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A SEQUENTIAL IMPLICIT FUNCTION THEOREM FOR THE CHORDS ITERATION

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ABSTRACT. In this paper we study the local convergence of the method

$$0 \in f(p, x_k) + A(x_{k+1} - x_k) + F(x_{k+1}),$$

in order to find the solution of the generalized equation

$$\text{find} \quad x \in X \quad \text{such that} \quad 0 \in f(p, x) + F(x).$$

We first show that under the strong metric regularity of the linearization of the associated mapping and some additional assumptions regarding dependence on the parameter and the relation between the operator A and the Jacobian $\nabla_x f(\bar{p}, \bar{x})$, we prove linear convergence of the method which is uniform in the parameter p. Then we go a step further and obtain a sequential implicit function theorem describing the dependence of the set of sequences of iterates of the parameter.

1. Introduction

Generalized equations function as an abstract model of wide variety of variational problems such as linear and nonlinear complementarity problems, systems of nonlinear equations, first-order necessary conditions for nonlinear programming etc. They are also widely used in engineering and economics. More information about these applications and many other will appear in [7].

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The local convergence and the stability of the method of chords for solving generalized equations of the form

$$\text{find} \quad x \in X \quad \text{such that} \quad 0 \in f(x) + F(x), \tag{1}$$

is presented in [9] and [10] by R. T. Marinov. He studies the iterative procedure

$$0 \in f(x_k) + A(x_{k+1} - x_k) + F(x_{k+1}),$$

where f is a function and F is a set-valued mapping acting from a Banach space X to a linear normed space Y and $A \in L(X, Y)$. The generalized equations were introduced by S. M. Robinson in the late 20th century as a general tool for describing, analyzing , and solving different problems in a unified manner; for a survey of earlier results see [11]. For example, when $F = \{0\}$, (1) is an equation; when F is the positive orthant in $\mathbb{R}^n$, (1) is a system of inequalities; when F is the normal cone to a convex and closed set in X, (1) represents variational inequalities. For other examples, the reader could refer to [3].

With the aim of approximating a solution to the generalized equation

$$\text{find} \quad x \in X \quad \text{such that} \quad 0 \in f(p, x) + F(x), \tag{2}$$

for a fixed value of the parameter p of the Banach space P in [4] and [6] A. L. Dontchev and R. T. Rockaffelar choose a starting point x_0 and generate a sequence $\{x_k\}_{k=0}^{\infty}$, iteratively for $k = 0, 1, 2, \ldots$ by taking x_{k+1} to be a solution to the auxiliary generalized equation

$$0 \in f_k(p, x_{k+1}) + F(x_{k+1}),$$

where

$$f_k(p, x) = f(p, x_k) + \nabla_x f(p, x_k)(x - x_k).$$

This iteration reduces to the standard Newton's method for solving a nonlinear equations when F is the zero mapping.

Newton's type method has been a basic tool for proving various inverse function type theorems. "Generalized" Newton method leads to modern set-valued extensions of these theorems in terms of metric regularity (see, in particular [5], [1]). A wide overview of Newton's methods for nonsmooth equations and generalized equations is available in the books by Klatte and Kummer [8] and Facchinei and Pang [7].

In this paper we study the local convergence of the method

$$0 \in f(p, x_k) + A(x_{k+1} - x_k) + F(x_{k+1}), \tag{3}$$

in order to find the solution of the generalized equation (2). We show that this method is convergent to the value $s(p)$ of the Lipschitz continuous localization of the solution mapping, provided that the mapping

$$f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x})(\cdot - \bar{x}) + F(\cdot)$$

is strongly metrically regular at $\bar{x}$ for 0 with associated Lipschitz continuous single valued localization σ around 0 for $\bar{x}$ of the inverse

$$[f(\bar{p},\bar{x}) + \nabla_x f(\bar{p},\bar{x})(\cdot - \bar{x}) + F(\cdot)]^{-1},$$

$A \in L(X,Y)$ is such that $8.\,\mathrm{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\| < 1$, f is Lipschitz continuous with respect to p uniformly with x and $\nabla_x f$ is Lipschitz continuous with respect to x uniformly with p. We establish an implicit function theorem, using implicit mapping that involves sequences of iterates, obtained by method (3), as elements of a sequence space. We show that the single-valued graphical localization of this mapping is Lipschitz localization and prove some estimates for the Lipschitz modulus of this localization.

This work is organized as follows. In Section 2 we collect some definitions and results that we will need afterwards. In Section 3 we establish an implicit function theorem for chords iteration.

Throughout this paper all the norms are denoted by $\|\cdot\|$. The closed ball centered at x with radius a is denoted by $\mathbb{B}_a(x)$.

2. Preliminaries

In this paper we work with Banach spaces P for p, X for x and Y for the range of f considering generalized equation of the form (2). We assume that f is continuously Fréchet differentiable in x with derivative denoted by $\nabla_x f(p,x)$ and both $f(p,x)$ and $\nabla_x f(p,x)$ depend continuously on (p,x). Then *graph* of F is the set $\mathrm{gph}\, F = \{(x,y) \in X \times Y \mid y \in F(x)\}$ and the *inverse* of F is the mapping $F^{-1}\colon Y \rightrightarrows X$ defined by $F^{-1}(y) = \{x \mid y \in F(x)\}$.

The *solution mapping* associated with the generalized equation (2) is the potentially set-valued mapping $S\colon P \rightrightarrows X$ defined by

$$S\colon p \mapsto \{x \mid f(p,x) + F(x) \ni 0\} \qquad \text{for} \quad p \in P. \tag{4}$$

Graphical localization of S at $\bar{p}$ for $\bar{x}$, where $\bar{x} \in S(\bar{p})$ is a set-valued mapping with its graph having the form $(Q \times U) \cap \mathrm{gph}\, S$ for some neighborhoods Q of $\bar{p}$ and U of $\bar{x}$. The localization is single-valued when this mapping reduces to a function for Q into U. If it is not only single-valued but Lipschitz continuous on Q, we speak of *Lipschitz localization.*

A function $f\colon X \to Y$ is said to be *Lipschitz continuously* relative to a set D, if $D \subset \mathrm{int}\,\mathrm{dom}\, f$ and there exists a constant $k \geq 0$ (a Lipschitz constant) such that

$$\|f(x') - f(x'')\| \leq k\|x' - x''\| \qquad \text{for all} \quad x', x'' \in D. \tag{5}$$

It is said to be a Lipschitz continuous around $\bar{x}$ when this holds for some neighborhood D of $\bar{x}$. The *Lipschitz modulus* of f at $\bar{x}$, denoted $\mathrm{lip}(f;\bar{x})$ is the infimum of the set of values k for which there exists a neighborhood D of $\bar{x}$ such that (5) holds. Equivalently,

$$\operatorname{lip}(f;\bar{x}) := \limsup_{\substack{x',x''\to\bar{x}\\ x'\neq x''}} \frac{\|f(x')-f(x'')\|}{\|x'-x''\|}.$$

Further, a function $f\colon P\times X\to Y$ is said to be Lipschitz continuous with respect to x uniformly in p around $(\bar{p},\bar{x})\in\operatorname{int}\operatorname{dom} f$ when there are neighborhoods Q of $\bar{p}$ and U of $\bar{x}$ along with a constant k and such that

$$\|f(p,x')-f(p,x'')\|\le k\|x'-x''\| \qquad \text{for all} \quad x',x''\in U \quad \text{and} \quad p\in Q.$$

Accordingly, the partial *uniform Lipschitz modulus* has the form

$$\widehat{\operatorname{lip}}_x(f;(\bar{p},\bar{x})) := \limsup_{\substack{x',x''\to\bar{x}\\ x'\neq x''\\ p\to\bar{p}}} \frac{\|f(p,x')-f(p,x'')\|}{\|x'-x''\|}.$$

A mapping $F\colon X\rightrightarrows Y$ with $(\bar{p},\bar{x})\in\operatorname{gph} F$ is called *strongly regular* at $\bar{x}$ for $\bar{y}$ if its inverse F^{-1} has a Lipschitz localization at $\bar{y}$ for $\bar{x}$.

For the convenience of the reader we repeat the relevant material from [4] and [6] without proofs, thus making our exposition self-contained.

Theorem 2.1 (Implicit function theorem)**.** ([4]) *For the generalized equation* (2) *and its solution mapping S in* (4)*, let $\bar{p}$ and $\bar{x}$ be such that $\bar{x}\in S(\bar{p})$. Assume that f is Lipschitz continuous with respect to p uniformly in x at $(\bar{p},\bar{x})$, that is,*

$$\widehat{\operatorname{lip}}_p(f;(\bar{p},\bar{x}))<\infty,$$

and that the inverse G^{-1} of the mapping

$$G(x)=f(\bar{p},\bar{x})+\nabla_x f(\bar{p},\bar{x})(x-\bar{x})+F(x) \qquad \textit{for which} \quad G(\bar{x})\ni 0, \tag{6}$$

has a Lipschitz continuous localization σ at 0 for $\bar{x}$.

Then the mapping S has a Lipschitz localization s at $\bar{p}$ for $\bar{x}$ with

$$\operatorname{lip}(s;\bar{p})\le\operatorname{lip}(\sigma;0).\widehat{\operatorname{lip}}_p(f;(\bar{p},\bar{x})).$$

In the case of the generalized equation (2) with $P=Y$ and $f(p,x)=g(x)-p$ for a function $g\colon X\to Y$, so that

$$S(p)=\big\{x \mid p\in g(x)+F(x)\big\}=(g+F)^{-1}(p), \tag{7}$$

the property of both the mapping G in Theorem 2.1 as well as the mapping $g+F$ translates as strong regularity and the theorem may be transformed to the following form.

Theorem 2.2 (Inverse version). ([4]) *In the framework of the solution mapping (7), consider any pair $(\bar{p}, \bar{x})$ with $\bar{x} \in S(\bar{p})$. Then the mapping $g + F$ is strongly regular at $\bar{x}$ for $\bar{p}$ if and only if its partial linearization*

$$x \mapsto G(x) = g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) + F(x)$$

is strongly regular at $\bar{p}$ for $\bar{x}$. In addition, if s and σ are the associated Lipschitz localizations of $(g + F)^{-1}$ and G^{-1} respectively, then

$$\operatorname{lip}(s - \sigma; \bar{p}) = 0.$$

This implies in particular that $\operatorname{lip}(s; \bar{p}) = \operatorname{lip}(\sigma; \bar{p})$.

Theorem 2.3 (Stability of strong regularity under perturbation). ([4]) *Consider a mapping $T\colon X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ such that, for a positive constant k and neighborhoods U of $\bar{x}$ and V of $\bar{y}$, the mapping $y \mapsto T^{-1}(y) \cap U$ is a Lipschitz continuous function on V with Lipschitz constant k. Then for every positive constant μ with $k\mu < 1$ there exists neighborhoods $U' \subset U$ of $\bar{x}$ and $V' \subset V$ of $\bar{y}$ such that for every function $h\colon X \to Y$ which is Lipschitz continuous on U with Lipschitz constant μ the mapping $(h + T)^{-1}(y) \cap U'$ is Lipschitz continuous function on $h(\bar{x}) + V'$ with Lipschitz constant $k/(1 - k\mu)$.*

The following property of Theorem 2.3 is utilized in the paper.

Corollary 2.4. ([4]) *For mapping $F\colon X \rightrightarrows Y$ and point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, let F be strongly regular at $\bar{x}$ for $\bar{y}$ with associated Lipschitz localization s of F^{-1} at $\bar{y}$ for $\bar{x}$. Consider also a function $r\colon P \times X \to Y$ such that $r(p, \bar{x})$ is continuous at $\bar{p}$ and*

$$\operatorname{lip}(s; \bar{y}).\widehat{\operatorname{lip}}_x(r; (\bar{p}, \bar{x})) < 1.$$

Then for each

$$\gamma > \frac{\operatorname{lip}(s; \bar{y})}{1 - \operatorname{lip}(s; \bar{y}) \cdot \widehat{\operatorname{lip}}_x(r; (\bar{p}, \bar{x}))}$$

there exist neighborhoods U of $\bar{x}$, V of $\bar{y}$ and Q of $\bar{p}$ such that for every $p \in Q$ the mapping $y \mapsto (r(p, \cdot) + F)^{-1}(y) \cap U$ is Lipschitz continuous function on $r(\bar{p}, \bar{x}) + V$ with Lipschitz constant γ.

Throughout the paper we use the following result.

Theorem 2.5 (Contraction mapping principle). ([2]) *Let X be a complete metric space with metric ρ. Consider a point $\bar{x} \in X$ and a function $\Phi\colon X \to X$ for which there exist scalars $a > 0$ and $\lambda \in [0, 1)$ such that:*

(a) $\rho(\Phi(\bar{x}), \bar{x}) \leq a(1 - \lambda)$;

(b) $\rho(\Phi(x'), \Phi(x'')) \leq \lambda \rho(x', x'')$ *for every* $x', x'' \in \mathbb{B}_a(\bar{x})$.

Then there is a unique $x \in \mathbb{B}_a(\bar{x})$ satisfying $x = \Phi(x)$, that is, Φ has a unique fixed point in $\mathbb{B}_a(\bar{x})$.

3. Main result

In this section we study the iterative procedure (3) with given starting point x_0, in order to approximate a solution to the generalized equation (2). We reconceive chords iteration as an inclusion, the solution of which gives an infinite sequence $\xi = \{x_1, x_2, \dots, x_k, \dots\}$ of the Banach space $l_\infty(X)$ with elements $x_k \in X, \quad k = 1, 2, \dots$, instead of just an element in X. The norm on $l_\infty(X)$ is

$$\|\xi\|_\infty = \sup_{k\geq 1} \|x_k\|.$$

Define a mapping $\Xi\colon X \times P \rightrightarrows l_\infty(X)$ by

$$\Xi:\ (u,p) \mapsto \Big\{\xi \in l_\infty(X) \mid \bigcap_{k=0}^{\infty} f(p, x_k) + A(x_{k+1} - x_k) + F(x_{k+1}) \ni 0 \text{ with } x_0 = u\Big\}, \tag{8}$$

whose value for a given (u,p) is the set of all sequences $\{x_k\}_{k=1}^{\infty}$ generated by chords iteration (3) for p start from u. If $\bar{x}$ is a solution of (2) for $\bar{p}$, the constant sequence $\bar{\xi} = \{\bar{x}, \bar{x}, \dots, \bar{x}, \dots\}$ satisfies $\bar{\xi} \in \Xi(\bar{p}, \bar{x})$.

Theorem 3.1 (Uniformly convergence of chords iteration)**.** *In the framework of the generalized equation* (2) *with solution mapping S in* (4)*, let $\bar{x} \in S(\bar{p})$. Assume that*

$$\widehat{\operatorname{lip}}_p(f;(\bar{p},\bar{x})) + \widehat{\operatorname{lip}}_x(\nabla_x f;(\bar{p},\bar{x})) < \infty$$

and the mapping G in (6) *be strongly regular at $\bar{x}$ for 0 with associated Lipschitz localization σ of the inverse G^{-1} at 0 for $\bar{x}$ and $A \in L(X,Y)$ is such that* $8.\operatorname{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\| < 1$.

Then for every γ, such that

$$\frac{1}{3} > \gamma > \frac{2\operatorname{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\|}{1 - 2\operatorname{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\|}, \tag{9}$$

there exist neighborhoods Q of $\bar{p}$ and U of $\bar{x}$ such that for every $p \in Q$ and $u \in U$ there is exactly one sequence $\xi(u,p)$ with components $x_1, \dots, x_k, \dots$, all belonging to U and generated by chords method (3) *starting from u for the value p of the parameter. This sequence is convergent to the value $s(p)$ of the Lipschhiz localization s of the solution mapping S at $\bar{p}$ for $\bar{x}$ whose existence is claimed in* Theorem 2.1*; moreover the convergence is q-linear with constant γ, that is*

$$\|x_{k+1} - s(p)\| \leq \gamma \|x_k - s(p)\| \qquad \textit{for all} \quad k \geq 0. \tag{10}$$

In other words, the mapping Ξ in (8) *has a single valued graphical localization ξ at $(\bar{p},\bar{x})$ for $\bar{\xi}$; moreover, for u close to $\bar{x}$ and p close to $\bar{p}$ the value $\xi(u,p)$ of this localization is sequence which converges to the associated solution $s(p)$ for p in the sense of* (10).

Proof. Let $A \in L(X,Y)$ be such that $8.\,\mathrm{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\| < 1$ and denote the constant $\omega := 2\|\nabla_x f(\bar{p},\bar{x}) - A\|$. We choose γ as in (9). Let $\kappa > \mathrm{lip}(\sigma;0)$ be such that $\frac{\kappa\omega}{1-\kappa\omega} < \gamma < \frac{1}{3}$ and $\mu > \widehat{\mathrm{lip}}_x(\nabla_x f;(\bar{p},\bar{x}))$. Next we choose $\varepsilon > \omega$ such that $\kappa\varepsilon < 1$ and

$$\frac{\kappa\omega}{1-\kappa\varepsilon} < \gamma < \frac{1}{3}. \tag{11}$$

Then the assumed strong regularity of the mapping G in (6) at $\bar{x}$ for 0 and the choice of κ mean that there exist positive constants α' and b' such that the mapping $y \mapsto \sigma(y) = G^{-1}(y) \cap \mathbb{B}_{\alpha'}(\bar{x})$ is Lipschitz continuous function on $\mathbb{B}_{b'}(0)$ with Lipschitz constant κ. Along with the mapping G in (6) consider the parametrized mapping

$$x \mapsto G_{p,w}(x) = f(p,w) + \nabla_x f(p,w)(x-w) + F(x).$$

Note that $G_{p,w}(x) = r(p,w;x) + G(x)$, where the function

$$x \mapsto r(p,w;x) = f(p,w) + \nabla_x f(p,w)(x-w) - f(\bar{p},\bar{x}) - \nabla_x f(\bar{p},\bar{x})(x-\bar{x})$$

is affine, and hence Lipschitz continuous, with Lipschitz constant

$$\eta(p,w) = \|\nabla_x f(p,w) - \nabla_x f(\bar{p},\bar{x})\|.$$

Now let κ' be such that $\kappa > \kappa' > \mathrm{lip}(\sigma;0)$ and let $\chi > 0$ satisfy

$$\chi\,\kappa' < 1 \qquad \text{and} \qquad \frac{\kappa'}{1-\chi\,\kappa'} < \kappa.$$

Applying Corollary 2.4 to the mapping $G_{p,w}$ and noting that $r(\bar{p},\bar{x};\bar{x}) = 0$, we obtain that there are positive constants $\alpha \le \alpha'$ and $b \le b'$ such that for p and w satisfying $\eta(p,w) \le \chi$ the mapping $G_{p,w}^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is Lipschitz continuous function on $\mathbb{B}_b(0)$ with Lipschitz constant κ. We denote this function by $\varphi(p,w;\cdot)$.

By the continuity of $\nabla_x f(p,x)$ near $(\bar{p},\bar{x})$, there exist positive constants c and a such that

$$\|\nabla_x f(p,\bar{x}) - A\| < \|\nabla_x f(\bar{p},\bar{x}) - A\| + \|\nabla_x f(p,\bar{x}) - \nabla_x f(\bar{p},\bar{x})\| < \omega$$

whenever $p \in \mathbb{B}_c(\bar{p})$. Make a and c smaller if necessary so that $a \le \alpha$ and

$$\|\nabla_x f(p,x') - \nabla_x f(p,x'')\| \le \mu\|x' - x''\|, \tag{12}$$

for $x', x'' \in \mathbb{B}_a(\bar{x})$ and $p \in \mathbb{B}_c(\bar{p})$. Now we can apply Theorem 2.1 and further adjust the constants a and c so that the truncation $S(p) \cap \mathbb{B}_a(\bar{x})$ of the solution mapping S in (4) is function s, which is Lipschitz continuous on $\mathbb{B}_c(\bar{p})$.

Next, take a even smaller if necessary so that

$$\frac{43}{8}\mu a^2 + 2a\omega \le b, \qquad \frac{1}{2}\mu a + \omega \le \varepsilon, \qquad \frac{7}{4}\kappa\mu a + \kappa\omega < \frac{1}{3}(1-\kappa\varepsilon). \tag{13}$$

The first and the third inequality in (13) allow us to choose $\delta > 0$ satisfying

$$\delta + \frac{1}{8}\mu a^2 \le b, \qquad \kappa\delta + \frac{3}{2}\kappa\mu a^2 + \kappa a\omega \le a(1-\kappa\varepsilon). \tag{14}$$

Then make c even smaller if necessary so that

$$\|s(p) - \bar{x}\| \le \frac{a}{2} \quad \text{and} \quad \|f(p,\bar{x}) - f(\bar{p},\bar{x})\| \le \delta \qquad \text{for} \quad p \in \mathbb{B}_c(\bar{p}). \tag{15}$$

Summarizing, we have found constants a, b and c such that for each $p \in \mathbb{B}_c(\bar{p})$ and $w \in \mathbb{B}_a(\bar{x})$ the function $\varphi(p, w; \cdot)$ is Lipschitz continuous on $\mathbb{B}_b(0)$ with constant κ, and also the conditions (12)–(15) are satisfied.

We frequently use an estimate for smooth functions obtained by simple calculus. For the standard equality

$$f(p,u) - f(p,v) = \int_0^1 \nabla_x f(p, v + t(u-v))(u-v)\,\mathrm{d}t,$$

which yields

$$\begin{aligned}
&\|f(p,u) - f(p,v) - \nabla_x f(p,v)(u-v)\| \\
&= \left\| \int_0^1 \nabla_x f(p, v+t(u-v))(u-v)\,\mathrm{d}t - \nabla_x f(p,v)(u-v) \right\| \\
&\le \mu \int_0^1 t\,\mathrm{d}t\, \|u-v\|^2,
\end{aligned}$$

we have from (12) that for all $u, v \in \mathbb{B}_a(\bar{x})$ and $p \in \mathbb{B}_c(\bar{p})$

$$\|f(p,u) - f(p,v) - \nabla_x f(p,v)(u-v)\| \le \frac{1}{2}\mu\|u-v\|^2. \tag{16}$$

Moreover,

$$\begin{aligned}
\|f(p,u) &- f(p,v) - A(u-v)\| \\
&\le \|f(p,u) - f(p,v) - \nabla_x f(p,\bar{x})(u-v)\| \\
&\quad + \|\nabla_x f(p,\bar{x})(u-v) - A(u-v)\| \\
&\le \|f(p,u) - f(p,v) - \nabla_x f(p,v)(u-v)\| \\
&\quad + \|\nabla_x f(p,v)(u-v) - \nabla_x f(p,\bar{x})(u-v)\| \\
&\quad + \|\nabla_x f(p,\bar{x}) - A\|\|u-v\|.
\end{aligned}$$

Then for all $u, v \in \mathbb{B}_a(\bar{x})$ and $p \in \mathbb{B}_c(\bar{p})$ we have

$$\begin{aligned}
\|f(p,u) &- f(p,v) - A(u-v)\| \\
&\le \frac{1}{2}\mu\|u-v\|^2 + \mu\|v-\bar{x}\|\|u-v\| + \omega\|u-v\|.
\end{aligned} \tag{17}$$

Fix $p \in \mathbb{B}_c(\bar{p})$ and $w \in \mathbb{B}_a(\bar{x})$ and consider the function

$$x \mapsto g(p, w; x) := -f(p, w) - A(x - w) + f(p, s(p)) + \nabla_x f(p, s(p))(x - s(p)). \quad (18)$$

Recall that here $s(p) = S(p) \cap \mathbb{B}_a(\bar{x})$ for all $p \in \mathbb{B}_c(\bar{p})$. For any $x \in \mathbb{B}_a(\bar{x})$ using (12), (15) and (16), we have

$$\begin{aligned}
\|g(p, w; x)\| &\leq \|f(p, s(p)) - f(p, w) - \nabla_x f(p, w)(s(p) - w)\| \\
&\quad + \|\nabla_x f(p, w)(x - w) + \nabla_x f(p, w)(s(p) - x) \\
&\quad + \nabla_x f(p, s(p))(x - s(p)) - A(x - w)\| \\
&\leq \frac{1}{2}\mu \|s(p) - w\|^2 + \|\nabla_x f(p, w) - A\| \|x - w\| \\
&\quad + \|\nabla_x f(p, s(p)) - \nabla_x f(p, w)\| \|x - s(p)\| \\
&\leq \frac{1}{2}\mu \|s(p) - w\|^2 + \|x - w\| (\|\nabla_x f(p, w) - \nabla_x f(p, \bar{x})\| \\
&\quad + \|\nabla_x f(p, \bar{x}) - A\|) + \mu \|s(p) - w\| \|x - s(p)\| \\
&\leq \frac{1}{2}\mu \frac{9}{4} a^2 + 2a(\mu \|w - \bar{x}\| + \omega) + \mu \frac{3}{2} a \frac{3}{2} a \\
&= \frac{43}{8}\mu a^2 + 2a\omega.
\end{aligned}$$

Then from the first inequality in (13),

$$\|g(p, w; x)\| \leq b. \quad (19)$$

Using (15) and (16) we come to

$$\begin{aligned}
&\|f(\bar{p}, \bar{x}) - f(p, s(p)) - \nabla_x f(p, s(p))(\bar{x} - s(p))\| \\
&\leq \|f(\bar{p}, \bar{x}) - f(p, \bar{x})\| + \|f(p, \bar{x}) - f(p, s(p)) - \nabla_x f(p, s(p))(\bar{x} - s(p)\| \\
&\leq \delta + \frac{1}{2}\mu \|\bar{x} - s(p)\|^2 \leq \delta + \frac{1}{8}\mu a^2.
\end{aligned}$$

Then using the first inequality in (14), we have

$$\|f(\bar{p}, \bar{x}) - f(p, s(p)) - \nabla_x f(p, s(p))(\bar{x} - s(p))\| \leq b. \quad (20)$$

Hence, remembering that $p \in \mathbb{B}_c(\bar{p})$ and $s(p) \in \mathbb{B}_a(\bar{x})$, both $g(p, w; x)$ and $f(\bar{p}, \bar{x}) - f(p, s(p)) - \nabla_x f(p, s(p))(\bar{x} - s(p))$ are in the domain of $\varphi(p, s(p); \cdot)$ where this function is Lipschitz continuous with Lipschitz constant κ.

We now choose $p \in \mathbb{B}_c(\bar{p})$ and $u \in \mathbb{B}_a(\bar{x})$, and construct a sequence $\xi(u, p)$ generated by iteration (3) starting from u for the value p of the parameter, whose existence, uniqueness and convergence is claimed in the statement of the theorem.

If $u = s(p)$ there is nothing to prove, so assume $u \neq s(p)$. Our first step is to show that, for the function g defined in (18), the mapping

$$\Phi_0 \colon x \mapsto \varphi(p, s(p); g(p, u; x)),$$

has a unique fixed point in $\mathbb{B}_a(\bar{x})$. Using the equality

$$\bar{x} = \varphi(p, s(p); -f(\bar{p}, \bar{x}) + f(p, s(p)) + \nabla_x f(p, s(p))(\bar{x} - s(p))),$$

(19), (20), the Lipschitz continuity of $\varphi(p, s(p); \cdot)$ in $\mathbb{B}_b(0)$ with constant κ, (15) and (17), we have

$$\begin{aligned}
&\|\bar{x} - \Phi_0(\bar{x})\| \\
&= \|\varphi(p, s(p); -f(\bar{p}, \bar{x}) + f(p, s(p)) + \nabla_x f(p, s(p))(\bar{x} - s(p))) - \varphi(p, s(p); g(p, u; \bar{x}))\| \\
&\le \kappa\| - f(\bar{p}, \bar{x}) + f(p, s(p)) + \nabla_x f(p, s(p))(\bar{x} - s(p)) \\
&\quad + f(p, u) + A(\bar{x} - u) - f(p, s(p)) - \nabla_x f(p, s(p))(\bar{x} - s(p))\| \\
&\le \kappa\|f(p, u) - f(\bar{p}, \bar{x}) + A(\bar{x} - u)\| \\
&\le \kappa\|f(p, \bar{x}) - f(\bar{p}, \bar{x})\| + \kappa\|f(p, \bar{x}) - f(p, u) - A(\bar{x} - u)\| \\
&\le \kappa\delta + \kappa\frac{1}{2}\mu\|\bar{x} - u\|^2 + \kappa\mu\|u - \bar{x}\|\|\bar{x} - u\| + \kappa\omega\|\bar{x} - u\| \\
&\le \kappa\delta + \frac{3}{2}\kappa\mu a^2 + \kappa\omega a.
\end{aligned}$$

Now using the second inequality in (14), we come to

$$\|\bar{x} - \Phi_0(\bar{x})\| \le a(1 - \kappa\varepsilon). \tag{21}$$

Further, for any $v', v'' \in \mathbb{B}_a(\bar{x})$, by (12), (15), (18) and the Lipschitz continuity of $\varphi(p, s(p); \cdot)$, we obtain

$$\begin{aligned}
&\|\Phi_0(v') - \Phi_0(v'')\| \\
&\quad = \|\varphi(p, s(p); g(p, u; v')) - \varphi(p, s(p); g(p, u; v''))\| \\
&\quad \le \kappa\|g(p, u; v') - g(p, u; v'')\| \\
&\quad = \kappa\|\nabla_x f(p, s(p))(v' - v'') - A(v' - v'')\| \\
&\quad \le \kappa\|\nabla_x f(p, s(p)) - A\|\|v' - v''\| \\
&\quad \le \kappa\big(\|\nabla_x f(p, s(p)) - \nabla_x f(p, \bar{x})\| + \|\nabla_x f(p, \bar{x}) - A\|\big)\|v' - v''\| \\
&\quad \le \kappa(\mu\|s(p) - \bar{x}\| + \omega)\|v' - v''\| \le \left(\frac{1}{2}\kappa\mu a + \kappa\omega\right)\|v' - v''\|.
\end{aligned}$$

By the second inequality in (13)

$$\|\Phi_0(v') - \Phi_0(v'')\| \le \kappa\varepsilon\|v' - v''\|. \tag{22}$$

Now utilizing (21) and (22) to the contraction mapping principle, we conclude that there exists a fixed point $x_1 \in \Phi_0(x_1) \cap \mathbb{B}_a(\bar{x})$ which translates to $g(p, u; x_1) \in G_{p,s(p)}(x_1)$ or, equivalently,

$$0 \in f(p, u) + A(x_1 - u) + F(x_1)$$

meaning that x_1 is obtained by iteration (3) from u for p, and there is no more than just one such iterate in $\mathbb{B}_a(\bar{x})$.

Now we will show that x_1 satisfies the estimate

$$\|x_1 - s(p)\| \le \gamma \|u - s(p)\|.$$

Using (11), we can take a smaller if necessary so that

$$\frac{\frac{7}{4}\kappa\mu a + \kappa\omega}{1 - \kappa\varepsilon} \le \gamma < \frac{1}{3}. \tag{23}$$

Let $w_0 = \gamma\|u - s(p)\|$. Then $w_0 > 0$ and $w_0 \le \frac{1}{3}(a + \frac{a}{2}) = \frac{a}{2}$, hence $\mathbb{B}_{w_0}(s(p)) \subset \mathbb{B}_a(\bar{x})$. We apply again the contraction mapping principle to the mapping Φ_0 but now on $\mathbb{B}_{w_0}(s(p))$. Noting that $s(p) = \varphi(p, s(p); 0)$ and using (17) and (19), we have

$$\begin{aligned}
&\|s(p) - \Phi_0(s(p))\| \\
&= \|\varphi(p, s(p); 0) - \varphi(p, s(p); g(p, u; s(p)))\| \\
&= \kappa \|f(p, s(p)) - f(p, u) - A(s(p) - u)\| \\
&\le \frac{1}{2}\kappa\mu\|s(p) - u\|^2 + \kappa\mu\|u - \bar{x}\|\|s(p) - u\| + \kappa\|\nabla_x f(p, \bar{x}) - A\|\|s(p) - u\| \\
&\le \left(\frac{7}{4}\kappa\mu a + \kappa\omega\right)\|s(p) - u\|.
\end{aligned}$$

Then from (23), we come to

$$\|s(p) - \Phi_0(s(p))\| \le \gamma\|s(p) - u\|(1 - \kappa\varepsilon) = w_0(1 - \kappa\varepsilon). \tag{24}$$

Since $\mathbb{B}_{w_0}(s(p)) \subset \mathbb{B}_a(\bar{x})$, from (22) we obtain

$$\|\Phi_0(v') - \Phi_0(v'')\| \le \kappa\varepsilon\|v' - v''\|, \qquad \text{for any} \quad v', v'' \in \mathbb{B}_{w_0}(s(p)). \tag{25}$$

Hence the contraction mapping principle applied to the function Φ_0 on the ball $\mathbb{B}_{w_0}(s(p))$ yields that there exists x_1' in this ball such that $x_1' = \Phi_0(x_1')$. But the fixed point x_1' of Φ_0 in $\mathbb{B}_{w_0}(s(p))$ must then coincide with the unique fixed point x_1 of Φ_0 in the larger set $\mathbb{B}_a(\bar{x})$. Hence, the fixed point x_1' of Φ_0 on $\mathbb{B}_a(\bar{x})$ satisfies

$$\|x_1 - s(p)\| \le \gamma\|u - s(p)\|$$

which for $x_0 = u$ means that (10) holds for $k = 0$.

We now proceed by induction. If the claim holds for $k = 1, 2, \ldots, n$, we define $\Phi_n \colon x \mapsto \varphi(p, s(p); g(p, x_n; x))$ and replace u by x_n we will obtain again (21) and (22). In this way we have that the function Φ_n has a unique fixed point x_{n+1} in $\mathbb{B}_a(\bar{x})$. This means that $g(p, x_n; x_{n+1}) \in G_{p,x_n}(x_{n+1})$ and hence x_{n+1} is the unique iterate from x_n for p which is in $\mathbb{B}_a(\bar{x})$. Next by employing again the contraction mapping principle as in (24) and (25) to Φ_n but now on the ball $\mathbb{B}_{w_n}(s(p))$ for $w_n = \gamma\|x_n - s(p)\|$ we obtain that x_{n+1} is at distance w_n from $s(p)$.

Now we may conclude that for any starting point $u \in \mathbb{B}_a(\bar{x})$ and any $p \in \mathbb{B}_c(\bar{p})$, there is a unique sequence $\xi(u,p)$ starting from u and generated by chords method (3) for p whose components are contained in $\mathbb{B}_a(\bar{x})$. □

Next we will proof that the single valued localization ξ of the mapping Ξ at $(\bar{p},\bar{x})$ for $\bar{\xi}$ is Lipschitz continuous near $(\bar{p},\bar{x})$.

Theorem 3.2 (Implicit function theorem)**.** *In addition to the assumptions of* Theorem 3.1 *suppose that $A \in L(X,Y)$ is such that*

$$28 \cdot \operatorname{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\| < 1.$$

Then the single valued localization ξ of the mapping Ξ at $(\bar{p},\bar{x})$ for $\bar{\xi}$ is Lipschitz continuous near $(\bar{p},\bar{x})$, moreover having

$$\widehat{\operatorname{lip}}_u(\xi;(\bar{x},\bar{p})) = 0 \qquad \text{and} \qquad \widehat{\operatorname{lip}}_p(\xi;(\bar{x},\bar{p})) \le 2\operatorname{lip}(\sigma;0)\cdot\widehat{\operatorname{lip}}_p(f;(\bar{p},\bar{x})). \quad (26)$$

P r o o f. Let $A \in L(X,Y)$ be such that $28.\operatorname{lip}(\sigma;0)\|\nabla_x f(\bar{p},\bar{x}) - A\| < 1$ and $\omega := 2\|\nabla_x f(\bar{p},\bar{x}) - A\|$. Let $\kappa > 2\operatorname{lip}(\sigma;0)$ be such that $7\kappa\omega < 1$.

Along with the mapping G in (6) consider the parametrized mapping

$$x \mapsto H_{p,w}(x) = f(p,w) + A(x-w) + F(x).$$

For the function

$$x \mapsto r(p,w;x) = A(x-w) - \nabla_x f(\bar{p},\bar{x})(x-\bar{x}) + f(p,w) - f(\bar{p},\bar{x})$$

we have $r(\bar{p},\bar{x};\bar{x}) = 0$ and

$$\begin{aligned}\|r(p,w;x') - r(p,w;x'')\| &= \|A(x'-x'') - \nabla_x f(\bar{p},\bar{x})(x'-x'')\| \\ &\le \|A - \nabla_x f(\bar{p},\bar{x})\|\|x'-x''\| \le \frac{\omega}{2}\|x'-x''\|.\end{aligned}$$

This means that the function $r(p,w;x)$ is Lipschitz continuous with Lipschitz constant $\frac{\omega}{2}$. Note that

$$H_{p,w}(x) = r(p,w;x) + G(x).$$

Now let $\kappa' = \frac{\kappa}{2} > \operatorname{lip}(\sigma;0)$. As we have already mentioned in the proof of Theorem 3.1 the assumed strong regularity of the mapping G in (6) at $\bar{x}$ for 0 and the choice of κ' mean that there exist positive constants α' and b' such that the mapping $y \mapsto \sigma(y) = G^{-1}(y) \cap \mathbb{B}_{\alpha'}(\bar{x})$ is Lipschitz continuous function on $\mathbb{B}_{b'}(0)$ with Lipschitz constant κ'. It is easy to see that

$$\frac{2\kappa'}{2-\omega\kappa'} < \kappa.$$

Applying Corollary 2.4 to the mapping $H_{p,w}$ we obtain that there are positive constants $\alpha \le \alpha'$, $b \le b'$ and c such that for $p \in \mathbb{B}_c(\bar{p})$ and $w \in \mathbb{B}_\alpha(\bar{x})$, the mapping $y \mapsto H_{p,w}^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is Lipschitz continuous function on $\mathbb{B}_b(0)$ with Lipschitz constant κ. We denote this function by $\varphi(p, w; \cdot)$.

Then from the proof of Theorem 3.1 the truncation $S(p) \cap \mathbb{B}_\alpha(\bar{x})$ of the solution mapping in (4) is Lipschitz continuous function on $\mathbb{B}_c(\bar{p})$ and its values are in $\mathbb{B}_{a/2}(\bar{x})$.

For any positive $a' \le a$, by adjusting the size of the constant c and take a starting point $u \in \mathbb{B}_{a'}(\bar{x})$ so we can arrange that, for any $p \in \mathbb{B}_c(\bar{p})$ all elements x_k of the sequence $\xi(u, v)$ are in $\mathbb{B}_{a'}(\bar{x})$. Indeed, by taking $\delta > 0$ to satisfy(14) with a replaced by a' and then choosing c so that (15) holds for the new δ and for a' we are sure that all requirements for a will hold for a' as well and hence all chords iterates x_k will be at distance a' from $\bar{x}$.

Let

$$\nu > \widehat{\mathrm{lip}}_p(f; (\bar{p}, \bar{x})).$$

Choose a positive $d \le a/2$ and make c smaller if necessary, so that for every $p, p' \in \mathbb{B}_c(\bar{p})$ and $w \in \mathbb{B}_d(\bar{x})$ we have

$$\|f(p, w) - f(p', w)\| \le \nu \|p - p'\|. \tag{27}$$

Using (16), (17) and (15) we obtain that for every $x \in \mathbb{B}_d(\bar{x})$, every $p, p' \in \mathbb{B}_c(\bar{p})$ and every $u, u' \in \mathbb{B}_d(\bar{x})$, we have

$$\begin{aligned}
&\|f(p', u') + A(x - u') - f(p, u) - \nabla_x f(p, u)(x - u)\| \\
&\le \|f(p', u') - f(p', x) + A(x - u')\| + \|f(p', x) - f(p, x)\| \\
&\quad + \|f(p, x) - f(p, u) - \nabla_x f(p, u)(x - u)\| \\
&\le \frac{1}{2}\mu \|x - u'\|^2 + \mu \|x - \bar{x}\| \|x - u'\| + \omega \|x - u'\| + \delta + \frac{1}{2}\mu \|x - u\|^2 \\
&\le \frac{3}{2}\mu a^2 + a\omega + \delta.
\end{aligned}$$

Using the first inequality in (14), we come to

$$\|f(p', u') + A(x - u') - f(p, u) - \nabla_x f(p, u)(x - u)\| \le b. \tag{28}$$

Choose $\tau > \omega$ such that $\kappa\tau < \frac{1}{7}$. Make d smaller if necessary so that

$$2\mu d + \omega \le \tau. \tag{29}$$

Since

$$\frac{\kappa\tau}{1 - \kappa\tau} < \frac{1}{6}$$

we can take c smaller in order to have

$$\frac{\kappa\tau(2d) + 2(\kappa\nu(2c) + 2\kappa d\tau)}{1 - \kappa\tau} \le d. \tag{30}$$

We have already proved that there is a unique sequence $\xi(u, p) = (x_1, \ldots, x_k, \ldots)$ for $p, p' \in \mathbb{B}_c(\bar{p}),\ \ u, u' \in \mathbb{B}_d(\bar{x}),\ \ (p, u) \ne (p', u')$, generated by chords iteration

(3) starting from u whose components are all in $\mathbb{B}_d(\bar{x})$ and hence in $\mathbb{B}_{a/2}(\bar{x})$. For this sequence, taking $x_0 = u$, we have that for all $k \geq 0$

$$x_{k+1} = \varphi(p, x_k; 0) := (f(p, x_k) + A(\cdot - x_k) + F(\cdot))^{-1}(0) \cap \mathbb{B}_\alpha(\bar{x}). \tag{31}$$

Let

$$\gamma_0 = \frac{\kappa\tau\|u - u'\| + \kappa\nu\|p - p'\| + 2\kappa d\tau}{1 - \kappa\tau}.$$

By (30) we get that $\gamma_0 \leq d$ and then $\mathbb{B}_{\gamma_0}(x_1) \subset \mathbb{B}_a(\bar{x})$. Consider the function

$$\Psi_0 \colon x \mapsto \varphi(p, u; -f(p', u') - A(x - u') + f(p, u) + \nabla_x f(p, u)(x - u)).$$

Applying (28), the Lipschitz continuity of $\varphi(p, u; \cdot)$ on $\mathbb{B}_b(0)$, (17), (29) and (27), we obtain

$$\begin{aligned}
&\|x_1 - \Psi_0(x_1)\| \\
&= \|\varphi(p, u; 0) - \varphi(p, u; -f(p', u') - A(x_1 - u') + f(p, u) + \nabla_x f(p, u)(x_1 - u))\| \\
&\leq \kappa\|f(p, u) - f(p', u') - A(u - u') - A(x_1 - u) + \nabla_x f(p, u)(x_1 - u)\| \\
&\leq \kappa\|f(p, u) - f(p, u') - A(u - u')\| + \kappa\|f(p, u') - f(p', u')\| \\
&\quad + \kappa\|\nabla_x f(p, u)(x_1 - u) - A(x_1 - u)\| \\
&\leq \kappa\frac{1}{2}\mu\|u - u'\|^2 + \kappa\mu\|u' - \bar{x}\|\|u - u'\| + \kappa\omega\|u - u'\| \\
&\quad + \kappa\nu\|p - p'\| + \kappa\|x_1 - u\|\|\nabla_x f(p, u) - A\| \\
&\leq \|u - u'\|\left(\frac{1}{2}\kappa\mu 2d + \kappa\mu d + \kappa\omega\right) + \kappa\nu\|p - p'\| \\
&\quad + 2\kappa d\big(\|\nabla_x f(p, \bar{x}) - A\| + \|\nabla_x f(p, u) - \nabla_x f(p, \bar{x})\|\big) \\
&\leq (2\kappa\mu d + \kappa\omega)\|u - u'\| + \kappa\nu\|p - p'\| + 2\kappa d(\omega + \mu d) \\
&\leq \kappa\tau\|u - u'\| + \kappa\nu\|p - p'\| + 2\kappa d\tau.
\end{aligned}$$

Now using the assumption for γ_0 we come to

$$\|x_1 - \Psi_0(x_1)\| \leq \gamma_0(1 - \kappa\tau). \tag{32}$$

For $v, v' \in \mathbb{B}_{\gamma_0}(x_1)$, using (12) we have

$$\begin{aligned}
\|\Psi_0(v) - \Psi_0(v')\| &\leq \kappa\|\nabla_x f(p, u)(v - v') - A(v - v')\| \leq \kappa\|\nabla_x f(p, u) - A\|\|v - v'\| \\
&\leq \kappa\big(\|\nabla_x f(p, u) - \nabla_x f(p, \bar{x})\| + \|\nabla_x f(p, \bar{x}) - A\|\big)\|v - v'\| \\
&\leq \kappa(\mu d + \omega)\|v - v'\|.
\end{aligned}$$

Using (29) we come to

$$\|\Psi_0(v) - \Psi_0(v')\| \leq \kappa\tau\|v - v'\|. \tag{33}$$

Hence, by (32), (33) and the contraction mapping principle, there is a unique x_1' in $\mathbb{B}_{\gamma_0}(x_1)$ such that

$$x_1' = \varphi(p, u; -f(p', u') - A(x_1' - u') + f(p, u) + \nabla_x f(p, u)(x_1' - u))$$

which means

$$f(p', u') - A(x_1' - u') + F(x_1') \ni 0,$$

that is, x_1' is the unique chords iterate from u' for p' satisfying

$$\|x_1' - x_1\| \le \gamma_0.$$

Since $\gamma_0 \le d$, we obtain that $x_1' \in \mathbb{B}_a(\bar{x})$ and then x_1' is the unique chords iteration from u' for p' which is in $\mathbb{B}_a(\bar{x})$.

By induction, we construct a sequence $\xi' = \{x_1', \dots, x_k', \dots\} \in \Xi(p', u')$ such that the distance from x_k' to the corresponding component x_k of ξ satisfies for $k = 2, 3, \dots$ the estimate

$$\|x_k' - x_k\| \le \gamma_k := \frac{\kappa\tau\|x_{k-1}' - x_{k-1}\| + \kappa\nu\|p - p'\| + 2\kappa d\tau}{1 - \kappa\tau}. \tag{34}$$

Suppose that for some $n > 1$ we have found $x_2', x_3', \dots x_n'$ with this property. First observe that

$$\gamma_k \le \left[\frac{\kappa\tau}{1 - \kappa\tau}\right]^{k+1} \|u - u'\| + \frac{\kappa\nu\|p - p'\| + 2\kappa d\tau}{1 - \kappa\tau} \sum_{i=0}^{k} \left(\frac{\kappa\tau}{1 - \kappa\tau}\right)^i$$

from which we get the estimate that, for all $k = 0, 2, \dots, n - 1$,

$$\gamma_k \le \frac{\kappa\tau}{1 - \kappa\tau}\|u - u'\| + \frac{\kappa\nu\|p - p'\| + 2\kappa d\tau}{1 - 2\kappa\tau}. \tag{35}$$

We obtain through (30) that $\gamma_k \le d$ for all k and consequently $x_k' \in \mathbb{B}_d(x_k) \subset \mathbb{B}_a(\bar{x})$.

To show that x_{n+1}' is chords iterate from x_n' for p', we proceed in the same way as in obtaining x_1' from u' for p'. Consider the function

$$\Psi_k \colon x \mapsto \varphi(p, x_k; -f(p', x_k') - A(x - x_k') + f(p, x_k) + \nabla_x f(p, x_k)(x - x_k)).$$

By replacing Ψ_0 by Ψ_k, u by x_k, u' by x_k', and x_1 by x_{k+1} in (31) and (32), we get

$$\|x_{k+1} - \Psi_k(x_{k+1})\| \le \kappa\tau\|u - u'\| + \kappa\nu\|p - p'\| + 2\kappa d\tau \le \gamma_k(1 - \kappa\tau)$$

and

$$\|\Psi_k(v) - \Psi_k(v')\| \le \kappa\tau\|v - v'\| \qquad \text{for any} \quad v, v' \in \mathbb{B}_{\gamma_k}(x_{k+1}).$$

Then, by the contraction mapping principle there is a unique fixed point x_{k+1}' in $\mathbb{B}_{\gamma_k}(x_{k+1})$ with $x_{k+1}' = \Psi_k(x_{k+1}')$, which gives us

$$f(p', x_k') - A(x_{k+1}' - x_k') + F(x_{k+1}') \ni 0.$$

Moreover, since $\gamma_k \le d$, we have that $x_{k+1}' \in \mathbb{B}_a(\bar{x})$.

We have constructed a sequence $x'_1, x'_2, \dots, x'_k, \dots$, generated by chords iteration for p, starting from u', and whose components are in $\mathbb{B}_a(\bar{x})$. This sequence is the value $\xi(u', p')$ of the single-valued localization ξ whose value $\xi(u, p)$ is the sequence $x_1, x_2, \dots, x_k, \dots$. Taking into account (34) and (35), we come to the estimate

$$\|\xi(u,p) - \xi(u',p')\|_\infty \le O(\tau)\|u - u'\| + \kappa\nu\|p - p'\| + O(\tau).$$

Since τ can be chosen arbitrarily small, this yields (26). □

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