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q-SZÁSZ OPERATORS WHICH PRESERVE x^2

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ABSTRACT. In the present paper we construct q-Szász operators that preserve the third test function e_2. Rate of global convergence is obtained in the frame of weighted spaces. Furthermore, we obtain a Voronovskaja type theorem for these operators.

1. Introduction

J. P. King [1] has presented an example of linear and positive operators $V_n\colon \mathbb{C}\,[0,1] \to \mathbb{C}\,[0,1]$ given as follows

$$V_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(r_n^*(x))^k\,(1-r_n^*(x))^{n-k}, \qquad 0 \le x \le 1,$$

where $r_n^*(x)\colon\ [0,1] \to [0,1]$ are defined by

$$r_n^*(x) = \begin{cases} x^2, & n = 1, \\ -\dfrac{1}{2\,(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4\,(n-1)^2}}, & n = 2, 3, \dots \end{cases}$$

This sequence preserves the test functions e_0, e_2 and $V_n\,(e_1;x) = r_n^*(x)$ holds. Replacing r_n^* by e_1, we recover Bernstein polynomials. Further results regarding V_n operator have been recently obtained by H. Gonska and P. Piţul [2]. Approximation results on the Szász-Mirakjan operators preserving e_2 have been investigated by O. Duman and M. A. Özarslan [3]. In [4] O. Agratini indicated a general technique to construct sequences of univariate operators of discrete

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type with the same property as in King's example, i.e., their degree of exactness is null, but they reproduce the third function of the celebrated criterion. This way gives the modified variants of Szász, Baskakov and Bernstein-Chlodovsky operators. L. Rempulska and K. Tomczak [5] studied certain operators preserving e_2 that include the Stancu operator. On the other hand, recently King type q-Bernstein operators were studied by N. I. Mahmudov [6] and King type q-Szász operators by O. Agratini and O. Doğru [7] and N. I. Mahmudov [8]. Notice that different type q-Szász-Mirakjan operators were studied in [11], [9], [10],

The goal of this article is to construct and investigate a variant of q-Szász-Mirakjan operators in the case $q > 1$, studied in [10], which preserve the functions e_0 and e_2. Constructed operator has a better rate of convergence than the classical King type Szász-Mirakjan operators studied in [3].

The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce King type q-Szász operators and evaluate their moments. In Section 3, we study convergence properties of our operators for functions with polynomial growth. The main tool is a weighted modulus of smoothness. Furthemore we give a Voronovskaja-type asymptotic formula.

2. Construction and estimation of moments

Throughout the paper we employ the standard notations of q-calculus.

q-integer and q-factorial are defined by

$$[n]_q := \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \in \mathbb{R}^+ \setminus \{1\}, \\ n & \text{if } q = 1 \end{cases} \quad \text{for } n \in \mathbb{N} \qquad \text{and} \qquad [0]_q = 0,$$

$$[n]_q! := [1]_q[2]_q \dots [n]_q \quad \text{for } n \in \mathbb{N} \qquad \text{and} \qquad [0]_q! = 1.$$

For integers $0 \le k \le n$ q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

If $|q| > 1$, or $0 < |q| < 1$ and $|z| < \frac{1}{1-q}$, the q-exponential function $e_q(x)$ was defined by Jackson, see [16],

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}. \tag{1}$$

If $|q| > 1$, $e_q(z)$ is an entire function and

$$e_q(z) = \prod_{j=0}^{\infty} \left(1 + (q-1)\frac{z}{q^{j+1}}\right), \qquad |q| > 1. \tag{2}$$

There is another q-exponential function which is entire when $0 < |q| < 1$ and which converges when $|z| < \frac{1}{|1-q|}$ if $|q| > 1$. To obtain it we must invert the base in (1), i.e. $q \to \frac{1}{q}$:

$$E_q(z) := \mathrm{e}_{1/q}(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{[k]_q!}.$$

We immediately obtain from (2) that

$$E_q(z) = \prod_{j=0}^{\infty} \left(1 + (1-q)\, zq^j\right), \qquad 0 < |q| < 1.$$

We set

$$s_{nk}(q; x) = \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k\, x^k}{[k]_q!} e_q\left(-[n]_q q^{-k} x\right), \qquad n = 1, 2, \ldots \tag{3}$$

Let $B_m[0,\infty)$ be the set of all functions f satisfying the condition that $|f(x)| \le M_f(1 + x^m)$, $x \in [0,\infty)$, $m > 0$ with some constant M_f depending on f. Let us introduce

$$C_m[0,\infty) := B_m[0,\infty) \cap C[0,\infty),$$

$$C_m^*[0,\infty) := \left\{ f \in C_m[0,\infty) :\ \exists \lim_{x\to\infty} \tfrac{|f(x)|}{1+x^m} < \infty \right\},$$

$$C_B[0,\infty) := \left\{ f \in C[0,\infty) : f \text{ is bounded on } [0,\infty) \right\}.$$

These spaces are endowed with the norm

$$\|f\|_m := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1 + x^m}.$$

In order to introduce a King type q-Szász-Mirakjan operators, we present a construction due to N. I. Mahmudov [10].

Definition 1. Let $q > 1$ and $n \in \mathbb{N}$. For $f\colon [0,\infty) \to \mathbb{R}$ we define the Szász-Mirakjan operator based on the q-integers

$$M_{n,q}(f; x) := \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n]_q}\right) \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left(-[n]_q\, q^{-k} x\right). \tag{4}$$

Moments $M_{n,q}(e_m;x)$ are of particular importance in the theory of approximation by positive operators. From (4) we easily derive the following recurrence formula and explicit formulas for moments $M_{n,q}(e_m;x)$, $m=0,1,2,3,4$.

Lemma 2. ([10]) *Let $q>1$. The following recurrence formula holds*

$$M_{n,q}(e_{m+1};x)=x\sum_{j=0}^{m}\binom{m}{j}\frac{q^j}{[n]_q^{m-j}}M_{n,q}\left(e_j;q^{-1}x\right). \tag{5}$$

Lemma 3. ([10]) *Let $q>1$. We have*

$$M_{n,q}(e_0;x)=1,\qquad M_{n,q}(e_1;x)=x,\qquad M_{n,q}(e_2;x)=x^2+\frac{x}{[n]_q},$$
$$M_{n,q}(e_3;x)=x^3+\frac{2+q}{[n]_q}x^2+\frac{1}{[n]_q^2}x,$$
$$M_{n,q}(e_4;x)=x^4+\left(3+2q+q^2\right)\frac{x^3}{[n]_q}+\left(3+3q+q^2\right)\frac{x^2}{[n]_q^2}+\frac{1}{[n]_q^3}x.$$

Now we give an explicit formula for the moments $M_{n,q}(e_m;x)$, as a q-analogue of a result of Becker, see [13: Lemma 3].

Lemma 4. ([10]) *For $q>1$, $m\in\mathbb{N}$ there holds*

$$M_{n,q}(e_m;x)=\sum_{j=1}^{m}\mathbb{S}_q(m,j)\frac{x^j}{[n]_q^{m-j}}, \tag{6}$$

where

$$\begin{aligned}&\mathbb{S}_q(m+1,j)=[j]_q\,\mathbb{S}_q(m,j)+\mathbb{S}_q(m,j-1), \qquad m\geq 0,\quad j\geq 1,\\ &\mathbb{S}_q(0,0)=1,\qquad \mathbb{S}_q(m,0)=0,\quad m>0,\qquad \mathbb{S}_q(m,j)=0,\quad m<j.\end{aligned} \tag{7}$$

In particular $M_{n,q}(t^m;x)$ is a polynomial of degree m without a constant term.

We transform the operators defined at (4) in order to preserve the quadratic function e_2. Defining the functions

$$v_{n,q}(x)=\frac{-1+\sqrt{1+4[n]_q^2x^2}}{2[n]_q},\qquad x\geq 0, \tag{8}$$

we consider the linear and positive operators

$$\begin{aligned}M_{n,q}^{*}(f;x)&:=M_{n,q}\left(f;v_{n,q}(x)\right)\\ &=\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{[n]_q}\right)\frac{1}{q^{k(k-1)/2}}\frac{\left([n]_qv_{n,q}(x)\right)^k}{[k]_q!}e_q\left(-[n]_qq^{-k}v_{n,q}(x)\right).\end{aligned} \tag{9}$$

Lemma 5. *The operators defined at* (9) *verify the following identities*

$$M_{n,q}^{*}\left(e_0;x\right)=1,\qquad M_{n,q}^{*}\left(e_1;x\right)=v_{n,q}(x),\qquad M_{n,q}^{*}\left(e_2;x\right)=x^2,$$

$$M_{n,q}^{*}\left(e_3;x\right)=v_{n,q}^{3}(x)+\frac{2+q}{[n]_q}v_{n,q}^{2}(x)+\frac{1}{[n]_q^2}v_{n,q}(x)$$

$$M_{n,q}^{*}\left(e_4;x\right)=v_{n,q}^{4}(x)+\frac{3+2q+q^2}{[n]_q}v_{n,q}^{3}(x)+\frac{3+3q+q^2}{[n]_q^2}v_{n,q}^{2}(x)+\frac{1}{[n]_q^3}v_{n,q}(x).$$

P r o o f. The proof is based on the formulas of Lemma 3. □

Lemma 6. *Let* $v_{n,q}$, $n\in\mathbb{N}$ *be defined by* (8), *where* $q>1$. *The following statements are true.*

(i) $v_{n,q}\left(0\right)=0$;

(ii) $0\le v_{n,q}(x)\le x$;

(iii) $x-v_{n,q}(x)$ *is strictly increasing in* x *and*

$$x-v_{n,q}(x)\le\frac{1}{2[n]_q},$$

$$M_{n,q}^{*}\big((e_1-xe_0)^2;x\big)\le\frac{x}{[n]_q}.$$

P r o o f. For proving (iii) we can consider the function $h\colon[0,\infty)\to\mathbb{R}$, $h(x)=x-v_{n,q}(x)$. It is clear that h is strictly increasing and

$$\begin{aligned}0\le h(x)&=\frac{1}{2[n]_q}\left(2[n]_qx+1-\sqrt{1+4[n]_q^2x^2}\right)\\&=\frac{1}{2[n]_q}\frac{4[n]_qx}{2[n]_qx+1+\sqrt{1+4[n]_q^2x^2}}\\&\le\frac{2x}{2[n]_qx+\sqrt{4[n]_q^2x^2}}=\frac{1}{2[n]_q}.\end{aligned}$$

It follows that

$$M_{n,q}^{*}\big((e_1-xe_0)^2;x\big)=2x\left(x-v_{n,q}(x)\right)\le\frac{x}{[n]_q}.$$ □

Lemma 7. *Assume that* $q>1$. *For every* $x\in[0,\infty)$ *there hold*

$$M_{n,q}^{*}\left(e_1-xe_0;x\right)=v_{n,q}(x)-x,\tag{10}$$

$$M_{n,q}^{*}\big((e_1-xe_0)^2;x\big)=2x\left(x-v_{n,q}(x)\right),\tag{11}$$

$$M_{n,q}^{*}\left((e_1-xe_0)^4;x\right)=\frac{1}{[n]_q^3}v_{n,q}(x)+\frac{1}{[n]_q^2}\left(\left(3+3q+q^2\right)v_{n,q}^2(x)-4xv_{n,q}(x)\right)$$
$$+\frac{1}{[n]_q}\left(\left(3+2q+q^2\right)v_{n,q}^3(x)-4\left(2+q\right)xv_{n,q}^2(x)\right)$$
$$+\frac{1}{[n]_q}\left(6x^2v_{n,q}(x)\right)+\left(x-v_{n,q}(x)\right)^4. \tag{12}$$

P r o o f. Let us prove the explicit formula (12) for $M_{n,q}^{*}\left((e_1-xe_0)^4;x\right)$.

$$\begin{aligned}
&M_{n,q}^{*}\left((e_1-xe_0)^4;x\right)\\
&\quad=M_{n,q}^{*}\left(e_4;x\right)-4xM_{n,q}^{*}\left(e_3;x\right)+6x^2M_{n,q}^{*}\left(e_2;x\right)-4x^3M_{n,q}^{*}\left(e_1;x\right)+x^4\\
&\quad=v_{n,q}^4(x)+\frac{3+2q+q^2}{[n]_q}v_{n,q}^3(x)+\frac{3+3q+q^2}{[n]_q^2}v_{n,q}^2(x)+\frac{1}{[n]_q^3}v_{n,q}(x)\\
&\qquad-4x\left(v_{n,q}^3(x)+\frac{2+q}{[n]_q}v_{n,q}^2(x)+\frac{1}{[n]_q^2}v_{n,q}(x)\right)\\
&\qquad+6x^2\left(v_{n,q}^2(x)+\frac{1}{[n]_q}v_{n,q}(x)\right)-4x^3v_{n,q}(x)+x^4\\
&\quad=\frac{1}{[n]_q^3}v_{n,q}(x)+\frac{1}{[n]_q^2}\left(\left(3+3q+q^2\right)v_{n,q}^2(x)-4xv_{n,q}(x)\right)\\
&\qquad+\frac{1}{[n]_q}\left(\left(3+2q+q^2\right)v_{n,q}^3(x)-4\left(2+q\right)xv_{n,q}^2(x)+6x^2v_{n,q}(x)\right)\\
&\qquad+\left(v_{n,q}^4(x)-4xv_{n,q}^3(x)+6x^2v_{n,q}^2(x)-4x^3v_{n,q}(x)+x^4\right).
\end{aligned}$$
□

Corollary 8. *Assume that $q_n>1$, $q_n\to1$ as $n\to\infty$. We have*

$$\lim_{n\to\infty}[n]_{q_n}^2M_{n,q_n}^{*}\left((e_1-xe_0)^4;x\right)=3x^2.$$

P r o o f. The proof is based on the limit $\lim\limits_{n\to\infty}2[n]_{q_n}\left(x-v_{n,q_n}(x)\right)=1$ and on the formula (12). If $A_{n,q_n}(x)$, $B_{n,q_n}(x)$, $C_{n,q_n}(x)$, $D_{n,q_n}(x)$ first, second, third and fourth terms of the formula (12), respectively, then

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}^2M_{n,q_n}^{*}\left((e_1-xe_0)^4;x\right)\\
&\quad=\lim_{n\to\infty}[n]_{q_n}^2\left(A_{n,q_n}(x)+B_{n,q_n}(x)+C_{n,q_n}(x)+D_{n,q_n}(x)\right)\\
&\quad=0+3x^2+0+0=3x^2.
\end{aligned}$$
□

Lemma 9. *Let $m\in\mathbb{N}\cup\{0\}$ and $q>1$ be fixed. Then there exists a positive constant $K_m\left(q\right)$ such that*

$$\left\|M_{n,q}^{*}\left(1+e_m;x\right)\right\|_m\le K_m\left(q\right),\qquad n\in\mathbb{N}. \tag{13}$$

Moreover for every $f \in C_m^*[0,\infty)$ we have

$$\left\|M_{n,q}^*(f)\right\|_m \leq K_m(q)\left\|f\right\|_m, \qquad n \in \mathbb{N}. \tag{14}$$

Thus $M_{n,q}^*$ is a linear positive operator from $C_m^*[0,\infty)$ into $C_m^*[0,\infty)$ for any $m \in N \cup \{0\}$.

P r o o f. The inequality (13) is obvious for $m = 0$. Let $m \geq 1$. Then by (6) and Lemma 4 we have

$$\begin{aligned}\frac{1}{1+x^m} M_{n,q}^*(e_0 + e_m; x) &= \frac{1}{1+x^m} + \frac{1}{1+x^m}\sum_{j=1}^{m} \mathbb{S}_q(m,j)\,\frac{v_{n,q}^j(x)}{[n]^{m-j}} \\ &\leq 1 + \sum_{j=1}^{m} \mathbb{S}_q(m,j)\,\frac{x^j}{1+x^m} \\ &\leq 1 + \sum_{j=1}^{m} \mathbb{S}_q(m,j) = K_m(q),\end{aligned}$$

$K_m(q)$ is a positive constant depending on m and q. From this follows (13). On the other hand

$$\left\|M_{n,q}^*(f)\right\|_m \leq \left\|f\right\|_m \left\|M_{n,q}^*(1+e_m)\right\|_m$$

for every $f \in C_m^*[0,\infty)$. By applying (13), we obtain (14). □

3. Better error estimation

In this section we compute the rate of convergence of the operators $M_{n,q}^*$. Then, we show that the operator $M_{n,q}^*$ has better error estimation than that of the King type Szász-Mirakjan operators.

Theorem 10. *For every $f \in C_B[0,\infty)$, $n \in \mathbb{N}$ and $q > 1$, we have*

$$\left|M_{n,q}^*(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{2x\left(x - v_{n,q}(x)\right)}\right).$$

P r o o f. The proof is standard one, therefore we omit it. □

It is known that

$$\left|D_n^*(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{2x\left(x - v_n(x)\right)}\right),$$

where D_n^* is the King type Szász-Mirakjan defined in [3] and

$$v_n(x) = \frac{-1+\sqrt{1+4n^2x^2}}{2n}, \qquad x \geq 0.$$

It is clear that

$$x - v_{n,q}(x) = \frac{2x}{2[n]_q x + 1 + \sqrt{1 + 4[n]_q^2 x^2}}$$
$$\leq \frac{2x}{2nx + 1 + \sqrt{1 + 4n^2x^2}} = x - v_n(x).$$

This guarantees that $2x\left(x - v_{n,q}(x)\right) \leq 2x\left(x - v_n(x)\right)$ for $x \geq 0$, which corrects our claim.

4. Convergence of modified q-Szász operators

In this section we study the approximation properties of modified q-Szász operators.

The following result is a q-analogue of [14: Theorem 1]. It shows that $M_{n,q}^*(f;x)$ converges to f uniformly on $[0,\infty)$ as $n \to \infty$, whenever $f^*(x) := f\left(x^2\right)$ is uniformly continuous.

Theorem 11. *Let $f \in C_m^*[0,\infty)$ and let*

$$f^*(z) = f\left(z^2\right), \qquad z \in [0,\infty).$$

We have, for all $x \geq 0$,

$$\left|M_{n,q}^*(f;x) - f(x)\right| \leq 2\omega\left(f^*; \sqrt{\frac{1}{[n]_q}}\right). \tag{15}$$

Therefore, $M_{n,q_n}^*(f;x)$ converges to f uniformly on $[0,\infty)$ as $n \to \infty$, whenever f^* is uniformly continuous.

Proof. Let $x \in [0,\infty)$, and $f \in C_m^*[0,\infty)$ be fixed. Then $M_{n,q}^*(f;0) = f(0)$ in view of Lemma 6. By the definition of f^* we have

$$M_{n,q}^*(f;x) = M_{n,q}^*\big(f^*(\sqrt{e_1});x\big).$$

Thus we can write

$$\begin{aligned}
&|M_{n,q}^*(f;x) - f(x)| \\
&\quad = \left|M_{n,q}^*\left(f^*\left(\sqrt{e_1}\right);x\right) - f^*\left(\sqrt{x}\right)\right| \\
&\quad = \left|\sum_{k=0}^{\infty}\left(f^*\left(\sqrt{\frac{[k]_q}{[n]_q}}\right) - f^*\left(\sqrt{x}\right)\right) s_{n,k}\left(q; v_{n,q}(x)\right)\right|
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty}\left|\left(f^*\left(\sqrt{\frac{[k]_q}{[n]_q}}\right)-f^*\left(\sqrt{x}\right)\right)\right| s_{n,k}\left(q;v_{n,q}(x)\right)$$

$$\leq \sum_{k=0}^{\infty}\omega\left(f^*;\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right|\right) s_{n,k}\left(q;v_{n,q}(x)\right)$$

$$\leq \sum_{k=0}^{\infty}\omega\left(f^*;\frac{\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right|}{M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)}M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)\right)$$

$$\times\, s_{n,k}\left(q;v_{n,q}(x)\right).$$

Finally, from the inequality

$$\omega\left(f^*;\alpha\delta\right)\leq(1+\alpha)\,\omega\left(f^*;\delta\right),\qquad \alpha,\delta\geq 0,$$

we obtain

$$\begin{aligned}
&|M^*_{n,q}(f;x)-f(x)|\\
&\quad\leq \omega\big(f^*;M^*_{n,q}\big(|\sqrt{e_1}-\sqrt{x}e_0|;x\big)\big)\\
&\quad\quad\times\sum_{k=0}^{\infty}\left(1+\frac{\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right|}{M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)}\right) s_{n,k}\left(q;v_{n,q}(x)\right)\\
&\quad= 2\omega\left(f^*;M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)\right).
\end{aligned}$$

In order to complete the proof we need to show that we have for all $x>0$,

$$M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)\leq\sqrt{\frac{1}{[n]_q}}.$$

Indeed we obtain from the Cauchy-Schwarz and Lemma 6 inequality

$$\begin{aligned}
M^*_{n,q}\left(\left|\sqrt{e_1}-\sqrt{x}e_0\right|;x\right)&=\sum_{k=0}^{\infty}\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right| s_{n,k}\left(q;v_{n,q}(x)\right)\\
&=\sum_{k=0}^{\infty}\frac{\left|\frac{[k]_q}{[n]_q}-x\right|}{\sqrt{\frac{[k]_q}{[n]_q}}+\sqrt{x}} s_{n,k}\left(q;v_{n,q}(x)\right)\\
&\leq\frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\left|\frac{[k]_q}{[n]_q}-x\right| s_{n,k}\left(q;v_{n,q}(x)\right)
\end{aligned}$$

$$\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{\infty}\left|\frac{[k]_q}{[n]_q}-x\right|^2 s_{n,k}\left(q;v_{n,q}(x)\right)}$$

$$=\frac{1}{\sqrt{x}}\sqrt{M_{n,q}^{*}\left(\left(e_1-xe_0\right)^2;x\right)}$$

$$\leq\sqrt{\frac{1}{[n]_q}},$$

showing (15), and completing the proof. □

Next, we obtain a direct approximation theorem in $C_m^*[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f,\delta)$ does not tend to zero, as $\delta\to 0$. For every $f\in C_m^*[0,\infty)$ the weighted modulus of continuity is defined as follows

$$\Omega_m(f,\delta)=\sup_{x\geq 0,\ 0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^m}.$$

Lemma 12. ([15]) *Let $f\in C_m^*[0,\infty)$, $m\in\mathbb{N}$. Then*

(1) *$\Omega_m(f,\delta)$ is a monotone increasing function of δ,*

(2) $\lim_{\delta\to 0^+}\Omega_m(f,\delta)=0$,

(3) *for any $\alpha\in[0,\infty)$, $\Omega_m(f,\alpha\delta)\leq(1+\alpha)\Omega_m(f,\delta)$.*

The following is the main convergence result of the paper. In Theorem 13 we give an expression of the approximation error with the operators $M_{n,q}^*$ by means of Ω_m.

Theorem 13. *If $f\in C_m^*[0,\infty)$ then the inequality*

$$\left\|M_{n,q}^*(f)-f\right\|_{m+1}\leq k\Omega_m\left(f;\sqrt{\frac{1}{[n]_q}}\right)$$

holds, where k is a constant independent of f and n.

P r o o f. From the definition of $\Omega_m(f,\delta)$ and Lemma 12, with $\alpha=|t-x|\,\delta^{-1}$, considering the cases $t>x\geq 0$ and $x>t\geq 0$ separately, we may write

$$\begin{aligned}|f(t)-f(x)|&\leq\left(1+(x+|t-x|)^m\right)\Omega_m(f,|t-x|)\\&\leq\left(1+(x+|t-x|)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta)\end{aligned}$$

$$\le \left(1+(2x+t)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta)$$
$$=: \mu_x(t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta).$$

Then

$$|M^*_{n,q}(f;x)-f(x)|$$
$$\le \Omega_m(f,\delta)\left(M^*_{n,q}(\mu_x(\cdot);x)+M^*_{n,q}\left(\mu_x(\cdot)\frac{|e_1-xe_0|}{\delta};x\right)\right).$$

Applying the Cauchy-Schwartz inequality to the second term, we get

$$M^*_{n,q}\left(\mu_x(\cdot)\frac{|e_1-xe_0|}{\delta};x\right)$$
$$\le \left(M^*_{n,q}\left(\mu_x^2(\cdot);x\right)\right)^{1/2}\left(M^*_{n,q}\left(\frac{|e_1-xe_0|^2}{\delta^2};x\right)\right)^{1/2}.$$

Consequently

$$\left|M^*_{n,q}(f;x)-f(x)\right| \le \Omega_m(f,\delta)\Bigg(M^*_{n,q}(\mu_x(\cdot);x)+\left(M^*_{n,q}\left(\mu_x^2(\cdot);x\right)\right)^{1/2} \times\left(M^*_{n,q}\left(\frac{|e_1-xe_0|^2}{\delta^2};x\right)\right)^{1/2}\Bigg). \tag{16}$$

Notice that by Lemmas 9 and 4 there are positive constants $K_m(q)$ and $K^1_m(q)$ such that

$$M^*_{n,q}(\mu_x(\cdot);x)\le K_m(q)(1+x^m),$$
$$\left(M^*_{n,q}\left(\mu_x^2(\cdot);x\right)\right)^{1/2}\le K^1_m(q)(1+x^m). \tag{17}$$

Notice that, the second inequality follows from the first and $\mu_x^2(t) \le 2\left(1+(2x+t)^{2m}\right)$. Further, by Lemma 6, we get

$$\left(M^*_{n,q}\left(\frac{|e_1-xe_0|^2}{\delta^2};x\right)\right)^{1/2}\le\frac{1}{\delta}\sqrt{\frac{x}{[n]_q}}. \tag{18}$$

Now from (16), (17) and (18) we have

$$|M^*_{n,q}(f;x)-f(x)|$$
$$\le \Omega_m(f,\delta)\left(K_m(q)(1+x^m)+K^1_m(q)\frac{1}{\delta}\sqrt{\frac{x}{[n]_q}}(1+x^m)\right)$$

$$\leq \Omega_m(f,\delta)\left(K_m(q)(1+x^m)+K_m^1(q)K_1\frac{1}{\delta}\sqrt{\frac{1}{[n]_q}}\left(1+x^{m+1}\right)\right),$$

where

$$K_1=\sup_{x\geq 0}\frac{1+x^m+x+x^{m+1}}{1+x^{m+1}}.$$

If we take $\delta=\sqrt{\frac{1}{[n]_q}}$, then from the above inequality we obtain the desired result. $\square$

Next we prove Voronovskaja type result for M_{n,q_n}^*.

Theorem 14. *Assume that $q_n>1$, $q_n\to 1$ as $n\to\infty$. For any $f\in C_B[0,\infty)$ such that $f',f''\in C_B[0,\infty)$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(M_{n,q_n}^*(f;x)-f(x)\right)=\frac{1}{2}\left(xf''(x)-f'(x)\right)$$

uniformly on any $[0,A]$, $A>0$.

Proof. Let $f,f',f''\in C_B[0,\infty)$ and $x\in[0,\infty)$ be fixed. By the Taylor formula we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{19}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x)\in C_2^*[0,\infty)$ and $\lim_{t\to x}r(t;x)=0$. Applying M_{n,q_n}^* to (19) we obtain

$$\begin{aligned}{}_{q_n}\left(M_{n,q_n}^*(f;x)-f(x)\right) &= f'(x)[n]_{q_n}M_{n,q_n}^*(e_1-xe_0;x)\\ &\quad+\frac{1}{2}f''(x)[n]_{q_n}M_{n,q_n}^*\left((e_1-xe_0)^2;x\right)\\ &\quad+[n]_{q_n}M_{n,q_n}^*\left(r(\cdot;x)(e_1-xe_0)^2;x\right).\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&M_{n,q_n}^*\left(r(\cdot;x)(e_1-xe_0)^2;x\right)\\ &\qquad\leq\sqrt{M_{n,q_n}^*\left(r^2(\cdot;x);x\right)}\sqrt{M_{n,q_n}^*\left((e_1-xe_0)^4;x\right)}.\end{aligned}\tag{20}$$

Observe that $r^2(x;x) = 0$ and $r^2(\cdot\,;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 10 and Lemma 6 that

$$\lim_{n\to\infty} M^*_{n,q_n}\left(r^2(\cdot\,;x);x\right) = r^2(x;x) = 0 \tag{21}$$

uniformly with respect to $x \in [0,A]$. Now from (20), (21) and Corollary 8 we get immediately

$$\lim_{n\to\infty} [n]_{q_n} M^*_{n,q_n}\left(r(\cdot\,;x)(e_1 - xe_0)^2;x\right) = 0.$$

Finally, using Lemma 7, we obtain the assertion of our theorem. $\square$

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REFERENCES

[1] KING, J. P.: *Positive linear operators which preserve x^2*, Acta Math. Hungar. **99** (2003), 203–208.

[2] GONSKA, H.—PIŢUL, P.: *Remarks on an article of J. P. King*. In: Schr.reihe Fachbereichs Math. Univ. Duisburg-Essen 596, Univ. Duisburg-Essen, 2005, pp. 1–8.

[3] DUMAN, O.—ÖZARSLAN, M. A.: *Szász-Mirakjan type operators providing a better error estimation*, Appl. Math. Lett. **20** (2007), 181–188.

[4] AGRATINI, O.: *Linear operators that preserve some test functions*, Int. J. Math. Math. Sci. **2006**, Article ID94136, pp. 11, DOI10.1155/IJMMS.

[5] REMPULSKA, L.—TOMCZAK, K.: *Approximation by certain linear operators preserving x^2*, Turkish J. Math. **20** (2009), 273–281.

[6] MAHMUDOV, N. I.: *Korovkin-type theorems and applications*, Cent. Eur. J. Math. **7** (2009), 348–356.

[7] AGRATINI, O.—DOĞRU, O.: *Weighted approximation by q-Szasz-King operators*, Taiwanese J. Math. (To appear).

[8] MAHMUDOV, N. I.: *q-Szász-Mirakjan operators which preserve x^2*, J. Comput. Appl. Math. **235** (2011), 4621–4628.

[9] MAHMUDOV, N. I.: *On q-parametric Szász-Mirakjan operators*, Mediterr. J. Math. **7** (2010), 297–311.

[10] MAHMUDOV, N. I.: *Approximation properties of complex q-Szász-Mirakjan operators in compact disks*, Comput. Math. Appl. **60** (2010), 1784–1791.

[11] ARAL, A.—GUPTA, V.: *q-derivatives and applications to the q-Szasz Mirakyan operators*, Calcalo **43** (2006), 151–170.

[12] RADU, C.: *On statistical approximation of a general class of positive linear operators extended in q-calculus*, Appl. Math. Comput. **215** (2009), 2317–2325.

[13] BECKER, M.: *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. **27** (1978), 127–142.

[14] de la CAL, J.—CÁRCAMO, J.: *On uniform approximation by some classical Bernstein-type operators*, J. Math. Anal. Appl. **279** (2003), 625–638.

[15] LÓPEZ-MORENO, A.-J.: *Weighted simaltaneous approximation with Baskakov type operators*, Acta Math. Hungar. **104** (2004), 143–151.

[16] KAC, V.—CHEUNG, P.: *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

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