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EVERY SET IS "SYMMETRIC" FOR SOME FUNCTION

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ABSTRACT. Let $\langle h_n\rangle$ denote a sequence of positive real numbers. We show that for every set $A \subseteq \mathbb{R}$ there exists a function $f\colon \mathbb{R} \to \omega$ such that $A = \big\{x \in \mathbb{R} : \big(\exists\langle h_n\rangle\big)\big[h_n \searrow 0 \ \ \& \ \ (\forall\, n \in \mathbb{N})\big(f(x-h_n) = f(x+h_n) = f(x)\big)\big]\big\}$. This solves a problem of K. Ciesielski, K. Muthuvel and A. Nowik.

1. Notations and definitions

1.1. Symmetric operations defined on sets

Definition 1.1. For $A \subseteq \mathbb{R}$ a set of symmetric concentration is a set of points $x \in \mathbb{R}$ for which there exists a sequence of positive numbers $\langle h_n\rangle$ converging to 0 such that $x - h_n, x + h_n \in A$. We denote this set by $\mathsf{SC}(A)$.

Notations. For $A \subseteq \mathbb{R}$ let

$$A^c = \{x \in \mathbb{R}\colon x \notin A\} \quad \text{and} \quad \mathsf{SD}(A) = \mathsf{SC}(A) \cap A.$$

Notice that for every $X \subseteq \mathbb{R}$ there exists $Z \subseteq \mathbb{R}$ such that $\mathsf{SC}(Z) = X$ ([4]). To the contrast, we will show that not every subset of the real line is of the the form $\mathsf{SD}(A)$ for some $A \subseteq \mathbb{R}$ (Example 2). But it turned out that every set $X \subseteq \mathbb{R}$ is a countable union of sets of the form $\mathsf{SD}(A_n)$ for $n \in \omega$ (Corollary 3.1.1).

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1.2. Symmetric operations defined on functions

The following definitions can be found in [1], [2] or [4]:

Definition 1.2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly symmetrically continuous* at a point x if there exists a sequence $\langle h_n \rangle$ of positive numbers converging to 0 such that

$$\lim_{n\to\infty} f(x+h_n) - f(x-h_n) = 0.$$

For $f\colon \mathbb{R} \to \mathbb{R}$ let $S(f)$ denote the set of all points at which f is weakly symmetrically continuous.

Let us formulate a stronger definition than Definition 1.2:

Definition 1.3. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly symmetric* at $x \in \mathbb{R}$ if there exists a sequence $\langle h_n \rangle$ of positive numbers converging to 0 such that

$$\underset{n}{\forall}\ f(x-h_n) = f(x+h_n) = f(x).$$

Analogously, for $f\colon \mathbb{R} \to \mathbb{R}$ let $T(f)$ be the set of all $x \in \mathbb{R}$ such that f is weakly symmetric at x.

Theorem 1.1. ([4]) *Any set $A \subset \mathbb{R}$ is the set of points of weakly symmetricaly continuity for some function $f\colon \mathbb{R} \to \omega$, so using the previous notation we have:*

$$\underset{A\subseteq\mathbb{R}}{\forall}\ \underset{f\colon \mathbb{R}\to\omega}{\exists}\ S(f) = A.$$

Notice that for the sake of simplicity we abuse the notation from [1] and [2].

The aim of this paper is to prove an analogous result to Theorem 1.1 with $T(f)$ instead of $S(f)$. This solves a problem from [2] and [3].

2. Examples

Example 1. There exists a set $A \subseteq \mathbb{R}$ such that $\mathsf{SC}(A) = A^c$ and A does not contain an arithmetic progressions of length 3.

P r o o f. Let $\mathbb{R} = \{x_\alpha :\ \alpha < 2^\omega\}$. By induction on α we construct sets of reals $A^*_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$, $B^*_\alpha = \bigcup\limits_{\beta<\alpha} B_\beta$ for $\alpha \le 2^\omega$, and A_α, B_α for $\alpha < 2^\omega$ such that

(∗) $B^*_\alpha \subseteq \mathsf{SC}(A^*_\alpha)$, $B^*_\alpha = \{x_\beta\colon \beta < \alpha\} \setminus A^*_\alpha = \emptyset$, and A^*_α does not contain arithmetic progressions of length 3; hence $A^*_\alpha \cap \mathsf{SC}(A^*_\alpha) = \emptyset$.

Set $A_0 = \left\{x_0 \pm \frac{1}{n}\right\}$, $A^*_0 = \emptyset$, $B^*_0 = \{x_0\}$, $B_0 = \emptyset$. If $x_\alpha \in A^*_\alpha$, we let $A_\alpha = A^*_\alpha$, $B_\alpha = B^*_\alpha$. If $x_\alpha \notin A^*_\alpha$, by induction on n we select a sequence $h_n \searrow 0$ such that

$$x_\alpha \pm h_n \notin B^*_\alpha \cup \Big\{z :\ (\exists\, x)(\exists\, y)\Big[\Big(z = \tfrac{x+y}{2} \quad\vee\quad z = x + 2(y-x)\Big) \\ \&\ \ x, y \in A^*_\alpha \cup \{x_\alpha \pm h_k\colon k < n\}\Big]\Big\}$$

(such sequence h_n can be found since $|A^*_\alpha|, |B^*_\alpha| < 2^\omega$). We set $A_\alpha = A^*_\alpha \cup \{x_\alpha \pm h_n : n \in \omega\}$ and $B_\alpha = B^*_\alpha \cup \{x_\alpha\}$. Let us define $A = A^*_{2^\omega}$. $\square$

Example 2. There exists a set $A \subseteq \mathbb{R}$ such that there is no $Z \subseteq \mathbb{R}$ such that $\mathsf{SD}(Z) = A$.

P r o o f. The set A from Example 1 has this property. To see this it is enough to verify that if $\mathsf{SC}(A) = A^c$ and $A \subseteq \mathsf{SD}(Z)$, then $\mathsf{SC}(Z) = \mathbb{R}$.

If $A \subseteq \mathsf{SD}(Z)$, then $A \subseteq Z$, by definition of SD. Then $A \subseteq \mathsf{SC}(Z)$ and $A^c = \mathsf{SC}(A) \subseteq \mathsf{SC}(Z)$. Therefore $\mathsf{SC}(Z) = \mathbb{R}$. $\square$

On the other hand, let us recall the following result, which shows that in the case of SC the situation is different:

Theorem 2.1. ([4: Lemma 8]) *For every $X \subseteq \mathbb{R}$ there exists $Z \subseteq \mathbb{R}$ such that $\mathsf{SC}(Z) = X$.*

3. Main result

Our main result is Theorem 3.1. To prove it we prove the following two lemmas.

Lemma 3.1. *For each $a \in \mathbb{R}$ and every $X \subseteq \mathbb{Q} + a$ there exists a function $f\colon \mathbb{Q} + a \to \omega$ such that $T(f) = X$.*

P r o o f. Without loss of generality we can assume that $a = 0$.

Let us define a partially ordered set $\mathbb{P}$ as the set of all functions $s\colon A \to \omega$ such that

(1) $A \subseteq \mathbb{Q}$ is a finite set,

(2) if $x \in A$ and $h > 0$ are such that $x \pm h \in A$ and $s(x-h) = s(x) = s(x+h)$ then $x \in X$

ordered by $s \le t$ if $t \subseteq s$.

For each $q \in \mathbb{Q}$, the set $D_q = \big\{s \in \mathbb{P}\colon q \in \operatorname{dom}(s)\big\}$ is a dense subset of $\mathbb{P}$. Indeed, if $q \notin \operatorname{dom}(s)$, choose $n \in \omega \setminus \operatorname{ran}(s)$ and consider the extension $t = s \cup \{(q, n)\}$ of s. Then $t \in D_q$.

For each $q \in X$ and $m > 0$, the set

$$E^m_q = \Big\{s \in \mathbb{P}\colon \underset{r\in\mathbb{Q}}{\exists}\ 0 < r < \tfrac{1}{m}\ \ \&\ \ q \pm r, q \in \operatorname{dom}(s)\ \ \&\ \ s(q \pm r) = s(q)\Big\}$$

is a dense subset of $\mathbb{P}$. To see this let $s \in D_q$ be arbitrary, i.e., $q \in \operatorname{dom}(s)$. Choose $r \in \mathbb{Q}$ such that $0 < r < \frac{1}{m}$ and

(2′) $q \pm r \notin \operatorname{dom}(s) \cup \big\{\frac{x+y}{2}\colon x, y \in \operatorname{dom}(s)\big\} \cup \{2x - y\colon x, y \in \operatorname{dom}(s)\}$,

(2″) $2(q \pm r) - (q \mp h) \notin \operatorname{dom}(s)$.

Let $t = s \cup \{(q-r, s(q)), (q+r, s(q))\}$. The conditions $(2')$ and $(2'')$ guarantee that the extension t of s is a member of $\mathbb{P}$ and hence $t \in E_q^m$.

By Rasiowa-Sikorski theorem there is a generic filter $G \subseteq \mathbb{P}$ for $\{D_q \colon q \in \mathbb{Q}\} \cup \{E_m^q \colon m > 0 \ \ \& \ \ q \in X\}$ i.e.,

(i) if $s \in G$ and $s \leq t \in \mathbb{P}$, then $t \in G$;

(ii) if $s, t \in G$, then there is $u \in G$ such that $u \leq s$ and $u \leq t$;

(iii) $\underset{q \in \mathbb{Q}}{\forall} \ G \cap D_q \neq \emptyset$;

(iv) $\underset{q \in X}{\forall} \ \underset{m>0}{\forall} \ G \cap E_q^m \neq \emptyset$.

Let $f = \bigcup G$. Then f is a function by (ii), $\operatorname{dom}(f) = \mathbb{Q}$ by (iii), $X \subseteq T(f)$ by (2), and $T(f) \subseteq X$ by (iv). $\square$

Lemma 3.2. *There exists a function $p \colon \mathbb{R} \to \omega$ such that*

(1) $\underset{x,y \in \mathbb{R}}{\forall} \ x - y \in \mathbb{Q} \implies p(x) = p(y)$*;*

(2) *If $x - y \notin \mathbb{Q}$ then we do* <u>*not*</u> *have $p\left(\frac{x+y}{2}\right) = p(x) = p(y)$.*

P r o o f. As $\mathbb{R}/\mathbb{Q}$ and $\mathbb{R}$ are isomorphic linear spaces over $\mathbb{Q}$ (both bases have cardinality $\mathfrak{c}$), let $L \colon \mathbb{R}/\mathbb{Q} \to \mathbb{R}$ be any such isomorphism. Let $\{P_n : \ n \in \omega\}$ be a partition of $\mathbb{R}$ such that no P_n contains an arithmetic progression of length 3 (see for example the partition from [1: Thm 1.1]) and define a function $f \colon \mathbb{R} \to \omega$ by $f(x) = n \iff x \in P_n$. The function f has the property that if $x \neq y$ then we do not have $f(x) = f\left(\frac{x+y}{2}\right) = f(y)$. We define $p(x) = f(L([x]_{\mathbb{Q}}))$, where $[x]_{\mathbb{Q}}$ is a class of x in the quotient space $\mathbb{R}/\mathbb{Q}$. $\square$

Next theorem gives a solution to [2: Problem 6] and [3: Problem 2.6].

Theorem 3.1. *For every $X \subseteq \mathbb{R}$ there exists a function $f \colon \mathbb{R} \to \omega$ such that $T(f) = X$.*

P r o o f. We construct a function f from $\mathbb{R}$ to $\omega \times \omega$ instead of ω. Let us choose for every $[x]_{\mathbb{Q}} = B \in \mathbb{R}/\mathbb{Q}$ by virtue of Lemma 3.1 a function $f_B \colon B \to \omega$ such that $T(f_B) = B \cap X$. Define a function: $f \colon \mathbb{R} \to \omega \times \omega$ by $f(x) = (f_{[x]_{\mathbb{Q}}}(x), p(x))$, where p is from Lemma 3.2. We verify that f is a desired function. Combining Lemma 3.1 and condition (1) in Lemma 3.2 we see that $X \subseteq T(f)$ $(X \cap [x]_{\mathbb{Q}} \subseteq T(f)$ for all $x \in \mathbb{R})$ and condition (2) of Lemma 3.2 gives $T(f) \subseteq X$. $\square$

Corollary 3.1.1. *For every set $X \subseteq \mathbb{R}$ there is a countable partition of $\mathbb{R}$ into sets A_n such that $X = \bigcup\limits_{n \in \omega} \mathsf{SD}(A_n)$.*

P r o o f. Let $f \colon \mathbb{R} \to \omega$ be such that $T(f) = X$ and let $A_n = f^{-1}(\{n\})$. $\square$

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REFERENCES

[1] CIESIELSKI, K.—LARSON, L.: *Uniformly antisymmetric functions*, Real Anal. Exchange **19** (1993-94), 226–235.

[2] CIESIELSKI, K.—MUTHUVEL, K.—NOWIK, A.: *On nowhere weakly symmetric functions and functions with two-element range*, Fund. Math. **168** (2001), 119–130.

[3] NOWIK, A.—SZYSZKOWSKI, M.: *Points of weak symmetry*, Real Anal. Exchange **32** (2007), 563–568.

[4] SZYSZKOWSKI, M.: *Points of weak symmetric continuity*, Real Anal. Exchange **24** (1998-99), 807–813.

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