^\nu(1-x)^{n-k-\nu}, \tag{3.9}$$

with

$$\Delta f\left(\frac{\nu}{n}\right) = f\left(\frac{\nu+1}{n}\right) - f\left(\frac{\nu}{n}\right)$$

and the kth iterate of Δ denoted by Δ^k. From (3.9) it is easy to observe that

$$|D^kB_n(f,x)| \le (n)_k\cdot\omega_k\left(f,\frac{1}{n}\right), \tag{3.10}$$

which follows from the definition of the kth order modulus of continuity of f. It is well known (see [13]) that for all $\delta > 0$, $f \in C^k[0,1]$ we have

$$\omega_k(f,\delta) \le \delta^k\|f^{(k)}\|_{C[0,1]}.$$

The latter implies

$$\omega_k\left(B_nf,\frac{1}{n}\right) \le \frac{1}{n^k}\|B_n^{(k)}f\| \le \frac{(n)_k}{n^k}\cdot\omega_k\left(f,\frac{1}{n}\right), \tag{3.11}$$

where we have used (3.10). The proof of Theorem 2 is completed. □

Proof of Theorem 3. Using the properties of the modulus ω_s and the representation of the $(k+s)$th derivative of B_n-from (3.9) we compute

$$\begin{aligned}\omega_s\left(D^kB_nf,\frac{1}{n}\right) &\le \frac{1}{n^s}\cdot\|D^{k+s}B_nf\|\\ &\le \frac{(n)_{k+s}}{n^s}\cdot\omega_{k+s}\left(f,\frac{1}{n}\right) \le \frac{(n)_{k+s}}{n^s\cdot n^k}\cdot\omega_s\left(D^kf,\frac{1}{n}\right)\\ &\le 1\cdot\omega_s\left(D^kf,\frac{1}{n}\right).\end{aligned}$$

The proof of Theorem 3 is completed. □

Corollary 2. *For $k=1$ we get for all $n \ge 1$*

$$\omega_1\left(B_nf,\frac{1}{n}\right) \le 1\cdot\omega_1\left(f,\frac{1}{n}\right). \tag{3.12}$$

The last estimate improves the right hand side of (1.1) for the case $\delta = \frac{1}{n}$.

Corollary 3. *For $k = 2$ it follows, for all $n = 1, 2, \dots,$ that*

$$\omega_2\left(B_n f, \frac{1}{n}\right) \le \left(1 - \frac{1}{n}\right)\omega_2\left(f, \frac{1}{n}\right). \tag{3.13}$$

The following problem was formulated in [4]:

Problem 1. *Is it true for the Bernstein operator B_n that $f \in \mathrm{Lip}^*_M(\alpha, [0,1])$ implies $B_n f \in \mathrm{Lip}^*_M(\alpha, [0,1])$, $0 < \alpha \le 2$?*

The positive answer to this problem for the case $\alpha = 2$ was given by Adell and de la Cal in [1]. Indeed, from (3.10) we immediately get

$$\begin{aligned}\omega_2(B_n f, \delta) &\le \delta^2 \cdot \|B_n^{(2)} f\| \\ &\le \delta^2 \cdot n(n-1)\omega_2\left(f, \frac{1}{n}\right) \le \delta^2 \cdot n(n-1)M\frac{1}{n^2} \\ &\le M \cdot \delta^2.\end{aligned}$$

We are able to generalize the problem above, giving positive answer only with additional condition for δ, namely:

Theorem 4. *If*

$$\omega_k(f, \delta) \le M \cdot \delta^{k-\alpha}, \qquad 0 < \alpha < k \quad \text{and} \quad \delta \le \frac{1}{n}$$

then

$$\omega_k(B_n f, \delta) \le M \cdot \delta^{k-\alpha}.$$

Proof of Theorem 4. We compute as follows

$$\begin{aligned}\omega_k(B_n f, \delta) &\le \delta^k \|B_n^{(k)} f\| \le \delta^k \cdot n^k \cdot \omega_k(f, \frac{1}{n}) \\ &\le \delta^k n^k \cdot \frac{1}{n^k} \cdot n^\alpha \cdot M = \delta^{k-\alpha} \cdot (\delta \cdot n)^\alpha \cdot M \\ &\le M \cdot \delta^{k-\alpha}.\end{aligned}$$

The proof of Theorem 4 is completed. □

Remark 2. If $\alpha = 0$ in Theorem 4, i.e., f belongs to the saturation class for ω_k, then the statement holds for all $\delta \ge 0$. This may be verified by the following way

$$\begin{aligned}\omega_k(B_n f, \delta) &\le \delta^k \|B_n^{(k)} f\| \le \delta^k \cdot n^k \cdot \omega_k(f, \frac{1}{n}) \\ &\le M \cdot \delta^k.\end{aligned}$$

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