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CLEAN UNITAL ℓ-GROUPS

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ABSTRACT. A ring with identity is said to be *clean* if every element can be written as a sum of a unit and an idempotent. The study of clean rings has been at the forefront of ring theory over the past decade. The theory of partially-ordered groups has a nice and long history and since there are several ways of relating a ring to a (unital) partially-ordered group it became apparent that there ought to be a notion of a clean partially-ordered group. In this article we define a clean unital lattice-ordered group; we state and prove a theorem which characterizes clean unital ℓ-groups. We mention the relationship of clean unital ℓ-groups to algebraic K-theory. In the last section of the article we generalize the notion of clean to the non-unital context and investigate this concept within the framework of **W**-objects, that is, archimedean ℓ-groups with distinguished weak order unit.

1. Preliminaries

This paper contains the foundations of a general theory of unital ℓ-groups whose spaces of maximal convex ℓ-subgroups are boolean spaces (that is, compact zero-dimensional and Hausdorff). The notion of a clean unital ℓ-group will be of extreme importance. Examples of such spaces abound throughout the literature. Of the various equivalent conditions that we shall present one is most suitable from an algebraic point of view; it is an adaptation of an important property in the theory of rings. Recall that a ring is said to be *clean* if every element can be written as a sum of a unit and an idempotent. The study of clean rings has taken up much consideration as of late. For a history of clean rings we urge the reader to consult [12].

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In this section we present the concepts that will be used throughout the article. A *lattice-ordered group (ℓ-group)* is a group $(G, \cdot, 1)$ equipped with a lattice order, say $\leq$, such that whenever $g \leq h$, and $x, y \in G$, then $xgy \leq xhy$. We ought to point out that we use 1 for the identity element while we use the letter e to denote an arbitrary element of G.

The set
$$G^+ = \{g \in G : \ 1 \leq g\}$$
is the collection of positive elements of the group and is called the positive cone of G. Every ℓ-group has the property that it is generated (as a group) by its positive cone. In particular, letting $g^+ = g \vee 1$ and $g^- = g^{-1} \vee 1$ we have that $g^+, g^- \in G^+$ and $g = g^+(g^-)^{-1}$. We let $|g| = g^+g^- = g^+ \vee g^-$ denote the *absolute value of* g.

An *ℓ-subgroup of* G is a subgroup which is also a sublattice. If $H \leq G$ is an ℓ-subgroup of G, we say H is *convex* if $1 \leq g \leq h \in H$ implies that $g \in H$. The importance of convexity is that the set of cosets of such a ℓ-subgroup can be equipped with a partial order making it into a lattice; $gH \leq kH$ precisely if there is an $h \in H$ such that $g \leq kh$. Let $\mathfrak{C}(G)$ denote the set of all convex ℓ-subgroups of G. Since an arbitrary intersection of convex ℓ-subgroups is again a convex ℓ-subgroup it follows that $\mathfrak{C}(G)$ is a complete lattice under inclusion. Moreover, $\mathfrak{C}(G)$ is an algebraic frame (more on this later). A nice fact is that the join of two convex ℓ-subgroups is precisely the subgroup generated by the two subgroups. The convex ℓ-subgroup generated by an element $g \in G$ is denoted $G(g)$ and it is known that
$$G(g) = \big\{h \in G : \ |h| \leq |g|^n \ \text{ for some } \ n \in \mathbb{N}\big\}.$$
Moreover, $G(g) = G(|g|)$, and for $a, b \in G^+$, $G(a) \wedge G(b) = G(a \wedge b)$ and $G(a) \vee G(b) = G(a \vee b) = G(ab)$.

For a subset $X \subseteq G$ the polar of X is the set
$$X^\perp = \big\{g \in G : \ |g| \wedge |x| = 1 \ \text{ for all } \ x \in X\big\}.$$
For any subset $X \subseteq G$, $X^\perp \in \mathfrak{C}(G)$. When X is a singleton set, say $X = \{x\}$, we write $x^\perp$ instead of $\{x\}^\perp$. The positive elements $g, h \in G^+$ are said to be *disjoint* if $g \wedge h = 1$. For a given $1 \neq g \in G$, we recall that an element $e \in G$ is a *component of* g if the elements e and ge^{-1} are disjoint. In this case it follows, since disjoint elements commute, that g commutes with any of its components. The collection of components of a given positive element forms a boolean algebra under $\wedge$ and $\vee$.

An element $g \in G$ for which $g^\perp = \{1\}$ is called a *weak order unit*. If $g \in G$ and $G(g) = G$, then g is called a *strong order unit*. We ought to point out that we do not assume that our strong order units are positive. It is straightforward to check that a strong order unit is in fact a weak order unit, but not conversely. When G possesses a positive strong order unit, say $u \in G^+$, we say G is a *unital ℓ-group*. We shall have several occasions to write (G, u) is a unital ℓ-group.

If $P \in \mathfrak{C}(G)$ is proper and $g \wedge h \in P$ implies that either $g \in P$ or $h \in P$, then we call P a *prime* subgroup. It is a fact that a proper convex ℓ-subgroup P is a prime subgroup if and only if whenever g and h are disjoint, then either g or h belongs to P. We denote the collection of prime subgroups of G by $\mathrm{Spec}(G)$. A typical Zorn's Lemma argument guarantees that for any $1 < g$ there is a convex ℓ-subgroup maximal with respect to not containing g. Such a convex ℓ-subgroup is called a *value of* g, and the collection of values of g is denoted by $\mathrm{Yos}(g)$ and is called the *Yosida space of* g. Every value of g is in fact a prime subgroup. Thus prime subgroups exist. Moreover, it is known that $\bigcap \mathrm{Spec}(G) = \{1\}$. Again by Zorn's Lemma minimal prime subgroups exist.

For a unital ℓ-group (G, u) it is prudent that we point out that the values of u are precisely the maximal proper convex ℓ-subgroups of G, and so as is customary we write $\mathrm{Max}(G)$ instead of $\mathrm{Yos}(u)$. $\mathrm{Max}(G)$ can be equipped with the hull-kernel topology. Recall that the collection of all sets of the form

$$\mathcal{U}(a) = \big\{V \in \mathrm{Max}(G) : \ a \notin V\big\}$$

forms a base for the open sets of the hull-kernel topology. It is well-known that, with regards to the hull-kernel topology, $\mathrm{Max}(G)$ is a compact Hausdorff space. Observe that the operator $\mathcal{U}$ satisfies the following properties:

Lemma 1.1. *For all $g \in G$ and $a, b \in G^+$ the following hold.*

i) $\mathcal{U}(g) = \mathcal{U}(|g|)$;
ii) $\mathcal{U}(a) \cap \mathcal{U}(b) = \mathcal{U}(a \wedge b)$;
iii) $\mathcal{U}(a) \cup \mathcal{U}(b) = \mathcal{U}(a \vee b) = \mathcal{U}(ab) = \mathcal{U}(ba)$.

We denote $\mathrm{Max}(G) \smallsetminus \mathcal{U}(a)$ by $\mathcal{V}(a)$.

Our basic references for the theory of lattice-ordered groups are [6] and [7]. Recall that G is said to be *archimedean* if for every $g, h \in G^+$, whenever $g^n \leq h$ for all $n \in \mathbb{N}$, then $g = 1$. See [6: Chapter 10] for properties of archimedean ℓ-groups. For instance, Theorem 53.3 is the result that all archimedean ℓ-groups are abelian.

2. The inverse topology on $\mathrm{Max}(G)$

Throughout this section we assume that (G, u) is a unital ℓ-group and $u \in G^+$. We begin by pointing out that the operator $\mathcal{V}$ satisfies some interesting properties. For convenience sake we let $r(G) = \bigcap\{M \in \mathrm{Max}(G)\}$ and call this the *radical of* G.

Lemma 2.1. *For all $g \in G$ and $a, b \in G^+$ the following hold.*

i) $\mathcal{V}(g) = \mathcal{V}(|g|)$;

ii) $\mathcal{V}(a) \cap \mathcal{V}(b) = \mathcal{V}(a \vee b) = \mathcal{V}(ab) = \mathcal{V}(ba)$;

iii) $\mathcal{V}(a) \cup \mathcal{V}(b) = \mathcal{V}(a \wedge b)$;

iv) $\mathcal{V}(g) = \emptyset$ *if and only if g is a strong order unit,*

v) $\mathcal{V}(g) = \mathrm{Max}(G)$ *if and only if $g \in r(G)$.*

By condition ii) it follows that the collection $\{\mathcal{V}(a) : a \in G^+\}$ forms a base for a topology on $\mathrm{Max}(G)$. We call this topology the *inverse topology on* $\mathrm{Max}(G)$ and denote it by $\mathrm{Max}(G)^{-1}$. (For information on the inverse topology of the maximal ideal space of a commutative ring with identity the reader should consult [10], and information on the inverse topology on the space of minimal prime subgroups of a lattice-ordered group please consult [9].) If $P, Q \in \mathrm{Max}(G)$ are distinct points then there are elements $p \in P^+ \smallsetminus Q$ and $q \in Q^+ \smallsetminus P$. It follows that $P \in \mathcal{V}(p) \smallsetminus \mathcal{V}(q)$ and $Q \in \mathcal{V}(q) \smallsetminus \mathcal{V}(p)$ and so $\mathrm{Max}(G)^{-1}$ is a T_1-space. Our aim is to show that $\mathrm{Max}(G)^{-1}$ is a zero-dimensional Hausdorff space. We begin by recalling the Riesz Decomposition Theorem for ℓ-groups.

Theorem 2.2 (**The Riesz Decomposition Theorem**). *Let G be an ℓ-group. Suppose $1 \le x, g_1, \dots, g_n$ and $x \le g_1 \cdots g_n$, then there exist $x_1, \dots, x_n \in G^+$ for which $x_i \le g_i$ for each $i = 1, \dots, n$ and $x = x_1 \cdots x_n$.*

Proposition 2.3. *Suppose (G, u) is a unital ℓ-group. The inverse topology on $\mathrm{Max}(G)$ is a zero-dimensional Hausdorff topology. Moreover, the inverse topology on $\mathrm{Max}(G)$ is finer than the hull-kernel topology on $\mathrm{Max}(G)$.*

P r o o f. Let $g \in G^+$ and define

$$R_g = \{x \in G^+ : \ \mathcal{V}(x) \cap \mathcal{V}(g) = \emptyset\}.$$

If $P \in \mathcal{V}(g)$, then for any $x \in R_g$, $P \in \mathcal{U}(x)$. Conversely, suppose that $P \in \mathcal{U}(x)$ for every $x \in R_g$. If $g \notin P$, then the convex ℓ-subgroup generated by P and $G(g)$, say S, must be all of G. Therefore, $u \in S$. By an application of the Riesz Decomposition and the Triangle Inequality, it follows that $u = x_1 \cdots x_k$ for appropriate $x_i \in P \cup G(g)$. Then $\emptyset = \mathcal{V}(u) = \mathcal{V}(x_1 \cdots x_k) = \mathcal{V}(x_1) \cap \cdots \cap \mathcal{V}(x_k)$.

Partition the set $T = \{x_1, \ldots, x_k\}$ into two sets T_1 and T_2, where $x_i \in T_1$ if and only if $x_i \in P$, and set $T_2 = T \smallsetminus T_1$. Set $T_1 = \{y_1, \ldots, y_r\}$. For each $x \in T_2$, $x \in G(g)$ and so $\mathcal{V}(g) \subseteq \mathcal{V}(x)$. It follows that $\mathcal{V}(y_1 \cdots y_r) \cap \mathcal{V}(g) = \emptyset$. Therefore, $y_1 \cdots y_r \in R_g$ and so by hypothesis $y_1 \cdots y_r \notin P$. This contradicts that each $y_i \in P$ and therefore so is the product. Consequently, $\mathcal{V}(g) = \bigcap\limits_{x \in R_g} \mathcal{U}(x)$. Therefore, every basic open set is an intersection of closed sets, and hence clopen; $\operatorname{Max}(G)^{-1}$ is zero-dimensional. We already pointed out that $\operatorname{Max}(G)^{-1}$ satisfies the T_1-separation axiom therefore it is Hausdorff.

As for the last statement a basic open set of the hull-kernel topology is of the form $\mathcal{U}(a)$ for some $a \in G^+$. Since these sets are clopen in the inverse topology it follows that every open set relative to the hull-kernel topology is open relative to the inverse topology. □

Corollary 2.4. *Suppose (G, u) is a unital ℓ-group. For any pair of distinct maximal convex ℓ-subgroups, say P and Q, the set PQ contains a strong order unit.*

Proof. Since $\operatorname{Max}(G)^{-1}$ is Hausdorff and $P, Q \in \operatorname{Max}(G)$ are distinct there are disjoint basic open sets, say $\mathcal{V}(a)$, $\mathcal{V}(b)$, such that $P \in \mathcal{V}(a)$ and $Q \in \mathcal{V}(b)$. Since $\mathcal{V}(ab) = \mathcal{V}(a) \cap \mathcal{V}(b) = \emptyset$ then $ab \in PQ$ is a strong order unit. □

Theorem 2.5. *For a unital ℓ-group G the following statements are equivalent.*

1. *The inverse topology on* $\operatorname{Max}(G)$ *is compact.*
2. *For every $g \in G^+$ there is an $h \in G^+$ such that gh is a strong order unit while $g \wedge h \in r(G)^+$.*
3. $\operatorname{Max}(G) = \operatorname{Max}(G)^{-1}$.

Proof. The proof that (3) implies (1) is patent. Conversely, suppose that $\operatorname{Max}(G)^{-1}$ is compact. Since $\mathcal{U}(g)$ is clopen in the inverse topology it is compact and so $\mathcal{U}(g) = \mathcal{V}(g_1) \cup \cdots \cup \mathcal{V}(g_n) = \mathcal{V}(g_1 \cdots g_n)$ for appropriate $g_1, \ldots, g_n \in G^+$. Thus, for each $g \in G^+$ there is an $h \in G^+$ such that $\mathcal{V}(g) = \mathcal{U}(h)$. Consequently, every basic open set of the inverse topology is open relative to the hull-kernel topology and so the two topologies are the same. Observe also that by what we have just demonstrated together with (iv) and (v) of Lemma 2.1, (3) implies (2). That (2) implies (3) also follows from (2) together with (iv) and (v) of Lemma 2.1. □

When the conditions of Theorem 2.5 are satisfied then the hull-kernel topology on $\operatorname{Max}(G)$ is compact, zero-dimensional, and Hausdorff, i.e. $\operatorname{Max}(G)$ is a boolean space. The next section characterizes when this happens in general.

3. Clean unital ℓ-groups

The notion of a clean object first arose in the theory of rings (see [13]); a ring is called clean if every element is the sum of a unit and an idempotent. Below is the first application of this notion to the theory of ordered groups. It is our aim to characterize a clean unital ℓ-group. For rings the idempotents play an integral part, while for unital ℓ-groups the components of a strong order unit play the central role. In general, we say a positive element $e \in G$ is a *component* of G if there is another positive element, say $f \in G^+$, such that $e \wedge f = 1$ and ef is a strong order unit. Notice that since disjoint elements commute it follows that in this case $ef = fe = e \vee f$. We let $B(G)$ denote the collection of all components of G.

For a fixed positive strong order unit v, if $e \in G^+$ satisfies $e \wedge ve^{-1} = 1$, then we say e is a v-component of G. We denote the set of v-components of G by $B(G, v)$. Observe that

$$B(G) = \bigcup \big\{ B(G, v) : \ v \ \text{ is a strong order unit of } \ G \big\}.$$

Definition 3.1. The unital ℓ-group (G, u) is said to be a *u-clean* ℓ-group if for every $g \in G$ there exists a strong order unit $v \in G$ and a u-component of G, say e, such that $g = ve$.

It would appear that we are defining left u-clean ℓ-groups as it is not clear that the definition is left-right symmetric. But one of the byproducts of Theorem 3.5 is that the notion is, in fact, left-right symmetric. First some examples.

Here are some examples of u-clean ℓ-groups.

1) Suppose (G, u) is a totally-ordered unital ℓ-group. Then G is a clean unital ℓ-group. To see this let $g \in G$ and, without loss of generality, we suppose g is not a strong order unit. Consider gu^{-1}. If $g \leq 1$, then $gu^{-1} \leq u^{-1}$ and hence it is a strong order unit of G. If $1 \leq g$, then since g is not a strong order unit $g^3 \leq u$ and hence $g \leq g^{-1}ug^{-1}$. Next,

$$u \leq ug \leq ug^{-1}ug^{-1} = (ug^{-1})^2$$

whence both ug^{-1} and its inverse gu^{-1} are strong order units. Thus in both cases $g = (gu^{-1})u$ is a u-clean decomposition of g. Consequently, a totally-ordered unital ℓ-group is u-clean for any positive unit u of G.

2) Any finite direct product of unital ℓ-groups is a unital ℓ-group. We leave the proof of this to the interested reader.

Proposition 3.2. *Every (unital) homomorphic image of a clean unital ℓ-group is clean.*

P r o o f. Suppose (G,u) and (H,v) are unital ℓ-groups and $\phi\colon G \to H$ is a surjective ℓ-group homomorphism with $\phi(u)=v$. Suppose further that (G,u) is a clean ℓ-group. Let $h \in H$. Then there is some $g \in G$ for which $\phi(g)=h$. Write $g = we$ where e is a component of u and w is a strong order unit. Then there is some $n \in \mathbb{N}$ such that $u \le |w|^n$; $v \le |\phi(w)|^n$ and so $\phi(w)$ is a strong order unit. We claim that a component of u maps to a component of $\phi(u)=v$. To see this observe that

1) $\phi(e) \wedge v\phi(e)^{-1} = \phi(e) \wedge \phi(u)\phi(e)^{-1} = \phi(e \wedge ue^{-1}) = \phi(1_G) = 1_H$, and
2) $\phi(e) \vee \phi(ue^{-1}) = \phi(u) = v$.

Therefore, $h = \phi(g) = \phi(w)\phi(e)$ is a clean decomposition of h. Consequently, (H,v) is a clean unital ℓ-group. $\square$

Lemma 3.3. *Suppose (G,u) is a unital ℓ-group. Any clopen subset of* $\mathrm{Max}(G)$ *is of the form $\mathcal{U}(e)$ for some component $e \in B(G,u)$. In particular, if $v \in G^+$ is a strong order unit and f is a component of v, then $\mathcal{U}(f)=\mathcal{U}(e)$ for some component e of u.*

P r o o f. Let K be a clopen subset of $\mathrm{Max}(G)$. Since $\mathrm{Max}(G)$ is a compact Hausdorff space it follows that K is compact and since it is open it is of the form $\mathcal{U}(g')$ for some $g' \in G^+$. Similarly, $\mathrm{Max}(G) \smallsetminus K = \mathcal{U}(h')$ for some $h' \in G^+$. Notice that by Lemma 1.1 $g'h'$ is an order unit of G. Also, $g' \wedge h' \in r(G)^+$. Consider $\mathcal{U}(g'(g' \wedge h')^{-1})$. If $N \in \mathcal{U}(g')$, then $N \in \mathcal{U}(g'(g' \wedge h')^{-1})$; and conversely. Therefore, $\mathcal{U}(g') = U(g'(g' \wedge h')^{-1})$. Similarly, $\mathcal{U}(h') = \mathcal{U}(h'(g' \wedge h')^{-1})$. Now, letting $g = g'(g' \wedge h')^{-1}$ and $h = h'(g' \wedge h')^{-1}$ we find that gh is an order unit while $g \wedge h = 1$. Moreover, $\mathcal{U}(g) = K$.

Next, choose n such that $u \le (gh)^n$. Since $g \wedge h = 1$ it follows that g and h commute. Therefore, $u \le g^n h^n$. By the Riesz Decomposition Property there are $u_1, u_2 \in G^+$ such that $u = u_1u_2$ and $u_1 \in G(g), u_2 \in G(h)$. Observe that since $g \wedge h = 1$ it follows that $u_1 \wedge u_2 = 1$. Furthermore, $\mathcal{U}(u_1) = \mathcal{U}(g) = K$. $\square$

Definition 3.4. Recall that a frame is complete lattice, say L, for which the following strengthened distributive law holds: for all $a \in L$ and $S \subseteq L$

$$a \wedge \bigvee S = \bigvee \{a \wedge s : \ s \in S\}.$$

For an ℓ-group G, its lattice $\mathfrak{C}(G)$ is an example of an algebraic frame. Recall that $c \in L$ is said to be compact if whenever $c \le \bigvee S$ for some $S \subseteq L$, then there is a finite number of elements in S, say $s_1, s_2, \ldots, s_n \in S$ such that $c \le s_1 \vee \cdots \vee s_n$. The collection of compact elements of L is denoted by $\mathfrak{K}(L)$. A frame L is called *algebraic* if every element is the supremum of compact elements. An algebraic frame is said to be *coherent* whenever the finite meet of compact elements is compact. For a unital ℓ-group G, its frame of convex ℓ-groups is a coherent frame.

Suppose L is a frame. Banaschewski [2] considered the following condition on a coherent frame, which he termed *weakly zero-dimensional*: if whenever $a, b \in L$ and $1 = a \vee b$ then there exits $c, d \in \mathfrak{K}(L)$ such that $c \leq a, d \leq b$, $c \wedge d = 0$ and $c \vee d = 1$. Banaschewski used this condition to characterize clean rings via their frames of radical ideals.

We now come to the main theorem of this article, a characterization of clean unital ℓ-groups. For commutative rings with identity there are several characterizations of clean rings involving different concepts; see [12: Theorem 1.7].

Theorem 3.5. *Let (G, u) be a unital ℓ-group. The following statements are equivalent.*

1. *(G, v) is a clean unital ℓ-group for every order unit $v \in G^+$.*
2. *The collection $\{\mathcal{U}(e) : e \in B(G)\}$ forms a base for the hull-kernel topology on* $\mathrm{Max}(G)$*.*
3. *The collection $\{\mathcal{U}(e) : e \in B(G, u)\}$ forms a base for the hull-kernel topology on* $\mathrm{Max}(G)$*.*
4. $\mathrm{Max}(G)$ *is a boolean space.*
5. *For each $g \in G$ there is an $e \in B(G, u)$ such that $\mathcal{V}(g) \subseteq \mathcal{U}(g)$ while $\mathcal{V}(ug^{-1}) \cap \mathcal{U}(e) = \emptyset$.*
6. *$\mathfrak{C}(G)$ is a weakly zero-dimensional frame.*
7. *For every pair of distinct maximal convex ℓ-subgroups, say M and N, there is a component of u in exactly one of them.*
8. *(G, u) is a clean unital ℓ-group.*

Proof.

(1) $\implies$ (2). Suppose (G, v) is a clean unital ℓ-group for all order units $v \in G^+$. Suppose $M \in \mathrm{Max}(G)$ and $M \in \mathcal{U}(g)$. Notice that, without loss of generality, we may assume that $g \in G^+$. By an argument similar to the proof of Proposition 2.3 there is some $m \in M^+$ such that mg is an order unit. Choose an order unit $v \in G$ and a component e of mg such that $g = ve$. We claim that $M \in \mathcal{V}(e) \subseteq \mathcal{U}(g)$. Suppose $e \notin M$, then $mge^{-1} \in M$ (by primality). Thus, $v = ge^{-1} \in M$, a contradiction. Therefore, $e \in M$. For any $N \in \mathcal{V}(e)$, then $g \notin N$ since otherwise $w \in N$. Thus, $M \in \mathcal{V}(e) \subseteq \mathcal{U}(g)$, whence the collection $\{\mathcal{U}(e) : e \in B(G)\}$ is a base for the hull-kernel topology on $\mathrm{Max}(G)$.

(2) $\iff$ (3). This follows from Lemma 3.3.

(2) $\implies$ (4). This is patent as for each $e \in B(G)$, $\mathcal{U}(e)$ is a clopen subset of the compact Hausdorff space $\mathrm{Max}(G)$.

(4) $\implies$ (5). Let $g \in G$. Since the sets $\mathcal{V}(g) \cap \mathcal{V}(ug^{-1})$ are disjoint closed subsets of the compact Hausdorff space $\mathrm{Max}(G)$, (4) ensures that there is a basic clopen set K for which $\mathcal{V}(g) \subseteq K$ and $\mathcal{V}(ug^{-1}) \cap K = \emptyset$. By Lemma 3.3, $K = \mathcal{U}(e)$ for some $e \in B(G,u)$.

(5) $\implies$ (1). Let $g \in G$ and fix $v \in G^+$ an order unit. By (5), there is some component $e \in B(G,u)$ such that $\mathcal{V}(g) \subseteq \mathcal{U}(e)$ while $\mathcal{U}(e) \cap \mathcal{V}(ug^{-1}) = \emptyset$. By Lemma 3.3 we may assume that e is a component of v. Consider ge^{-1}. If $N \in \mathrm{Max}(G)$ and $ge^{-1} \in N$. In the case that $e \in N$, then $g \in N$ and so $N \in \mathcal{V}(e) \cap \mathcal{V}(g)$, a contradiction. Thus, $ve^{-1} \in N$ and so $vN = eN = gN$. Consequently, $vg^{-1} \in N$, thus $N \in \mathcal{V}(vg^{-1}) \cap \mathcal{U}(e)$, a contradiction. It follows that ge^{-1} is an order unit. Since $g = (ge^{-1})e$ is a clean expression of g we conclude that G is v-clean.

(3) $\implies$ (6). Suppose $G(g) \vee G(h) = G$ for $g, h \in G^+$. Since $G(g \vee h) = G(g) \vee G(h) = G$ there is some natural $n \in \mathbb{N}$ such that $u \leq (g \vee h)^n$ and so by the Riesz Decomposition Theorem $u = u_1 \cdots u_n$ for $1 \leq u_i \leq g \vee h$.

It follows that $\mathcal{V}(g) \cap \mathcal{V}(h) = \emptyset$ are disjoint closed subsets of $\mathrm{Max}(G)$ and therefore there is a clopen set of the form $\mathcal{V}(e)$ with e a component of u separating $\mathcal{V}(g)$ and $\mathcal{V}(h)$, say $V(g) \subseteq \mathcal{V}(e)$ and $\mathcal{V}(h) \subseteq \mathcal{V}(ue^{-1})$. Set $g' = g \wedge e$ and $h' = h \wedge (ue^{-1})$. Notice that $g' \wedge h' = 1$. Furthermore, $G(g') \vee G(h') = G$. Consequently, $\mathfrak{C}(G)$ is a weakly zero-dimensional frame.

(6) $\implies$ (4). Suppose $M \in \mathrm{Max}(G)$ and $M \in \mathcal{U}(h)$ for $h \in G^+$. Then $G(h) \vee M = G$ and so there are principal convex ℓ-subgroups $G(e) \leq G(h)$ and $G(f) \leq M$ such that $G(e) \vee G(f) = G$ while $G(e) \wedge G(f) = 1$. Without loss of generality, $e, f \in G^+$. We leave it to the interested reader to check that $M \in \mathcal{V}(f) \subseteq \mathcal{U}(h)$. It follows that $\mathrm{Max}(G)$ is a boolean space.

(3) $\implies$ (7). If $M, N \in \mathrm{Max}(G)$ are distinct points then there is some $e \in B(G,u)$ such that $M \in \mathcal{U}(e)$ while $N \notin \mathcal{U}(e)$. It follows that $e \in N \smallsetminus M$.

(7) $\implies$ (4). Suppose $V \subseteq \mathrm{Max}(G)$ and $M \in \mathrm{Max}(G) \smallsetminus V$. For each $N \in V$ there is some $e_N \in B(G,u)$ for which e_N is in exactly one of M or N. By primality, either $e_N \in N \smallsetminus M$ or $ue_N^{-1} \in N \smallsetminus M$. Without loss of generality we may assume that $e_N \in B(G,u)$ and $e_N \in N \smallsetminus M$. It follows that $\{\mathcal{V}(e_N)\}_{N \in V}$ is an open cover of V. Therefore, by compactness there is a finite subcover of V, say $V \subseteq \mathcal{V}(e_{N_1}) \cup \cdots \cup \mathcal{V}(e_{N_k}) = \mathcal{V}(e_{N_1} \vee \cdots \vee e_{N_k})$. It follows that $M \in \mathcal{U}(e_{N_1} \vee \cdots \vee e_{N_k})$ while $\mathcal{U}(e_{N_1} \vee \cdots \vee e_{N_k}) \cap V = \emptyset$. Consequently, the collection of clopen subsets of $\mathrm{Max}(G)$ forms a base for the hull-kernel topology on $\mathrm{Max}(G)$.

(1) $\Longrightarrow$ (8). Obvious.

(8) $\Longrightarrow$ (7). Let $M, N \in \mathrm{Max}(G)$ be distinct maximal convex ℓ-subgroups. Choose $g \in M^+ \smallsetminus N$. By (8) choose $e \in B(G,u)$ and an order unit $v \in G$ such that $g = ve$. It follows that $e \notin N$. We aim to show that $e \in M$. Otherwise, $e^{-1}u \in M$. Then $vu = vee^{-1}u = ge^{-1}u \in M$. However, $u \leq vu$ and thus vu is a strong order unit. Consequently, $e \in M$. □

Remark 3.6. At this point we are unable to construct a proof of the equivalence of (8) and (6) from the previous theorem without using the existence of prime ideals; a choice-free proof. For more information on this topic we urge the readers to check [2].

Remark 3.7. It is true that for any compact topological space X, the ℓ-group of continuous $\mathbb{Z}$-valued functions on X is a clean ℓ-group. Interestingly, it is not a clean ring. Moreover, if G is any hyper-archimedean ℓ-group, then G is a clean unital ℓ-group. To those familiar with algebraic K-theory it follows that for any commutative ring with identity, say R, the Grothendieck ℓ-group $K_0(R)$ (which is isomorphic to $C(\mathrm{Spec}(R), \mathbb{Z})$, is a clean unital ℓ-group. We posit the question of when is $K_0(R)$ a clean ℓ-group for a, not necessarily commutative, ring R.

We conclude this section with a look at MV-algebras. We dispense with the formal definitions of an MV-algebra; instead the reader is urged to peruse the literature. We do point out that the category of MV-algebras is naturally equivalent to the category of abelian unital ℓ-groups. In particular, for each abelian unital ℓ-group, say (G,u), the set

$$\Gamma(G,u) = \{g \in G : \ 1 \leq g \leq u\}$$

is an MV-algebra. Conversely, given an MV-algebra, say A, there is an abelian unital ℓ-group (G,u) for which A and $\Gamma(G,u)$ are MV-isomorphic; Γ is known as the Mundici functor. [3: Proposition 30] characterizes those MV-algebras A for which $\mathrm{Max}(A)$ is a boolean space. It follows that for an abelian unital ℓ-group (G,u), G is a clean ℓ-group if and only if every pure ideal of $\Gamma(G,u)$ is generated by idempotents.

4. Tidy ℓ-groups

In this final section we work in the category **W** consisting of archimedean ℓ-groups with designated weak order unit (G,u) and ℓ-group homomorphisms which preserve the unit. We consider a weakening of the definition of clean ℓ-group that is appropriate when the group is not unital. We begin by recalling the Yosida Representation. First, recall that for a fixed weak order unit $u \in G^+$,

$\mathrm{Yos}(u)$ denotes the set of all values of u equipped with the hull-kernel topology. $\mathrm{Yos}(u)$ is a compact Hausdorff space with respect to the hull-kernel topology. When u is a fixed weak order unit we shall write YG instead of $\mathrm{Yos}(u)$. And when (G, u) is a $\mathbf{W}$-object it is customary to write $\mathrm{coz}(g)$ instead of $\mathcal{U}(g)$ and call such a set a *cozero set*; the complement of a cozero set (in $\mathrm{Yos}(u)$) is called a *zeroset*. The collection of cozero sets of G is denoted by $\mathrm{coz}\, G$.

Let $\overline{\mathbb{R}}$ denote the two-point compactification of $\mathbb{R}$, namely $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. For a Tychonoff space X, that is X is completely regular and Hausdorff, define

$$D(X) = \big\{f\colon X \to \overline{\mathbb{R}} \mid f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is a dense subset of } X\big\}.$$

$D(X)$ is a lattice under pointwise operations, however the pointwise sum of two elements need not exist. For $f, g \in D(X)$, the sum $f + g$ is defined on the dense set $f^{-1}\mathbb{R} \cap g^{-1}\mathbb{R}$, though it need not extend to a continuous function on X. $D(X)$ is a group under pointwise addition precisely when X is a quasi F-space, that is, every dense cozero subset of X is C^*-embedded. The following representation theorem has many incarnations, among them [14] and [8: Theorem 2.7].

Theorem 4.1 (**Yosida Representation Theorem**). *Let (G, u) be a $\mathbf{W}$-object. Then there is an ℓ-isomorphism of G onto an ℓ-group $\widehat{G} \subset D(YG)$ such that $\widehat{G}$ separates the points of YG and $u \mapsto \mathbf{1}$.*

Henceforth, we identify any $\mathbf{W}$-object (G, u) with its image in $D(YG)$.

Definition 4.2. Let (G, u) be a $\mathbf{W}$-object. We call $g \in G$ *u-tidy* if it may be written as the sum $g = v + e$ where v is a weak order unit and e is a component of u. If every element of G is u-tidy, then we say G is u-tidy. When it is clear which weak order unit we are referring to we shall drop any mention of it in the name tidy, e.g. we shall say the the $\mathbf{W}$-object (G, u) is tidy.

Lemma 4.3. *Let (G, u) be a $\mathbf{W}$-object. Then $\mathrm{coz}\, G$ is a base for the topology on YG. Furthermore, if F_0, F_1 are disjoint closed subsets of YG then there is a $g \in G$ with value 0 on F_0 and value 1 on F_1. We may choose $0 \le g \le u$. In particular, if $K \subseteq YG$ is a clopen subset, then the characteristic function χ_K belongs to G.*

Lemma 4.4. *Let (G, u) be a $\mathbf{W}$-object and $e \in B(G, u)$. Then in the Yosida representation $e = \chi_K$ the characteristic function defined on a clopen subset $K \subseteq YG$.*

We can characterize tidy $\mathbf{W}$-objects in a similar way to clean unital ℓ-groups.

Theorem 4.5. *The* **W***-object* (G,u) *is tidy if and only if* YG *is zero-dimensional.*

Proof.

Necessity: Suppose (G,u) is tidy. Let $p \in \operatorname{coz}(a)$ for some $p \in YG$ and $a \in G$. Select $h \in G$ with $0 \leq h \leq u$ for which $h(p) = 1$ and $h(q) = 0$ for all $q \in Z(a)$. Then the sets $Z_0 = h^{-1}\big[0, \frac{1}{3}\big]$ and $Z_1 = h^{-1}\big[\frac{2}{3}, 1\big]$ are disjoint closed subsets of YG satisfying

$$p \in \operatorname{int} Z_1 \subseteq Z_1$$

and

$$Z(a) \subseteq \operatorname{int} Z_0.$$

Let $g \in G$ with $0 \leq g \leq u$ and value i on Z_i $(i = 1,2)$. By hypothesis g is tidy and hence we may write $g = v + e$ where v is a weak order unit and $e = \chi_K$ for some clopen subset K of YG. Let $K' = YG \smallsetminus K$ so that K and K' form a clopen partition of YG.

First we claim that $p \in K'$. Otherwise, $p \in K \cap \operatorname{int} Z_1$ and for any $q \in K \cap \operatorname{int} Z_1$ we get that $1 = g(q) = u(q) + e(q) = u(q) + 1$ whence $K \cap \operatorname{int} Z_1 \subseteq Z(v)$. Therefore $Z(v)$ is not co-dense contradicting our choice of weak-order unit v. Consequently, $p \in K'$.

Next, we demonstrate that $K' \cap \operatorname{int} Z_0 = \emptyset$. Otherwise, since $Z_0 \subseteq Z(g)$ we have for any $q \in K' \cap \operatorname{int} Z_0$

$$0 = g(q) = v(q) + e(q) = v(q).$$

Once again this contradicts our choice of weak order unit v. We conclude that $p \in K' \subseteq \operatorname{coz}(a)$, whence YG is zero-dimensional.

Sufficiency: Suppose YG is zero-dimensional and let $g \in G$. The disjoint closed sets $Z(g)$ and $Z(g-u)$ can be separated by a clopen set; that is, there exists a clopen set $K \subseteq YG$ such that $Z(g-u) \subseteq K$ and $K \cap Z(g) = \emptyset$. Write $K' = YG \setminus K$ and let $v = g - \chi_{K'}$. By the corollary we have that $v \in G$. Clearly, $g = v + \chi_{K'}$. We need only show that v is a weak-order unit.

Consider the following string of set equalities:

$$\begin{aligned} Z(v) &= \big\{q \in YG :\ v(q) = 0\big\} \\ &= \big\{q \in YG :\ g(q) = \chi_{K'}\big\} \\ &= (K \cap Z(g)) \cup (K' \cap Z(g-u)) \\ &= \emptyset \end{aligned}$$

It follows that v is a weak-order unit and, since g was arbitrarily chosen, G is tidy. $\square$

Example 4.6. Here is an example of a **W**-object G with two weak order units, say $u, v \in G^+$, for which $\mathrm{Yos}(u)$ is zero-dimensional yet $\mathrm{Yos}(v)$ is not. Thus, G is u-tidy but not v-tidy. Notice the difference between this example and Theorem 3.5.

Let $X = \mathbb{N}$ and let K be a compactification of X for which $K \setminus X$ is homeomorphic to $[0, 1]$. That such a space exists follows from [11: Theorem 2.2]. Let $u \in C(X)$ be the function defined by $u(n) = \frac{1}{n}$. Define G as follows:

$$G = \big\{ f \in C(X) : \text{ there is some } g \in C(K) \text{ such that } g|_X = uf \big\}.$$

Observe that G is an archimedean ℓ-group. Also, the function $f(n) = n$ belongs to G. By [8: 7.1], since the **W**-objects (G, u) and $(C(K), 1)$ are isomorphic, then $\mathrm{Yos}(u)$ is homeomorphic to K, which is not zero-dimensional since it contains a subspace which is not zero-dimensional. On the other hand $\mathrm{Yos}(u)$ is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$ which is zero-dimensional.

Remark 4.7. In the paper [5] the authors show that a hull class in **W** contains the class of all objects of the form $(C(X), 1)$ with X compact extremally disconnected if and only if the class of compact Hausdorff spaces X for which $(C(X), 1)$ belongs to the hull class is a covering class in the category of compact Hausdorff spaces. Therefore, if the class of tidy **W**-objects were a hull class then the class of boolean spaces would be a hull class, but this not so, as is well known.

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