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CHARACTERIZATION OF SUMMABILITY METHODS WITH HIGH INDICES

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ABSTRACT. In this paper we establish a set of necessary and sufficient conditions in order that $|C,0|_k \implies |R,p_n|_s$ and $|R,p_n|_k \implies |C,0|_s$ for the case $1 < k \leq s < \infty$. As a corollary, we obtain that a crucial assumption of [BOR, H.: *A new result on the high indices theorem*, Analysis **29** (2009), 403–405] is omitted and that the other one is not only sufficient but also necessary for his consequence to hold.

1. Introduction

Let $\sum a_v$ be a given infinite series with partial sums (s_n). By σ_n^α we denote n-th Cesàro means of order α, $\alpha > -1$, of the sequences (s_n). The series $\sum a_v$ is said to be absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \tag{1.1}$$

[7]. Since $\sigma_n^0 = s_n$, the summability $|C,0_k$is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1} |a_n|^k < \infty. \tag{1.2}$$

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Let (p_n) be a sequence of positive real constants with $P_n = p_0+p_1+\cdots+p_n \to \infty$ as $n \to \infty$. The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (t_n) of the (R, p_n) Riesz means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_v$ is then said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

(see [13]).

If A and B are methods of summability, B is said to include A (written $A \implies B$) if every series summable by the method A is also summable by the method B. A and B said to be equivalent (written $A \iff B$) if each methods includes the other.

Problems on inclusion dealing absolute Cesàro and absolute weighted mean summabilities have been examined by many authors ([3–14]).

More recently, Bor [2] proved sufficient conditions for $|R, p_n| \iff |C, 0|_k$ as follows.

Theorem 1.1. *Let $k > 1$ and*

$$\sum_{n=v}^{\infty} \frac{n^{k-1}p_n^k}{P_n^k P_{n-1}} = O\left(\frac{v^{k-1}p_{v-1}^k}{P_{v-1}^k}\right). \tag{1.5}$$

If

$$P_{n+1} \geq dP_n. \tag{1.6}$$

where d is a constant such that $d > 1$, then $|R, p_n|_k \iff |C, 0|_k$.

2. Main results

In this paper we establish a set of necessary and sufficient conditions in order that $|R, p_n|_k \implies |C, 0|_s$ and $|C, 0|_k \implies |R, p_n|_s$ for the case $1 < k \leq s < \infty$. As a corollary, we obtain that a crucial assumption (1.5) is omitted, and (1.6) is not only sufficient but also necessary for Theorem 1.1 to hold. We make use of a result of Bennett [1], who has obtained necessary and sufficient conditions for factorable matrix to map $T\colon \ell^k \to \ell^s$. A factorable matrix T is one in which each entry $t_{nv} = b_n c_v$. A Riesz mean matrix is factorable.

It may be remarked that it will not be possible to extended our result by replacing (R, p_n) by a triangular matrix T, since necessary and sufficient conditions are not known for an arbitrary triangular matrix to map $T\colon \ell^k \to \ell^s$.

Our theorems read as follows.

Theorem 2.1. *Let $1 < k \le s < \infty$. Then $|R, p_n|_k \implies |C, 0|_s$ if and only if*

$$\left(\sum_{v=m-1}^{m} \frac{1}{v}\left(\frac{P_v P_{v-1}}{p_v}\right)^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{m+1} \frac{n^{s-1}}{P_{n-1}^s}\right)^{1/s} = O(1), \tag{2.1}$$

where k^* denotes the conjugate index of k, i.e., $\frac{1}{k} + \frac{1}{k^*} = 1$.

Theorem 2.2. *Let $1 < k \le s < \infty$. Then $|C, 0|_k \implies |R, p_n|_s$ if and only if*

$$\left(\sum_{m=1}^{v} \frac{P_{m-1}^{k^*}}{m}\right)^{1/k^*} \left(\sum_{n=v}^{\infty} \left(\frac{n^{1-1/s} p_n}{P_n P_{n-1}}\right)^{s}\right)^{1/s} = O(1), \tag{2.2}$$

where k^* denotes the conjugate index of k.

If $k = s$, then it is clear that (2.1) is equivalent to (1.6), and also (1.6) implies (2.2). In fact, (2.1) is satisfied if and only if

$$\frac{P_{m-1}}{m^{1/k^*}} \left(\left(\frac{P_{m-2}}{p_{m-1}}\right)^{k^*} + \left(\frac{P_m}{p_m}\right)^{k^*}\right)^{1/k*} \frac{m^{1/k^*}}{P_{m-1}} = O(1) \iff P_m = O(p_m),$$

and $P_m = O(p_m)$ is equivalent to (1.6) as well. On the other hand, if $P_m = O(p_m)$, i.e., (1.6) holds, then

$$\begin{aligned} \sum_{n=v}^{\infty} \left(\frac{n^{1-1/k} p_n}{P_n P_{n-1}}\right)^k &= O\left(\frac{1}{P_{v-1}^{k-1}}\right) \sum_{n=v}^{\infty} \frac{n^{k-1}}{P_{n-1}} \\ &= O\left(\frac{1}{P_{v-1}^{k}}\right) \sum_{n=v}^{\infty} \frac{n^{k-1}}{d^{n-v}} \\ &= O\left(\frac{v^{k-1}}{P_{v-1}^{k}}\right) \sum_{n=0}^{\infty} \frac{(1+n/v)^{k-1}}{d^{n}} \\ &= O\left(\frac{v^{k-1}}{P_{v-1}^{k}}\right), \end{aligned} \tag{2.3}$$

and

$$\sum_{m=1}^{v} \frac{P_{m-1}^{k^*}}{m} = O(1)\left(\sum_{m=1}^{v-1} \frac{P_{m-1}^{k^*}}{m(m+1)} + \frac{P_{v-1}^{k^*}}{v}\right) = O\left(\frac{P_{v-1}^{k^*}}{v}\right)$$

by Abel partial summation. Thus one can show that (2.2) is verified. Therefore we promptly get Theorem 1.1 under necessary and sufficient conditions in the following way.

Corollary 2.3. *Let $k \geq 1$. Then, $|C,0|_k \iff |R,p_n|_k$ if and only if condition* (1.6) *is satisfied.*

Now, if one takes $p_n = 1$ in Theorem 2.1 and Theorem 2.2, then condition (2.2) is satisfied by [12: Lemma 4.1], but condition (2.1), and so we get the following result of Flett [7].

Corollary 2.4. *Let $1 < k \leq s < \infty$. Then*

$$|C,0|_k \implies |C,1|_s \qquad \textit{but} \qquad |C,1|_k \not\Rightarrow |C,0_s|.$$

We can easily provide an example of Riesz mean satisfying Corollary 2.3. In fact, choose $p_n = x^n$ for $n = 0, 1, \dots$ where $x > 1$. A few calculations reveal that $P_v/p_v \to x/(x-1)$ as $v \to \infty$, and so the condition of Corollary 2.3 holds.

Proof of Theorem 2.1. It follows from (1.2) and (1.3) that

$$T_n = t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v. \tag{2.4}$$

Solving (2.4) for a_n gives

$$\frac{P_n P_{n-1}}{p_n} T_n = \sum_{v=1}^{n} P_{v-1} a_v$$

and

$$\frac{P_{n-1} P_{n-2}}{p_{n-1}} T_{n-1} = \sum_{v=1}^{n-1} P_{v-1} a_v \tag{2.5}$$

which implies that

$$a_n = \frac{1}{P_{n-1}} \left(\frac{P_n P_{n-1}}{p_n} T_n - \frac{P_{n-1} P_{n-2}}{p_{n-1}} T_{n-1} \right).$$

Define $a_n^* = n^{1-1/s} a_n$ and $T_n^* = n^{1-1/k} T_n$. Then

$$a_n^* = \frac{n^{1-1/s}}{P_{n-1}} \left(\frac{P_n P_{n-1}}{n^{1-1/k} p_n} T_n^* - \frac{P_{n-1} P_{n-2}}{(n-1)^{1-1/k}\, p_{n-1}} T_{n-1}^* \right) = \sum_{v=1}^{n} t_{nv} T_v^*.$$

where

$$t_{nv} = \begin{cases} 0, & v < n-1, \quad v > n \\ -\dfrac{n^{1-1/s}}{P_{n-1}} \left(\dfrac{P_v P_{v-1}}{v^{1-1/k} p_{v-1}} \right), & v = n-1 \\ \dfrac{n^{1-1/s}}{P_{n-1}} \left(\dfrac{P_v P_{v-1}}{v^{1-1/k} p_{v-1}} \right), & v = n. \end{cases} \tag{2.6}$$

Then, $|R,p_n|_k \implies |C,0|_s$ is equivalent to

$$\sum_{n=1}^{\infty} |T_n^*|^k < \infty \implies \sum_{n=1}^{\infty} |a_n^*|^s < \infty, \qquad \text{i.e.,} \quad T\colon \ell^k \to \ell^s,$$

where T is the matrix whose entries are defined by (2.6). From [1: Theorem 2(ii)], a factorable matrix with nonegative entrice $b_n c_v$ is a bounded operator from ℓ^k to ℓ^s iff

$$\left(\sum_{v=1}^{m} c_v^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} b_n^s\right)^{1/s} = O(1). \tag{2.7}$$

Applying (2.7) to the matrix T, we have that $T\colon \ell^k \to \ell^s$ iff the condition (2.1) holds, which completes the proof. □

Proof of Theorem 2.2. Let $a_n^* = n^{1-1/k} a_n$ and $T_n^* = n^{1-1/s} T_n$. Then, by (2.4), we have

$$T_n^* = \frac{n^{1-1/s} p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \frac{a_v^*}{v^{1-1/k}} = \sum_{v=1}^{n} h_{nv} a_v^* \tag{2.8}$$

where

$$h_{nv} = \begin{cases} \dfrac{n^{1-1/s} p_n}{P_n P_{n-1}} \left(\dfrac{P_{v-1}}{v^{1-1/k}}\right), & 1 \le v \le n \\ 0, & v > n. \end{cases}$$

The reminder is similar to the above, so is omitted. □

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