

MIXED-MEAN INEQUALITY FOR SUBMATRIX

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(Communicated by Sylvia Pulmannová)

ABSTRACT. For an $m \times n$ matrix $B = (b_{ij})_{m \times n}$ with nonnegative entries b_{ij} , let $B(k, l)$ denote the set of all $k \times l$ submatrices of B . For each $A \in B(k, l)$, let a_A and g_A denote the arithmetic mean and geometric mean of elements of A respectively. It is proved that if k is an integer in $(\frac{m}{2}, m]$ and l is an integer in $(\frac{n}{2}, n]$ respectively, then

$$\left(\prod_{A \in B(k, l)} a_A \right)^{\frac{1}{\binom{m}{k} \binom{n}{l}}} \geq \frac{1}{\binom{m}{k} \binom{n}{l}} \left(\sum_{A \in B(k, l)} g_A \right),$$

with equality if and only if b_{ij} is a constant for every i, j .

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1. Introduction

Let $x_1, \dots, x_n$ be positive real numbers, then the arithmetic-geometric mean inequality is

$$\frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n},$$

with equality if and only if $x_1 = \dots = x_n$.

There are many research articles devoted to the classical arithmetic-geometric mean inequality and its generalizations ([1], [2], [4], [9], [10]). The mixed-type arithmetic-geometric mean inequalities ([3], [5], [6], [7], [8], [11]) are one of the most important branch in these generalizations.

In [6], Kedlaya established the following mixed mean inequality, which was conjectured by F. Holland and whose inductive proof was given by T. Matsuda [8]:

2010 Mathematics Subject Classification: Primary 26B25.

Keywords: mixed mean, power mean, matrix, arithmetic-geometric mean inequality.

This work was supported by National Natural Science Foundation of China Grant No.11001014, the Fundamental Research Funds for the Central Universities Grant No. YX2010-29.

The arithmetic mean of the numbers

$$x_1, \sqrt{x_1 x_2}, \sqrt[3]{x_1 x_2 x_3}, \dots, \sqrt[n]{x_1 x_2 \dots x_n}$$

does not exceed the geometric mean of the numbers

$$x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + \dots + x_n}{n},$$

with equality if and only if $x_1 = \dots = x_n$.

In [3], Carlson established the following mixed mean inequality:

Let the arithmetic and geometric means of the real nonnegative numbers $x_1, \dots, x_n$ taken $n - 1$ at a time be denoted by

$$a_i = \frac{x_1 + \dots + x_n - x_i}{n - 1}, \quad g_i = \left(\frac{x_1 \dots x_n}{x_i} \right)^{\frac{1}{n-1}}.$$

Then for $n \geq 3$,

$$(a_1 \dots a_n)^{\frac{1}{n}} \geq \frac{g_1 + \dots + g_n}{n},$$

with equality if and only if $x_1 = \dots = x_n$.

In [7], Leng, Si and Zhu generalized the above result to any subsets and established another mixed arithmetic-geometric mean inequality:

Let X be a set of real nonnegative numbers $x_1, \dots, x_n$. For $A \subset X$, let a_A and g_A denote the arithmetic mean and geometric mean of all elements of A , respectively. If k is an integer in $(\frac{n}{2}, n]$, then

$$\left(\prod_{\substack{|A|=k \\ A \subset X}} a_A \right)^{\frac{1}{\binom{n}{k}}} \geq \frac{1}{\binom{n}{k}} \left(\sum_{\substack{|A|=k \\ A \subset X}} g_A \right), \quad (1.1)$$

with equality if and only if $x_1 = \dots = x_n$.

In this note, we established a new mixed arithmetic-geometric mean inequality for submatrix, which was an extension of the Carlson inequality and also an extension of (1.1).

Our main result is the following theorem.

THEOREM 1.1. For an $m \times n$ matrix $B = (b_{ij})_{m \times n}$ with nonnegative entries $b_{ij} \geq 0$, let $B(k, l)$ denote the set of all $k \times l$ submatrices of B . For each $A \in B(k, l)$, let a_A and g_A denote the arithmetic mean and geometric mean of elements of A respectively. If k is an integer in $(\frac{m}{2}, m]$ and l is an integer in $(\frac{n}{2}, n]$ respectively, then

$$\left(\prod_{A \in B(k, l)} a_A \right)^{\frac{1}{\binom{m}{k} \binom{n}{l}}} \geq \frac{1}{\binom{m}{k} \binom{n}{l}} \left(\sum_{A \in B(k, l)} g_A \right), \quad (1.2)$$

with equality if and only if b_{ij} is a constant for every i, j .

Remark 1. If $k \leq [\frac{m}{2}]$, for the matrix $B = (b_{ij})_{m \times n}$, taking $b_{11} = b_{12} = \dots = b_{1n} = b_{21} = b_{22} = \dots = b_{2n} = \dots = b_{k1} = b_{k2} = \dots = b_{kn} = 1$, $b_{ij} = 0$ for $k+1 \leq i \leq m$, then the right-hand side of (1.2) equals $1/\binom{m}{k}$, but the left-hand side is zero. If $l \leq [\frac{n}{2}]$, by the same argument, one can get a contradiction. Hence the statement of Theorem 1.1 fails for $k \leq [\frac{m}{2}]$ or $l \leq [\frac{n}{2}]$.

Remark 2. Taking $k = m$, $l = n$ in Theorem 1.1, inequality (1.2) is just the classical arithmetic-geometric mean inequality. For $m = 1$ or $n = 1$ in Theorem 1.1, inequality (1.2) is just the inequality (1.1).

Remark 3. The condition of our theorem is weaker than the ones of (1.1), because the infimum of $k \times l$ is $([\frac{m}{2}] + 1) \times ([\frac{n}{2}] + 1)$, which is less than $[\frac{mn}{2}] + 1$, the half of the element number of the matrix $B = (b_{ij})_{m \times n}$.

2. Proof of main results

Let X denote the finite set with positive real numbers $x_1, x_2, \dots, x_n$ and let A denote the subset of X with k elements $x_{i_1}, \dots, x_{i_k}$. The r power mean of the elements of A is denoted by

$$m_r(A) = \left[\frac{1}{k} (x_{i_1}^r + \dots + x_{i_k}^r) \right]^{\frac{1}{r}}, \quad (2.1)$$

where $r > 0$ is some real number. If $r = 1$ in the above equality, we get a_A , the arithmetic mean of the elements of A . Let $r \rightarrow 0$ in (2.1), we get g_A , the geometric mean of the elements of A .

We first established the following lemma.

LEMMA 2.1. *For any $m \times n$ matrix $X = (x_{ij})_{m \times n}$ with nonnegative entries $x_{ij} \geq 0$. Denote by $X_1, \dots, X_{\binom{m}{k}\binom{n}{l}}$ its all $k \times l$ -submatrices of X . If k is an integer in $(\frac{m}{2}, m]$ and l is an integer in $(\frac{n}{2}, n]$ respectively, then for any $1 \leq i \leq \binom{m}{k}\binom{n}{l}$*

$$m_r(X_i) = \left[\frac{1}{\binom{m}{k}\binom{n}{l}} \left[(m_r(X_i \cap X_1))^r + (m_r(X_i \cap X_2))^r + \dots \right. \right. \\ \left. \left. \dots + (m_r(X_i \cap X_{\binom{m}{k}\binom{n}{l}}))^r \right] \right]^{\frac{1}{r}}. \quad (2.2)$$

Here the intersection of $X_i \cap X_j$, $j = 1, \dots, \binom{m}{k}\binom{n}{l}$, is just the general set intersection.

Proof. It suffices to prove the equality

$$(m_r(X_i))^r = \frac{1}{\binom{m}{k}\binom{n}{l}} \left[(m_r(X_i \cap X_1))^r + (m_r(X_i \cap X_2))^r + \cdots \right. \\ \left. \cdots + \left(m_r \left(X_i \cap X_{\binom{m}{k}\binom{n}{l}} \right) \right)^r \right]. \quad (2.3)$$

Assume that $X_i = \{x_{i_1}, \dots, x_{i_{k \times l}}\}$, then the left hand of (2.3) is that

$$(m_r(X_i))^r = \frac{1}{k \times l} (x_{i_1}^r + \cdots + x_{i_{k \times l}}^r).$$

Now we need show that the right hand of (2.3) is also the mean of $\{x_{i_1}^r, \dots, x_{i_{k \times l}}^r\}$ with the same coefficient $\frac{1}{k \times l}$.

Assume that the right hand of (2.3) is $c_{i_1} x_{i_1}^r + \cdots + c_{i_{k \times l}} x_{i_{k \times l}}^r$. By the arbitrariness of X_i , we get that $c_{i_1} = \cdots = c_{i_{k \times l}}$.

Since $k > \frac{m}{2}$ and $l > \frac{n}{2}$, we have

$$X_i \cap X_j \neq \Phi, \quad j = 1, 2, \dots, \binom{m}{k}\binom{n}{l}.$$

Then the sum of all coefficients in $(m_r(X_i \cap X_j))^r$ is 1, $j = 1, 2, \dots, \binom{m}{k}\binom{n}{l}$. As a result, the sum of all coefficients in the right hand of (2.3) is

$$\frac{1}{\binom{m}{k}\binom{n}{l}} \underbrace{(1 + 1 + \cdots + 1)}_{\binom{m}{k}\binom{n}{l}} = 1,$$

i.e., $c_{i_1} + \cdots + c_{i_{k \times l}} = 1$. Therefore $c_{i_1} = \cdots = c_{i_{k \times l}} = \frac{1}{k \times l}$. $\square$

Remark 4. (2.2) is the generalization of related result in [7]. But the condition in Lemma is weaker, because the infimum of $k \times l$ is $(\lceil \frac{m}{2} \rceil + 1) \times (\lceil \frac{n}{2} \rceil + 1)$, which is less than $\lceil \frac{mn}{2} \rceil + 1$, the half of the element number of the matrix $X = (x_{ij})_{m \times n}$.

Remark 5. In (2.2), if $r = 1$, we have

$$a_{X_i} = \frac{1}{\binom{m}{k}\binom{n}{l}} \left(a_{X_i \cap X_1} + a_{X_i \cap X_2} + \cdots + a_{X_i \cap X_{\binom{m}{k}\binom{n}{l}}} \right). \quad (2.4)$$

In (2.2), if $r \rightarrow 0$, we have

$$g_{X_i} = \left(g_{X_i \cap X_1} \cdot g_{X_i \cap X_2} \cdots g_{X_i \cap X_{\binom{m}{k}\binom{n}{l}}} \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \quad (2.5)$$

2.1. Proof of Theorem

For an $m \times n$ matrix $B = (b_{ij})_{m \times n}$ with nonnegative entries $b_{ij} \geq 0$, let $B(k, l)$ denote the set of all $k \times l$ submatrices of B . For each element of $B(k, l)$, which is composed by the rows $(i_1, i_2, \dots, i_k)$ of B and by the columns $(j_1, j_2, \dots, j_l)$

of B , by the partial order of $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_l)$,¹ we could get a sequence of submatrices $B_1, B_2, \dots, B_{\binom{m}{k}\binom{n}{l}}$.

In Lemma 2.1, let X_i be B_i and X_j , $j = 1, 2, \dots, \binom{m}{k}\binom{n}{l}$, be corresponding to $B_1, B_2, \dots, B_{\binom{m}{k}\binom{n}{l}}$, then (2.4) is

$$a_{B_i} = \frac{1}{\binom{m}{k}\binom{n}{l}} \left(a_{B_i \cap B_1} + a_{B_i \cap B_2} + \dots + a_{B_i \cap B_{\binom{m}{k}\binom{n}{l}}} \right), \quad (2.6)$$

and (2.5) is

$$g_{B_i} = \left(g_{B_i \cap B_1} \cdot g_{B_i \cap B_2} \cdot \dots \cdot g_{B_i \cap B_{\binom{m}{k}\binom{n}{l}}} \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \quad (2.7)$$

By the arithmetic-geometric inequality, it follows that

$$a_{B_i \cap B_j} \geq g_{B_i \cap B_j}, \quad j = 1, 2, \dots, \binom{m}{k}\binom{n}{l}. \quad (2.8)$$

From (2.6) and (2.8), we infer that

$$a_{B_i} \geq \frac{1}{\binom{m}{k}\binom{n}{l}} \left(g_{B_i \cap B_1} + g_{B_i \cap B_2} + \dots + g_{B_i \cap B_{\binom{m}{k}\binom{n}{l}}} \right).$$

Therefore

$$\begin{aligned} \left(\prod_{A \in B(k,l)} a_A \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} &= \left(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}} a_{B_i} \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \\ &\geq \frac{1}{\binom{m}{k}\binom{n}{l}} \left(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}} \left(\sum_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i \cap B_j} \right) \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \end{aligned} \quad (2.9)$$

On the other hand, using the discrete case of Hölder's inequality in the form

$$\sum_{k=1}^n \left(\prod_{j=1}^m x_{jk} \right)^{\frac{1}{m}} \leq \left(\prod_{j=1}^n \left(\sum_{k=1}^m x_{jk} \right) \right)^{\frac{1}{m}},$$

where n, m are positive integers and $x_{jk} > 0$ ($j, k = 1, 2, \dots, m$), we obtain

$$\left(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}} \left(\sum_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i \cap B_j} \right) \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \geq \sum_{i=1}^{\binom{m}{k}\binom{n}{l}} \left(\prod_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i \cap B_j} \right)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \quad (2.10)$$

¹If we consider two elements of $B(k, l)$, B_m composed by the rows $(i_1, i_2, \dots, i_k)$ of B and by the columns $(j_1, j_2, \dots, j_l)$ of B and B_n composed by the rows $(i_1', i_2', \dots, i_k')$ of B and by the columns $(j_1', j_2', \dots, j_l')$ of B , then here the partial order $B_m \prec B_n$ means that $i_s < i_s'$ for some $1 \leq s \leq k$ or $i_s = i_s'$ for all $1 \leq s \leq k$ and $j_t < j_t'$ for some $1 \leq t \leq l$.

Combining (2.7), (2.9) and (2.10), it follows that

$$\begin{aligned} \left(\prod_{A \in B(k,l)} a_A \right)^{\frac{1}{\binom{m}{k} \binom{n}{l}}} &\geq \frac{1}{\binom{m}{k} \binom{n}{l}} \sum_{i=1}^{\binom{m}{k} \binom{n}{l}} \left(\prod_{j=1}^{\binom{m}{k} \binom{n}{l}} g_{B_i \cap B_j} \right)^{\frac{1}{\binom{m}{k} \binom{n}{l}}} \\ &= \frac{1}{\binom{m}{k} \binom{n}{l}} \sum_{i=1}^{\binom{m}{k} \binom{n}{l}} g_{B_i} = \frac{1}{\binom{m}{k} \binom{n}{l}} \left(\sum_{A \in B(k,l)} g_A \right), \end{aligned}$$

which is just the inequality (1.2).

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Received 27. 1. 2011

Accepted 12. 7. 2011

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