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ON ALGEBRAS WITH A GENERALIZED IMPLICATION

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ABSTRACT. We introduce the notion of gi-algebra as a generalization of dual BCK-algebra, and define the notions of strong, commutative and transitive gi-algebra, and then we show that an interval $\uparrow l = \{a \in P \mid l \leq a\}$ in a strong and commutative gi-algebra P is a lattice. Also, we define a congruence relation $\sim_D$ on a transitive gi-algebra P and show that the quotient set $P/\!\sim_D$ is a gi-algebra and a dual BCK-algebra.

1. Introduction

A $DBCK$-*algebra* is an algebraic system $(X, \cdot, 1)$ where X is a set, 1 is an element of X, and $\cdot$ is a binary operation on X satisfying the following axioms.

(D1) $(ab)((bc)(ac)) = 1$,

(D2) $a((ab)b) = 1$,

(D3) $aa = 1$,

(D4) $ab = 1$ and $ba = 1$ imply $a = b$,

(D5) $a1 = 1$.

The notion of $DBCK$-algebra was studied in [4, 12] as the dual concept of BCK-algebra ([8, 11, 15]), and it is a poset with the greatest element 1 satisfying the following properties.

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(DS) $a \leq b$ if and only if $ab = 1$,

(T) $ab \leq (bc)(ac)$,

(DE) $a(bc) = b(ac)$.

There are many algebras having the similar structure such as dual BCK-algebra, Hilbert algebra ([7–10]), implicative model ([13,14]) and implication algebra ([1–4,15]). Both Hilbert algebra and implicative model are equivalent. Also it is well known that a commutative Hilbert algebra is exactly an implication algebra ([10]) and the implication algebras are equivalent to the implicative $DBCK$-algebras ([4,15]).

If a binary operation "$\to$" on a Boolean algebra B is defined by $a \to b = \neg a \vee b$, $(B, \to)$ become one among above algebras. As a generalization of such operation, we consider a binary operation "$\circ$" on B defined by $a \circ b = \neg a \vee b \vee u_0$ for a fixed element u_0 of B. This operation has some basic properties of implication, especially, such as $b \leq a \circ b$ and $c \leq a \circ b \implies a \leq c \circ b$, but it satisfies neither $1 \circ a = a$ nor $a \circ b = b \circ a = 1 \implies a = b$. So we propose an algebraic structure equipped with such binary operation $\circ$ as an generalization of algebras that has the implication such as $\to$.

In this paper, we introduce the notion of gi-algebra which is a generalization of $DBCK$-algebra. In Section 2, we define the gi-algebra and the deductive system of it, and give some basic properties of them. In Section 3 and 4, we introduce the notions of strong, commutative and transitive gi-algebra, and we show that an interval $\uparrow l$ in a strong commutative gi-algebra P is a lattice. Also, we define a congruence relation $\sim_D$ on a transitive gi-algebra P and show that the quotient set $P/{\sim_D}$ is a gi-algebra and a $DBCK$-algebra.

2. gi-algebras

Definition 2.1. An *algebra with a generalized implication*, briefly *gi-algebra*, is a poset $(P, \leq)$ with a binary operation "$\cdot$" such that

(I1) $b \leq ab$,

(I2) $a \leq bc \implies b \leq ac$.

The gi-algebra defined in Definition 2.1 has the different notion with an implicative algebra of Rasiowa ([16]) that is an algebra $(P, \cdot, 1)$ where P is a poset with a greatest element 1 and $\cdot$ is a binary operation on P such that $x \leq y$ if and only if $x \cdot y = 1$. The Example 1(1) is a gi-algebra which is not implicative algebra, i.e., $1d = 1$ but $1 \not\leq d$. In Section 3, we deal a strong gi-algebra and it is an implicative algebra.

The following are examples of the gi-algebra.

Example 1.

(1) Let $P = \{a, b, c, d, 1\}$ be a poset with the Hasse diagram in Figure 1.

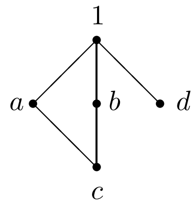

FIGURE 1. Hasse diagram of the poset P

We define a binary operation "$\cdot$" by the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	1
a	1	1	b	b	1
b	1	a	1	a	1
c	1	1	1	1	1
d	1	a	b	c	1

Then $(P, \cdot)$ is a *gi*-algebra.

(2) Let U be a non-empty set and $\mathcal{P}(U)$ the poset of all subsets of U and $X \in \mathcal{P}(U)$. We define a binary operation "$\circ_X$" by

$$A \circ_X B = A^C \cup B \cup X$$

for every $A, B \in \mathcal{P}(U)$. Then $(\mathcal{P}(U), \circ_X)$ is a *gi*-algebra.

The axioms (I1) and (I2) in the definition of *gi*-algebra are independent, as the following example shows.

Example 2. Let $Q = \{0, a, 1\}$ be a poset with the Hasse diagram in Figure 2.

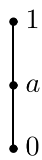

FIGURE 2. Hasse diagram of the poset Q

We define two binary operations "$\circ_1$" and "$\circ_2$" on $(Q, \leq)$, respectively, by the following:

$\circ_1$	1	a	0
1	1	1	a
a	1	1	a
0	1	1	1

$\circ_2$	1	a	0
1	1	0	0
a	1	a	0
0	1	1	1

Then $\circ_1$ satisfies the axioms (I1), but not (I2) because $a \leq 1 \circ_1 0$, but $1 \not\leq a = a \circ_1 0$. Also, $\circ_2$ satisfies the axioms (I2), but not (I1) because $a \not\leq 0 = 1 \circ_2 a$.

The following theorem can be easily verified from the properties (DS) and (DE) and the definition of $DBCK$-algebra.

Theorem 2.1. *Every $DBCK$-algebra is a gi-algebra.*

The converse of Theorem 2.1 is not true. In fact, the *gi*-algebra $(P, \cdot)$ of Example 1(1) is not a $DBCK$-algebra, because $1d = 1$ and $d1 = 1$, but $1 \neq d$.

Proposition 2.2. *A poset P is a gi-algebra if and only if P has the greatest element 1 and satisfies the following properties.*

(1) $aa = 1$,

(2) $a \leq bc \implies b \leq ac$.

Proof. Suppose that P be a *gi*-algebra. Then it has the property (2) from the definition of *gi*-algebra. To show the property (1), let $a \in P$. Then for every $b \in P$, $a \leq ba$ and $b \leq aa$ by (I1) and (I2). That is, aa is the greatest element in P. Hence P has the greatest element 1 and $1 = aa$ for every $a \in P$.

Conversely, suppose that P has the greatest element 1 satisfying the properties (1) and (2). Then we need only to show the property (I1). Let $a, b \in P$. Then $a \leq 1 = bb$ by (1), and this implies $b \leq ab$ by (2). Hence P is a *gi*-algebra. □

Proposition 2.3. *A gi-algebra P has the following properties: for any $a, b, c \in P$,*

(1) $a \leq b \implies ab = 1$,

(2) $a1 = 1$,

(3) $a(ba) = 1$,

(4) $a \leq (ab)b$,

(5) $a \leq b \implies bc \leq ac$,

(6) $((ab)b)b = ab$.

Proof.

(1) Let $a, b \in P$ with $a \leq b$. Since $b \leq 1b$ by (I1), $a \leq 1b$. This implies $1 \leq ab$ by (I2). Hence $ab = 1$.

(2) and (3) are clear from (I1) and (1) of this proposition.

(4) Let $a, b \in P$. Then $ab \leq ab$, and this implies $a \leq (ab)b$ by (I2).

(5) Let $a, b, c \in P$ with $a \leq b$. Since $b \leq (bc)c$ by (4) of this proposition, $a \leq (bc)c$. Hence $bc \leq ac$ by (I2).

(6) Let $a, b \in P$. Then it is clear that $ab \leq ((ab)b)b$ by (4) of this proposition. Also, since $a \leq (ab)b$, $((ab)b)b \leq ab$ by (5) of this proposition. Hence $((ab)b)b = ab$. □

Definition 2.2. Let $(P, \cdot)$ be a gi-algebra. A non-empty subset D of P is called a *deductive system* of P if

(1) $1 \in D$,

(2) $a \in D$ and $ab \in D$ imply $b \in D$.

A non-empty subset Q of P is called a *gi-subalgebra* of P if $ab \in Q$ for every $a, b \in Q$.

For any subset A of a poset P, we write $\uparrow A = \{x \in P \mid a \leq x \text{ for some } a \in A\}$ and $\uparrow a = \uparrow\{a\} = \{x \in P \mid a \leq x\}$.

Proposition 2.4. *If D is a deductive system of a gi-algebra P, then $\uparrow D = D$. In particular, $\uparrow a \subseteq D$ for every $a \in D$.*

Proof. Let D be a deductive system of P. Then it is clear $D \subseteq \uparrow D$. If $x \in \uparrow D$, then $a \leq x$ for some $a \in D$, and by Proposition 2.3(1), $ax = 1$ for some $a \in D$. Since $1 \in D$ and $a \in D$, $x \in D$ by definition of deductive system. It follows $\uparrow D \subseteq D$. Hence $\uparrow D = D$. It is clear that $\uparrow a \subseteq \uparrow D = D$ for every $a \in D$. $\square$

Proposition 2.5. *If D is a deductive system of a gi-algebra P, then*

(1) *$a \in D$ implies $ba \in D$ for every $b \in P$,*

(2) *D is a gi-subalgebra.*

Proof.

(1) Let $a \in D$ and $b \in P$. Then $a \leq ba$ by (I1), and This implies $ba \in \uparrow a \subseteq D$ by Proposition 2.4.

(2) If $a, b \in D$, then $ab \in D$ by (1) of this proposition. Hence D is a gi-subalgebra. $\square$

3. Strong and commutative gi-algebras

In a gi-algebra, the converse of Proposition 2.3(1) is not true. In fact, P of Example 1(1) is a gi-algebra and $1d = 1$, but $1 \not\leq d$.

Definition 3.1. A gi-algebra P is said to be *strong* if

(S) $ab = 1 \implies a \leq b$

for any $a, b \in P$.

The strong gi-algebras are equivalent to the dual weak BCK-algebras introduced by J. Cīrulis ([5]).

Proposition 3.1. *Let P be a gi-algebra. If P is strong, then $1a = a$ for every $a \in P$.*

P r o o f. Let $a \in P$. Then it is clear that $a \le 1a$ by (I1). Also, $1 \le (1a)a$ by Proposition 2.3(4). This implies $(1a)a = 1$, and $1a \le a$ by (S). Hence $1a = a$. $\square$

Definition 3.2. A gi-algebra P is said to be *commutative* if

(C) $(ab)b = (ba)a$

for every $a, b \in P$.

Theorem 3.2. *Let P be a gi-algebra. If P is strong and commutative, then $a \vee b = (ab)b$ for every $a, b \in P$, i.e., P is a join semi-lattice.*

P r o o f. Let $a, b \in P$. Then $(ab)b$ is an upper bound of a and b by (I1) and Proposition 2.3(4). Suppose that u is an upper bound of a and b. Then $a \le u$ and $b \le u$. This implies $bu = 1$ by Proposition 2.3(1), and $ub \le ab$ and $(ab)b \le (ub)b$ by Proposition 2.3(5). Since $(ub)b = (bu)u = 1u = u$ by (C) and Proposition 3.1, $(ab)b \le u$. Hence $(ab)b$ is the least upper bound of a and b. $\square$

The following example is a strong gi-algebra which is not join semi-lattice.

Example 3. Let $P = \{a, b, c, d, 1\}$ be a poset with the Hasse diagram in Figure 3.

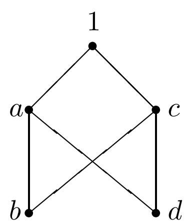

Figure 3. Hasse diagram of the poset P

We define a binary operation "$\cdot$" on $(P, \le)$ by the following:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	1	d
c	1	a	b	1	d
d	1	1	b	1	1

Then P is a strong gi-algebra which is not commutative, because $(ab)b = bb = 1 \ne a = 1a = (ba)a$. Also P has no $b \vee d$.

Theorem 3.3. *Let P be a strong and commutative gi-algebra and $l \in P$. If we define a map $'\colon P \to P$ by $a' = al$ for every $a \in P$, then it has the following properties: for every $a, b \in \uparrow l$.*

(1) $1' = l$ *and* $l' = 1$,

(2) $a \le b$ *implies* $b' \le a'$,

(3) $a'' = a$,

(4) $a \wedge b = (a' \vee b')'$,

(5) $(a \vee b)' = a' \wedge b'$ *and* $(a \wedge b)' = a' \vee b'$.

(6) $\uparrow l$ *is a lattice.*

Proof.

(1) For the greatest element 1, $1' = 1l = l$ by Proposition 3.1. For the element $l \in P$, $l' = ll = 1$ by Proposition 2.3(1).

(2) Let $a, b \in \uparrow l$. If $a \le b$, then $b' = bl \le al = a'$ by Proposition 2.3(5).

(3) Let $a \in \uparrow l$. Then $a'' = (al)l = a \vee l = a$ by Theorem 3.2.

(4) Let $a, b \in \uparrow l$. Then $a' \le a' \vee b'$ and $b' \le a' \vee b'$. This implies $(a' \vee b')' \le a'' = a$ and $(a' \vee b')' \le b'' = b$ by (2) and (3) of this theorem. Since $l \le (a' \vee b')l = (a' \vee b')'$, $(a' \vee b')' \in \uparrow l$. That is, $(a' \vee b')'$ is a lower bound of a and b in $\uparrow l$.

Suppose that c is a lower bound of a and b in $\uparrow l$. Then $c \le a$ and $c \le b$. This implies $a' \le c'$ and $b' \le c'$, hence $a' \vee b' \le c'$ implies $c = c'' \le (a' \vee b')'$ by (3) and (2) of this theorem. It follows that $(a' \vee b')'$ is the greatest lower bound of a and b in $\uparrow l$.

(5) Let $a, b \in \uparrow l$. Then $a', b' \in \uparrow l$, and $a' \wedge b' = (a'' \vee b'')' = (a \vee b)'$ and $(a \wedge b)' = (a' \vee b')'' = a' \vee b'$ by (4) and (3) of this theorem.

(6) It is clear from Theorem 3.2 and (4) of this theorem. □

The map $a \mapsto a'$ of Theorem 3.3 is, according to items (2) and (3), an antitone involution in $\uparrow l$. Upper semilattices with sectionally antitone involutions have been studied in [6]. An alternative proof of (4), (5) and (6) of Theorem 3.3 is mentioned in [5]. According to [5: Lemma 2.5], a terminal segment $\uparrow l$ satisfying the properties (1)–(3) of Theorem 3.3 has also the properties (4)–(5).

4. Transitive *gi*-algebra

Proposition 4.1. *Let P be a gi-algebra. Then the following are equivalent: for any $a, b, c \in P$,*

(T1) $ab \le (ca)(cb)$,

(T2) $ab \le (bc)(ac)$.

P r o o f. Suppose that P satisfies the property (T1). Then $bc \leq (ab)(ac)$ for any $a, b, c \in P$. Hence, by (I2), we have $ab \leq (bc)(ac)$ for any $a, b, c \in P$. In the similar way, we can show that (T2) implies (T1). □

Definition 4.1. A *gi*-algebra P is said to be *transitive* if it satisfies the equivalent axioms of Proposition 4.1.

The axioms (S) and (T2) are independent as the following examples show.

Example 4.

(1) Let $P = \{0, a, b, 1\}$ be a poset with the Hasse diagram in Figure 4.

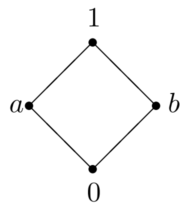

Figure 4. Hasse diagram of the poset P

We define a binary operation "$\cdot$" on $(P, \leq)$ by the following:

$\cdot$	1	a	b	0
1	1	a	b	0
a	1	1	b	a
b	1	a	1	b
0	1	1	1	1

Then it is a strong *gi*-algebra which is not transitive because $ba = a \not\leq b = ab = (a0)(b0)$.

(2) The *gi*-algebra P in Example 1(1) is transitive, which is not strong because $1d = 1$, but $1 \not\leq d$.

Proposition 4.2. *Let P be a gi-algebra. If P is transitive, then P satisfies the following property: for every $a, b, c \in P$,*

(E) $a(bc) = b(ac)$.

P r o o f. Suppose that P is transitive and $a, b, c \in P$. Then $b \leq (bc)c$ by Proposition 2.3(4) and $((bc)c)(ac) \leq b(ac)$ by Proposition 2.3(5). Since $a(bc) \leq ((bc)c)(ac)$ by (T2), $a(bc) \leq b(ac)$. Interchanging the role of a and b, we have $b(ac) \leq a(bc)$. Hence $a(bc) = b(ac)$. □

The converse of Proposition 4.2 is not true as the following example show.

Example 5. Let $P = \{1, a, b, 0\}$ be a poset with the Hasse diagram in Figure 5.

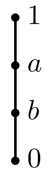

FIGURE 5. Hasse diagram of the poset P

We define a binary operation "$\cdot$" on $(P, \leq)$ by the following:

$\cdot$	1	a	b	0
1	1	a	b	a
a	1	1	a	1
b	1	1	1	1
0	1	1	1	1

Then it is a *gi*-algebra satisfying the property (E), but not transitive because $0b = 1 \not\leq a = 1a = (a0)(ab)$.

THEOREM 4.3. *Let P be a gi-algebra. then P is strong and transitive if and only if P is a $DBCK$-algebra.*

Proof. Suppose that P is strong and transitive. For any $a, b \in P$, $a1 = 1$ and $a((ab)b) = 1$ by (2) and (4) of Proposition 2.3, and $aa = 1$ by Proposition 2.2(1). Hence it satisfies (D5), (D2) and (D3). If $ab = 1$ and $ba = 1$, then $a \leq b$ and $b \leq a$ since P is strong. Hence $a = b$. For any $a, b, c \in P$, $ab \leq (bc)(ac)$ by (T2), and $(ab)((bc)(ac)) = 1$ by Proposition 2.3(1). Hence P is a $DBCK$-algebra.

Conversely, suppose that P is a $DBCK$-algebra. Then P is a *gi*-algebra by Theorem 2.1, and strong and transitive by (DS) and (D1). $\square$

Let P be a *gi*-algebra and D a deductive system of P. We define a binary relation $\sim_D$ on P by

$$a \sim_D b \iff [\, ab \in D \;\; \& \;\; ba \in D\,]$$

for any $a, b \in P$. Then $\sim_D$ is not equivalence relation in general. In Example 5, $D = \{1\}$ is a deductive system, but $\sim_D$ is not equivalence relation, because $a \sim_D 0$ and $0 \sim_D b$, but $a \not\sim_D b$.

THEOREM 4.4. *Let P be a transitive gi-algebra and D a deductive system of P. Then $\sim_D$ is a congruence relation on P.*

Proof. It is clear that $\sim_D$ is reflexive and symmetric. Suppose that $a \sim_D b$ and $b \sim_D c$ with $a, b, c \in P$. Then $bc \leq (ab)(ac)$ by (T1). Since $bc \in D$, $(ab)(ac) \in \uparrow D = D$ by Proposition 2.4. Since D is a deductive system and

$ab \in D$, $ac \in D$. Also, $ba \le (cb)(ca)$ implies $ca \in D$ by the same way. It follows $a \sim_D c$. Hence $\sim_D$ is an equivalence relation on P.

To show that $\sim_D$ is a congruence relation, let $a \sim_D b$ and $c \in P$. Then $ab \le (ca)(cb)$ by (T1). Since $ab \in D$, $(ca)(cb) \in \uparrow D = D$ by Proposition 2.4. Similarly, we have $(cb)(ca) \in D$. That is, $ca \sim_D cb$. Also, we have $ba \le (ac)(bc)$ by (T2). Since $ba \in D$, $(ac)(bc) \in \uparrow D = D$. Similarly, we have $(bc)(ac) \in D$. It follows $ac \sim_D bc$. Hence $\sim_D$ is a congruence relation on P. □

Let P be a transitive *gi*-algebra and D a deductive system of P. We can define a binary relation "$\preceq$" on the quotient set $P/{\sim_D}$ by

$$[a] \preceq [b] \iff \big[\big(\forall x \in [a]\big)\big(\exists y \in [b]\big)(x \le y)\big],$$

where $[a]$ and $[b]$ are equivalence classes with respect to $\sim_D$.

Theorem 4.5. *Let P be a transitive gi-algebra and D a deductive system of P. Then $(P/{\sim_D}, \preceq)$ is a poset.*

Proof. It is clear that $[a] \preceq [a]$ for every $[a] \in P/{\sim_D}$. Let $[a] \preceq [b]$ and $[b] \preceq [a]$. Then $a \le y$ for some $y \in [b]$. Since $y \sim_D b$, $1 = ay \sim_D ab$ by Proposition 2.3(1) and Theorem 4.4. This implies $1(ab) \in D$, and $ab \in D$ since D is a deductive system and $1 \in D$. Similarly, we have $ba \in D$ from $[b] \preceq [a]$. Hence $a \sim_D b$ and $[a] = [b]$.

To show that $\preceq$ is transitive, let $[a] \preceq [b]$ and $[b] \preceq [c]$. Then for each $x \in [a]$, $x \le y$ for some $y \in [b]$, and $y \le z$ for some $z \in [c]$. That is, for each $x \in [a]$, there exists $z \in [c]$ such that $x \le z$. Hence $[a] \preceq [c]$. Therefore $\preceq$ is a partial order on $P/{\sim_D}$. □

Let P be a transitive *gi*-algebra and D a deductive system of P. We define a binary operation "$\circ$" on the quotient set $P/{\sim_D}$ by

$$[a] \circ [b] = [ab]$$

for every $[a], [b] \in P/{\sim_D}$.

Proposition 4.6. *Let P be a transitive gi-algebra and D a deductive system of P. Then the poset $(P/{\sim_D}, \preceq)$ with the above binary operation $\circ$ satisfies the following properties.*

(1) *$[1]$ is the greatest element in $(P/{\sim_D}, \preceq)$,*
(2) *$[1] \circ [a] = [a]$,*
(3) *$[a] \preceq [b] \iff [a] \circ [b] = [1]$,*
(4) *$a \le b$ in $P \implies [a] \preceq [b]$,*
(5) *$[a] \circ ([b] \circ [c]) = [b] \circ ([a] \circ [c])$.*

P r o o f.

(1) Let $[a] \in P/{\sim_D}$. Then $x \leq 1$ for each $x \in [a]$, and $1 \in [1]$. Hence $[a] \preceq [1]$.

(2) Let $a \in P$. Since $a \leq 1a$ by (I1), $a(1a) = 1 \in D$ by Proposition 2.3(1). Also, since $1 \leq (1a)a$, $(1a)a = 1 \in D$. Hence $1a \sim_D a$, and $[1] \circ [a] = [1a] = [a]$.

(3) Suppose that $[a] \preceq [b]$. Since $a \in [a]$, $a \leq x$ for some $x \in [b]$, i.e., $x \sim_D b$. This implies $1 = ax \sim_D ab$ by Proposition 2.3(1) and Theorem 4.4. Hence $[a] \circ [b] = [ab] = [1]$. Conversely, suppose that $[ab] = [a] \circ [b] = [1]$ and $x \in [a]$. Since $x \sim_D a$ and $ab \sim_D 1$, we have $(xb)b \sim_D (ab)b \sim_D 1b$ by Theorem 4.4 and $[1b] = [1] \circ [b] = [b]$ by (2) of this proposition. Thus $(xb)b \in [b]$, and $x \leq (xb)b$ by Proposition 2.3(4). Hence $[a] \preceq [b]$.

(4) Let $a \leq b$ in P. Then $ab = 1$ by Proposition 2.3(1), and this implies $[a] \circ [b] = [ab] = [1]$. Hence $[a] \preceq [b]$ by (3) of this proposition.

(5) Let $a, b, c \in P$. Since P is transitive, $a(bc) = b(ac)$ by Proposition 4.2. Hence $[a] \circ ([b] \circ [c]) = [a(bc)] = [b(ac)] = [b] \circ ([a] \circ [c])$. □

Theorem 4.7. *Let P be a transitive gi-algebra and D a deductive system of P. Then the poset $(P/{\sim_D}, \preceq)$ is a gi-algebra with the binary operation $\circ$.*

P r o o f. The poset $(P/{\sim_D}, \preceq)$ has the greatest element $[1]$ by Proposition 4.6(1) and satisfies $[a] \circ [a] = [1]$ by Proposition 4.6(3). Let $[a] \preceq [b] \circ [c]$. Then $[1] = [a] \circ ([b] \circ [c]) = [b] \circ ([a] \circ [c])$ by (3) and (5) of Proposition 4.6. Hence $[b] \preceq [a] \circ [b]$ by Proposition 4.6(3). Hence $(P/{\sim_D}, \preceq, \circ)$ is a *gi*-algebra by Proposition 2.2. □

Theorem 4.8. *Let P be a transitive gi-algebra and D a deductive system od P. Then $(P/{\sim_D}, \circ)$ is a DBCK-algebra.*

P r o o f. It is clear that $(P/{\sim_D}, \preceq, \circ)$ is a strong *gi*-algebra by Proposition 4.6(3) and Theorem 4.7. Let $a, b, c \in P$. Then $ab \leq (ca)(cb)$ since P is transitive. It follows $[a]\circ[b] = [ab] \preceq [(ca)(cb)] = ([c]\circ[a])\circ([c]\circ[b])$ by Proposition 4.6(4). That is, $P/{\sim_D}$ is transitive. Hence $(P/{\sim_D}, \circ)$ is a $DBCK$-algebra by Theorem 4.3. □

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