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VECTOR FIELDS ON CERTAIN QUOTIENTS OF COMPLEX STIEFEL MANIFOLDS

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ABSTRACT. We consider quotients of complex Stiefel manifolds by finite cyclic groups whose action is induced by the scalar multiplication on the corresponding complex vector space. We obtain a description of their tangent bundles, compute their mod p cohomology and obtain estimates for their span (with respect to their standard differentiable structure). We compute the Pontrjagin and Stiefel-Whitney classes of these manifolds and give applications to their stable parallelizability.

1. Introduction

Let $W_{n,k}, 1 \le k < n$, denote the complex Stiefel manifold of unitary k-frames $(v_1, \ldots, v_k)$ in $\mathbb{C}^n$ where it is understood that $\mathbb{C}^n$ has the standard hermitian metric. One has the identification $W_{n,k} = U(n)/U(n-k)$ where $U(n)$ denotes the group of unitary transformations of $\mathbb{C}^n$ and $U(n-k)$ is imbedded in $U(n)$ as the subgroup that fixes the first k standard basis vectors $e_1, \ldots, e_k \in \mathbb{C}^n$.

One also has the complex projective Stiefel manifold $PW_{n,k}$ defined as the quotient of $W_{n,k}$ modulo the free action of the circle group $\mathbb{S}^1$ which acts via scalar multiplication: $z(v_1, \ldots, v_k) = (zv_1, \ldots, zv_k)$ for $(v_1, \ldots, v_k) \in W_{n,k}$ and $z \in \mathbb{S}^1$. Note that $PW_{n,k} = U(n)/(\mathbb{S}^1 \times U(n-k))$ where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is identified with the centre of $U(n)$. Observe that $\mathbb{S}^1 \times U(n-k) = U(1) \times U(n-k) \subset U(k) \times U(n-k) \subset U(n)$ where $U(1) \subset U(k)$ is the centre of $U(k)$ and $U(k) \times U(n-k)$ is the subgroup of $U(n)$ that stabilizes the complex vector

subspace $\mathbb{C}^k$, spanned by $e_1, \ldots, e_k$. Thus we get an equivalent description $PW_{n,k} = U(n)/(U(1) \times U(n-k))$.

We define $W_{n,k;m}$ to be the quotient of $W_{n,k}$ by the subgroup $\Gamma_m \subset \mathbb{S}^1$ of mth roots of unity. Thus $\dim W_{n,k;m} = \dim W_{n,k} = k(2n-k)$. The manifolds $W_{n,k;m}$ will be referred to as the *m-projective Stiefel manifolds*. Clearly $W_{n,k;m}$ is the coset space $U(n)/(\Gamma_m \times U(n-k))$ and the obvious quotient map $W_{n,k;m} \longrightarrow PW_{n,k}$ is the projection of a principal bundle with fibre and structure group $\mathbb{S}^1/\Gamma_m \cong \mathbb{S}^1$. Also the projection $W_{n,k} \longrightarrow W_{n,k;m}$ is a covering map with deck transformation group Γ_m. In particular $\pi_1(W_{n,k;m}) \cong \Gamma_m$ and the Euler characteristic $\chi(W_{n,k;m})$ vanishes. The manifold $W_{n,k;m}$ is orientable since Γ_m is a subgroup of the connected group $\mathbb{S}^1$ which preserves the orientation on $W_{n,k}$. Our aim in this paper is to initiate the study of the topology of $W_{n,k;m}$. In §2 we describe their tangent bundle and give (in Theorem 2.4) estimates for their span and stable span. Span and other related notions will be recalled in §2; see also [10]. We compute, in §3, the mod p cohomology of $W_{n,k;m}$. We also determine the height of the generator of $H^2(W_{n,k;m}; \mathbb{Z}) \cong \mathbb{Z}_m$. We show that, given n, k where $1 \le k < n-1$, $W_{n,k;m}$ is not stably parallelizable for all but finitely many values of m. See Theorem 3.5 for the precise statement. When $k = n-1$, $W_{n,n-1;m}$ is parallelizable, since $W_{n,n-1} \cong SU(n-1)$.

The case $k = 1$ corresponds to the (standard) lens space $L^n(m) = \mathbb{S}^{2n-1}/\Gamma_m$. The non-parallelizability of spheres $W_{n,1} = \mathbb{S}^{2n-1}$, $n \ne 1, 2, 4$, already implies non-parallelizability of the lens spaces $L^n(m)$ for any m. Kambe's [9: §4] result on immersion dimension for $L^n(p)$, p an odd prime, and the fact that $L^n(2) = \mathbb{R}P^{2n-1}$ are not stably parallelizable except when $n = 1, 2, 4$, implies that 'most' of $L^n(m)$, $m > 1$, are not *stably* parallelizable. From the celebrated work of Adams, we know that $\mathrm{span}(L^n(m)) \le \mathrm{span}(\mathbb{S}^{2n-1}) = \rho(2n) - 1$, where $\rho(n)$ is the Radon-Hurwitz number, defined as $\rho((2c+1)2^{4a+b}) = 8a + 2^b$, where $a, c \ge 0$ and $0 \le b \le 3$. See also [8] for lower bounds for span of lens spaces.

In view of this, we assume that $1 < k < n$ leaving out the case of lens spaces from consideration for the most part.

Our proofs involve standard arguments making use of well-known results and techniques. The description of the tangent bundle of $W_{n,k;m}$ relies on the description of the tangent bundle of $PW_{n,k}$ due to Lam [13]. Estimates for (stable) span involve well-known arguments such as those employed in the context of real projective Stiefel manifolds; see [10], [11]. The cohomology calculations involve spectral sequences and known results concerning the cohomology of Stiefel manifolds and of projective Stiefel manifolds (see [4] and [2]).

2. The tangent bundle of $W_{n,k;m}$

We describe below certain canonical vector bundles over the manifold $W_{n,k;m}$ and establish relations among them. We shall describe its tangent bundle and obtain lower bounds for their span and stable span.

Let $1 \leq k < n$ and let $m \geq 2$. Let $\Gamma_m \subset U(1)$ denote the group of mth roots of unity. Let $\pi_m \colon W_{n,k;m} \longrightarrow PW_{n,k}$ and $\pi_1 \colon W_{n,k} \longrightarrow PW_{n,k}$ be the canonical quotient maps. These are projections of principal bundles with structure groups $U(1)/\Gamma_m$ and $U(1)$ respectively. Let $p_m \colon W_{n,k} \longrightarrow W_{n,k;m}$ be the quotient map which is the universal covering projection with deck transformation group Γ_m. One also has the obvious covering projections $p_{m,l} \colon W_{n,k;l} \longrightarrow W_{n,k;m}$ whenever $l \mid m$. Note that $\pi_1 = \pi_m \circ p_m$ and $p_m = p_{m,l} \circ p_l$. We shall denote by $[v_1, \ldots, v_k]_m$ (or simply $[v_1, \ldots, v_k]$ when there is no danger of confusion) the element $\pi_m(v_1, \ldots, v_k) \in W_{n,k;m}$ where $(v_1, \ldots, v_k) \in W_{n,k}$. Also, $\pi_1(v_1, \ldots, v_k) \in PW_{n,k}$ will be denoted $[v_1, \ldots, v_k]_0$ (or more briefly $[v_1, \ldots, v_k]$).

Let $\zeta_{n,k}$ denote the complex line bundle over $PW_{n,k}$ associated to the principal $U(1)$-bundle $\pi_1 \colon W_{n,k} \longrightarrow PW_{n,k}$. Thus, the total space of $\zeta_{n,k}$ is the fibre product $W_{n,k} \times_{U(1)} \mathbb{C}$. It is isomorphic to the bundle over $PW_{n,k}$ whose fibre over a point $[v_1, \ldots, v_k]_0$ is the complex vector space $\mathbb{C}v_1 \subset \mathbb{C}^n$. Define $\xi_{n,k;m} := \pi_m^*(\zeta_{n,k})$ and let $\gamma_{n,k;m}$ be the complex line bundle associated to the principal $U(1)$-bundle obtained by extension of structure group via the character $\Gamma_m \subset U(1)$ of the Γ_m-bundle $W_{n,k} \longrightarrow W_{n,k;m}$. (When $m = 2$, $\xi_{n,k;m}$ is the complexification of the real line bundle associated to the double cover $W_{n,k} \longrightarrow W_{n,k;2}$.) Explicitly, $\gamma_{n,k;m}$ has total space the twisted product $W_{n,k} \times_{\Gamma_m} \mathbb{C}$ where Γ_m operates on $\mathbb{C}$ by scalar multiplication. We have the following lemma. We outline a proof, which is elementary, as the lemma will be used throughout. For any vector bundle η, η^l denotes the l-fold tensor product with itself and $l\eta$, the l-fold Whitney sum with itself.

LEMMA 2.1.

  (i) *The complex line bundle associated to the principal $U(1)/\Gamma_m \cong U(1)$-bundle with projection $W_{n,k;m} \longrightarrow PW_{n,k}$ is isomorphic to $\zeta_{n,k}^m$.*

  (ii) *One has an isomorphism $\xi_{n,k;m} \cong \gamma_{n,k;m}$ of complex line bundles over $W_{n,k;m}$.*

P r o o f.

  (i) The isomorphism $U(1)/\Gamma_m \longrightarrow U(1)$ is induced by the homomorphism $z \mapsto z^m$ of $U(1)$ onto itself. This homomorphism induces the map $\eta \mapsto \eta^m$, for any line bundle η associated to a principal $U(1)$-bundle. By definition, $\zeta_{n,k}$ is associated to the principal $U(1)$-bundle $\pi_1 \colon W_{n,k} \longrightarrow PW_{n,k}$. Since

$\pi_m \colon W_{n,k;m} \longrightarrow PW_{n,k}$ is the $U(1)/\Gamma_m$-bundle associated to π_1, it follows that the complex line bundle associated to the principal $U(1)/\Gamma_m \cong U(1)$-bundle π_m is $\zeta_{n,k}^m$.

(ii) By the very definition of $\gamma_{n,k;m}$, its total space has the description $E(\gamma_{n,k;m}) = \big\{ [x,z] \mid x \in W_{n,k}, \ z \in \mathbb{C} \big\}$ where $[x,z] = [x',z']$ if and only if $xg = x', g^{-1}z = z'$ for some $g \in \Gamma_m$. Also, one has $E(\xi_{n,k}) = \big\{ (p_m(v), tv_1) \mid t \in \mathbb{C}, \ x = (v_1, \ldots, v_k) \in W_{n,k} \big\}$. Consider the map $f \colon E(\xi_{n,k;m}) \longrightarrow E(\gamma_{n,k;m})$ defined as $(p_m(v), tv_1) \mapsto [v;t]$. It is readily checked that this is a well-defined continuous map that covers the identity map of the base space $W_{n,k;m}$, and is a linear isomorphism on each fibre. This completes the proof. $\qquad\square$

Observe that $\xi_{n,k;m}^l$, which corresponds to the character $\Gamma_m \longrightarrow U(1), z \mapsto z^l$, is non-trivial when $1 \leq l < m$. In particular it follows that the order of the class of $\xi_{n,k;m}$ in the Picard group $\mathrm{Pic}(W_{n,k;m})$ of $W_{n,k;m}$ is m. Indeed, $\xi_{n,k;m}$ is a generator of $\mathrm{Pic}(W_{n,k;m}) \cong \mathbb{Z}_m$. For, one has $H_1(W_{n,k;m}; \mathbb{Z}) \cong \pi_1(W_{n,k;m}) \cong \mathbb{Z}_m$. It is not difficult to see that $H_2(W_{n,k;m}; \mathbb{Z}) = 0$. Hence $\mathrm{Pic}(W_{n,k;m}) \cong H^2(W_{n,k;m}; \mathbb{Z}) \cong \mathbb{Z}_m$ by the universal coefficient theorem. The projection $\pi_m \colon W_{n,k;m} \longrightarrow PW_{n,k}$ induces a surjection $H^2(PW_{n,k}; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathbb{Z}_m \cong H^2(W_{n,k;m}; \mathbb{Z})$ and hence maps the generator $c_1(\zeta_{n,k})$ to the generator of $\mathbb{Z}_m$. By the naturality of Chern classes we see that $c_1(\xi_{n,k;m})$ is a generator of $H^2(W_{n,k;m}; \mathbb{Z}) \cong \mathbb{Z}_m$. Summarising we have

LEMMA 2.2. *The Picard group* $\mathrm{Pic}(W_{n,k;m})$ *is isomorphic to* $\mathbb{Z}_m$ *and is generated by* $\xi_{n,k;m}$.

The following isomorphism of complex vector bundles on $PW_{n,k}$ is well-known and is due to K.-Y. Lam [13]: $k\zeta_{n,k} \oplus \beta_{n,k} \cong n\varepsilon_{\mathbb{C}}$ where $\varepsilon_{\mathbb{C}}$ denotes the trivial complex line bundle and $\beta_{n,k}$ is the complex $(n-k)$-plane bundle whose fibre over $[v_1, \ldots, v_k]_0$ is the vector space $\{v_1, \ldots, v_k\}^\perp \subset \mathbb{C}^n$ where the orthogonal complement is taken with respect to the standard hermitian inner product on $\mathbb{C}^n$. Pulling back to $W_{n,k;m}$ via the projection $\pi_{n,k;m}$ we obtain an isomorphism

$$k\xi_{n,k;m} \oplus \beta_{n,k;m} \cong n\varepsilon_{\mathbb{C}} \tag{1}$$

of complex vector bundles over $W_{n,k;m}$ where $\beta_{n,k;m} := \pi_m^*(\beta_{n,k})$. Tensoring with the dual bundle $\xi_{n,k;m}^\vee \cong \xi_{n,k;m}^{m-1}$ we get $k\varepsilon_{\mathbb{C}} \oplus \xi_{n,k;m}^\vee \otimes_{\mathbb{C}} \beta_{n,k;m} \cong n\xi_{n,k;m}^\vee = n\xi_{n,k;m}^{m-1}$. Taking duals, we obtain

$$k\varepsilon_{\mathbb{C}} \oplus \xi_{n,k;m} \otimes_{\mathbb{C}} \beta_{n,k;m}^\vee \cong n\xi_{n,k;m}. \tag{2}$$

Recall from [13: Theorem 3.2] that the tangent bundle $\tau PW_{n,k}$ of $PW_{n,k}$ is isomorphic to the (real) vector bundle $k\zeta_{n,k}^\vee \otimes_{\mathbb{C}} \beta_{n,k} \oplus (k^2 - 1)\varepsilon_{\mathbb{R}}$. Since

$\pi_m \colon W_{n,k;m} \longrightarrow PW_{n,k}$ is a *principal* $\mathbb{S}^1$-bundle, we have

$$\tau W_{n,k;m} \cong k\xi^{\vee}_{n,k;m} \otimes_{\mathbb{C}} \beta_{n,k;m} \oplus k^2\varepsilon_{\mathbb{R}}. \tag{3}$$

In the above isomorphism, and in the sequel, we have used the same symbol to denote a complex vector bundle and its underlying real vector bundle, as there is no risk of confusion.

Remark 2.3. Assume that k is even, equivalently $W_{n,k;m}$ is even dimensional. Then $\tau W_{n,k;m}$ has a complex structure arising from the isomorphism of vector bundles given in (3). Thus $W_{n,k;m}$ admits an almost complex structure. Recall that, by the work of Wang [17], $W_{n,k} = SU(n)/SU(n-k)$ admits a complex structure invariant under the left action of $SU(n)$. When m divides n, Γ_m is contained in the centre Γ_n of $SU(n)$. In this case the action of Γ_m on $W_{n,k}$ preserves the complex structure. We conclude that $W_{n,k;m}$ admits a complex structure when it is even dimensional and $m|n$.

Using the isomorphism (1) and the fact that $\varepsilon_{\mathbb{C}} = 2\varepsilon_{\mathbb{R}}$, we obtain an isomorphism

$$\tau W_{n,k;m} \oplus k^2\varepsilon_{\mathbb{R}} \cong k(\xi^{\vee}_{n,k;m} \otimes_{\mathbb{C}} \beta_{n,k;m} \oplus k\varepsilon_{\mathbb{C}}) = nk\xi^{\vee}_{n,k;m} \tag{4}$$

of real vector bundles.

Recall that the *span* of a smooth manifold M is the maximum number $r \geq 0$ for which there exist r everywhere linearly independent vector fields on M. Equivalently span of M is the maximum number r such that $\tau M \cong r\varepsilon_{\mathbb{R}} \oplus \eta$ for some vector bundle η. The *stable span* of M is the maximum number s such that $\tau M \oplus t\varepsilon_{\mathbb{R}} \cong (s+t)\varepsilon_{\mathbb{R}} \oplus \theta$ for some vector bundle θ where $t > 0$. Indeed one may take $t = 1$ in the above definition of stable span. The rank of θ is then called the *geometric dimension* of τM. We denote the span of M by span(M). The notions of span, stable span, and geometric dimension can be extended in an obvious manner to any vector bundle. The reader may refer to [10] and [11] for a detailed discussion on the vector field problem, which asks for the determination of the span of a given smooth manifold.

THEOREM 2.4. *Suppose that $2 \leq k < n$ and $m \geq 2$. Then:*

(i) span$(W_{n,k;m}) >$ stablespan$(PW_{n,k}) \geq \dim(W_{n,k;m}) - 2n + 1$. *Moreover, when n is even,* span$(W_{n,k;m}) > \dim(W_{n,k;m}) - 2n + 3$.

(ii) span$(W_{n,k;m}) >$ stablespan$(W_{n,k-1;m})$.

(iii) $W_{n,n-1;m}$ *is parallelizable.*

P r o o f.

(i) Since π_m is a principal $\mathbb{S}^1$-bundle, one has the bundle isomorphism $\tau(W_{n,k;m}) \cong \pi^*(\tau PW_{n,k}) \oplus \varepsilon_{\mathbb{R}} = \pi^*(\tau PW_{n,k} \oplus \varepsilon_{\mathbb{R}})$. Hence span$(W_{n,k;m}) >$

stablespan($PW_{n,k}$). Now consider the projection $q\colon PW_{n,k} \longrightarrow \mathbb{C}P^{n-1}$ defined as $[v_1, \ldots, v_k] \mapsto [v_1]$. The stable tangent bundle $\tau PW_{n,k} \oplus (k^2 + 1)\varepsilon_{\mathbb{R}}$ is isomorphic to $nk\zeta_{n,k} = q^*(nk\zeta_{n,1})$. Clearly the bundle $nk\zeta_{n,1}$ over $\mathbb{C}P^{n-1}$ contains a trivial real vector bundle of rank $2(nk - (n-1))$. (See [7].) Therefore the stable span of $PW_{n,k}$ is at least $2nk - 2(n-1) - (k^2 + 1) = \dim W_{n,k;m} - 2n + 1$.

Let n be even. The complex 2-plane bundle $\zeta_{n,1} \oplus \zeta_{n,1}^\vee$ evidently admits a reduction of structure group to $SU(2) = Sp(1)$. Hence it is the underlying complex vector bundle of a quaternionic line bundle. Any such bundle can be classified by a map into the quaternionic projective space $\mathbb{H}P^r$ where $r = \lfloor (1/4)\dim_{\mathbb{R}}(\mathbb{C}P^{n-1}) \rfloor = n/2 - 1$. That is, there exists a continuous map $h\colon \mathbb{C}P^{n-1} \longrightarrow \mathbb{H}P^r$ such that $h^*(\omega) \cong \zeta_{n,1} \oplus \zeta_{n,1}^\vee$ where ω is the canonical quaternionic line bundle over $\mathbb{H}P^r$. The underlying real vector bundle $(nk/2)\omega$ admits $(2nk - 4r)\varepsilon_{\mathbb{R}}$ as a summand and so, working with underlying real vector bundles throughout, we have $nk\zeta_{n,1} = (nk/2)(q^*(\zeta_{n,1} \oplus \zeta_{n,1}^\vee)) = h^*((nk/2).\omega) = (2nk - 4r)\varepsilon_{\mathbb{R}} \oplus \eta$ for some real vector bundle η. As before, it follows that span($W_{n,k;m}$) > stablespan($PW_{n,k}$) $\geq \dim W_{n,k;m} - 2n + 3$.

(ii) Consider the fibre bundle projection $W_{n,k} \longrightarrow W_{n,k-1}$ with fibre $\mathbb{S}^{2n-2k+1}$. Since it is Γ_m-equivariant, we obtain a $\mathbb{S}^{2n-2k+1}$-bundle with projection $p\colon W_{n,k;m} \longrightarrow W_{n,k-1;m}$. Note that $p^*(\xi_{n,k-1;m}) = \xi_{n,k;m}$ and $p^*(\beta_{n,k-1;m}) = \beta_{n,k;m} \oplus \xi_{n,k;m}$. Write $k^2\varepsilon_{\mathbb{R}}$ as $(k-1)^2\varepsilon_{\mathbb{R}} \oplus (k-1)(\xi_{n,k;m}^\vee \otimes_{\mathbb{C}} \xi_{n,k;m}) \oplus \varepsilon_{\mathbb{R}}$. Substituting this in the expression (3) for $\tau W_{n,k;m}$ and observing that $k\xi_{n,k;m}^\vee \otimes \beta_{n,k;m} \oplus (k-1)\xi_{n,k;m}^\vee \otimes_{\mathbb{C}} \xi_{n,k;m} = (k-1)\xi_{n,k;m}^\vee \otimes (\beta_{n,k;m} \oplus \xi_{n,k;m}) \oplus \xi_{n,k;m}^\vee \otimes \beta_{n,k;m}$, we obtain that $\tau W_{n,k;m} \cong p^*(\tau W_{n,k;m-1}) \oplus \varepsilon_{\mathbb{R}} \oplus \xi_{n,k;m}^\vee \otimes \beta_{n,k;m} \cong p^*(\tau W_{n,k;m-1} \oplus \varepsilon_{\mathbb{R}}) \oplus \xi_{n,k;m}^\vee \otimes \beta_{n,k;m}$. Therefore span($W_{n,k;m}$) $\geq$ span($\tau W_{n,k-1;m} \oplus \varepsilon_{\mathbb{R}}$) > stablespan($W_{n,k-1;m}$) as asserted.

(iii) Note that $W_{n,n-1} \cong SU(n)$. Therefore $W_{n,n-1;m}$, being a quotient of a Lie group by a finite subgroup, is parallelizable. $\qquad\square$

We refer the reader to [18] and [8] for the span of lens spaces.

PROPOSITION 2.5. *Let $2 \leq k < n$ and let $m \geq 2$. One has*

$$\text{span}(W_{n,k;m}) = \text{stablespan}(W_{n,k;m})$$

in each of the following cases:

  (i) *k is even,*

  (ii) *n is odd, and,*

  (iii) *$n \equiv 2 \mod 4$.*

P r o o f. From (3) we obtain that span($W_{n,k;m}$) $\geq k^2 \geq 4$. Since $W_{n,k;m}$ is orientable, the first Stiefel-Whitney class $w_1(W_{n,k;m})$ vanishes. As observed already

in the introduction, the Euler characteristic $\chi(W_{n,k;m})$ vanishes. Furthermore, it follows from (4) that the Stiefel-Whitney classes $w_1(W_{n,k;m}), w_2(W_{n,k;m})$ vanish when nk is even. (We shall give a formula for the total Stiefel-Whitney class of $W_{n,k;m}$ in Proposition 3.4, which also implies that $w_i(W_{n,k;m}) = 0$, $i = 1, 2$, when nk is even.)

Our hypotheses on k and n imply that $d = \dim(W_{n,k;m}) = k(2n - k)$ is even when k is even, $d \equiv 1 \mod 4$ when both k and n are odd, and $d \equiv 3 \mod 8$ when k is odd and $n \equiv 2 \mod 4$. The proposition now follows from the work of Koschorke. More precisely, (i) follows from [12: Theorem 20.1] and, assuming, as we may, that k is odd, (ii) and (iii) follow, respectively, from [12: Corollaries 20.9, 20.10]. □

Remark 2.6. Recall that the generalized vector field problem asks for the determination of the geometric dimension of multiples of the Hopf bundle ξ_n over the real projective space $\mathbb{R}P^n$. When $m = 2$, $W_{n,1;2} = \mathbb{R}P^{2n-1}$ and the bundle $\xi_{n,1;2}$ is isomorphic, as a real vector bundle, to $2\xi_{n-1}$. Denoting by $p \colon W_{n,k;2} \longrightarrow \mathbb{R}P^{2n-1}$ the projection $[v_1, \ldots, v_k] \mapsto [v_1]$ we see that $\xi_{n,k;2} \cong p^*(2\xi_{2n-1})$. Therefore, using the bundle isomorphism (4), we have

$$\mathrm{stablespan}(W_{n,k;2}) \geq \mathrm{span}(2nk\xi_{2n-1}) - k^2. \tag{5}$$

Invoking Proposition 2.5 we obtain the following lower bound:

$$\mathrm{span}(W_{n,k;2}) \geq \mathrm{span}(2nk\xi_{2n-1}) - k^2 \tag{5}$$

when k is even, or n is odd, or $n \equiv 2 \mod 4$. Although the generalized vector field problem is yet to be solved completely, the precise value of the span of $r\xi_n$ is known from the work of Lam [14: Theorem 1.1] when $r = 8l + p$, $n = 8m + q$, $l \geq m \geq 0$, $\binom{l}{m}$ is odd, $0 \leq p, q \leq 7$. See also [15]. In many cases, (6) yields a better lower bound than Theorem 2.4(i).

We conclude this section with the following

PROPOSITION 2.7. *Let $m \geq 2$ be an integer. Let X be any topological space and let ξ be a complex line bundle over X such that*

(i) *ξ admits a reduction of structure group to $\mathbb{Z}_m$, and,*

(ii) *$n\xi$ admits k everywhere $\mathbb{C}$-linearly independent cross-sections. Then there exists a continuous map $f \colon X \longrightarrow W_{n,k;m}$ such that $f^*(\xi_{n,k;m}) \cong \xi$.*

P r o o f. Let $p \colon \widetilde{X} \longrightarrow X$ be a regular covering projection with deck transformation group $\Gamma_m \cong \mathbb{Z}_m$ such that ξ is isomorphic to the bundle with projection $E := \widetilde{X} \times_{\Gamma_m} \mathbb{C} \longrightarrow X$ where Γ_m acts on $\mathbb{C}$ via a character $\Gamma_m \longrightarrow \mathbb{C}^*$. The existence of such a covering is the content of (i). We identify the total space of ξ with E. Observe that ξ admits a hermitian metric: $(e, e') \mapsto z\bar{z}'$ is a hermitian

metric where $e = [x, z]$, $e' = [x, z'] \in E$, $x \in X$, $z, z' \in \mathbb{C}$. Consequently $\xi^\vee$ also admits a hermitian metric.

In view of (ii) and the existence of a hermitian metric on $n\xi$, we have a splitting $n\xi \cong k\varepsilon_\mathbb{C} \oplus \theta$. Taking duals, we get $n\xi^\vee \cong k\varepsilon_\mathbb{C} \oplus \theta^\vee$. Tensoring with ξ, we see that $n\varepsilon_\mathbb{C} = k\xi \oplus \eta$ where $\eta := \xi \otimes \theta^\vee$. Then η also admits a hermitian metric which is such that each copy of ξ and η are pairwise orthogonal.

For any hermitian vector bundle ν of rank n over X, one has an associated $W_{n,k;m}$-bundle, denoted $W_{n,k;m}(\nu)$, defined as the space of all Γ_m-equivalence classes of unitary k-frames in each fibre of ν. When ν is trivial, this is just the product bundle $X \times W_{n,k;m} \longrightarrow X$.

Now one has a cross-section $\sigma \colon X \longrightarrow X \times W_{n,k;m} = W_{n,k;m}(k\xi \oplus \eta)$ defined as follows: For any $x \in X$, let $\widetilde{x} \in p^{-1}(x)$ be any point in the fibre over $x \in X$. We identify $\widetilde{x}$ with $[\widetilde{x}, 1] \in \widetilde{X} \times_{\Gamma_m} \mathbb{C} = E(\xi)$. Then $\sigma(x) = [\widetilde{x}, \ldots, \widetilde{x}; 0] \in W_{n,k;m}(k\xi \oplus \eta)$ is well-defined and is independent of the choice of $\widetilde{x}$ in $p^{-1}(x)$. Since $p \colon \widetilde{X} \longrightarrow X$, is a local homeomorphism, it is immediate that σ is continuous.

The desired map $f \colon X \longrightarrow W_{n,k}$ is now obtained as the composition $pr_2 \circ \sigma$.

$\square$

Remark 2.8. An analogue of the above property for real projective Stiefel manifolds was established in [3]. A similar universal property for complex projective Stiefel manifolds was established in [2], under the additional assumption that X be a finite CW complex.

3. The mod p cohomology

In this section we shall describe the mod p cohomology of $W_{n,k;m}$ where p is a prime. Recall that $H^*(W_{n,k}; \mathbb{Z})$ is isomorphic to the exterior algebra $\Lambda_\mathbb{Z}(v_{2n-2k+1}, \ldots, v_{2n-1})$ where $v_q \in H^q(W_{n,k}; \mathbb{Z})$. This result is attributed to C. Ehresmann by Borel [4: Prop. 9.1]. It is customary to denote by $\Lambda_{\mathbb{Z}_p}(x_1, \ldots, x_k)$ any graded commutative algebra A over $\mathbb{Z}_p$ in which square-free monomials in $x_1, \ldots, x_r$ form a basis. (If p is odd, and all the generators x_j have odd degree, then A is isomorphic to the exterior algebra. However when $p = 2$, it need not be so.) This convention will be used in what follows.

NOTATIONS. Let $N := 2N'$ where $N' := N'_p = \min\limits_{n-k+1 \leq j \leq n} \left\{ j \mid \binom{n}{j} \not\equiv 0 \mod p \right\}$. (Note that the value of N' depends on n, k and p.) In what follows, we shall label (homogeneous) generators of a graded algebra by their degrees. Thus $|y_j| = j$ when $y_j \in H^*(X; R)$.

Theorem 3.1. *Suppose that $2 \leq k < n$ and $m \geq 2$.*

(i) *If p is any prime not dividing m, then*

$$p_m^* : \ H^*(W_{n,k;m}; \mathbb{Z}_p) \cong H^*(W_{n,k}; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(v_{2n-2k+1}, \ldots, v_{2n-1})$$

is an isomorphism of algebras.

(ii) *If p is a an odd prime that divides m, then*

$$H^*(W_{n,k;m}; \mathbb{Z}_p)$$
$$\cong H^*(\mathbb{S}^1; \mathbb{Z}_p) \otimes H^*(PW_{n,k}; \mathbb{Z}_p)$$
$$\cong \mathbb{Z}_p[y_2]/\langle y_2^{N'} \rangle \otimes \Lambda_{\mathbb{Z}_p}(y_1, y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1})$$

where N, N' are as defined above. (As usual, $\hat{\ }$ stands for omission of the variable.) Also $y_2 = c_1(\xi_{n,k;m}) \mod p$.

(iii) (a) *Suppose $m \equiv 2 \mod 4$. Then*

$$H^*(W_{n,k;m}; \mathbb{Z}_2)$$
$$= \mathbb{Z}_2[y_1]/\langle y_1^N \rangle \otimes \Lambda_{\mathbb{Z}_2}(y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1}).$$

(b) *Suppose that $m \equiv 0 \mod 4$. Then*

$$H^*(W_{n,k;m}; \mathbb{Z}_2)$$
$$= \mathbb{Z}_2[y_2]/\langle y_2^{N'} \rangle \otimes \Lambda_{\mathbb{Z}_2}(y_1, y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1}),$$
$$\text{where } y_1^2 = 0.$$

P r o o f.

(i) Let $\gamma \in \Gamma_m$. Recall that $p_m \colon W_{n,k} \longrightarrow W_{n,k;m}$ is the covering projection with deck transformation group Γ_m. The covering map $\gamma \colon W_{n,k} \longrightarrow W_{n,k}$ is homotopic to the identity since $\gamma \in U(n)$ and $U(n)$ is connected. It follows that Γ_m acts trivially on the cohomology groups of $W_{n,k}$. Since p does not divide m, $p_m^* \colon H^*(W_{n,k;m}; \mathbb{Z}_p) \longrightarrow H^*(W_{n,k}; \mathbb{Z}_p)^{\Gamma_m} = H^*(W_{n,k}; \mathbb{Z}_p)$ is an isomorphism. One knows that $H^*(W_{n,k}; \mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}(y_{2n-k+1}, \ldots, y_{2n-1})$ (see [4: Proposition 9.1]).

(ii) By definition, $\xi_{n,k;m} = \pi_m^*(\zeta_{n,k})$ where $\pi_m \colon W_{n,k;m} \longrightarrow PW_{n,k}$ is the projection of the principal $U(1)/\Gamma_m \cong \mathbb{S}^1$-bundle. (See §2.) Let $y_2 = c_1(\zeta_{n,k}) \in H^2(PW_{n,k}; \mathbb{Z}_p)$. We apply the Serre spectral sequence with $\mathbb{Z}_p$-coefficients to the principal $\mathbb{S}^1$-bundle with projection π_m. The differential $d \colon E_2^{0,1} \longrightarrow E_2^{2,0}$ maps the generator of $E_2^{0,1} \cong H^1(\mathbb{S}^1; \mathbb{Z}_p) \cong \mathbb{Z}_p$ to $c_1(\zeta_{n,k}^m) = my_2 \in H^2(PW_{n,k}; \mathbb{Z}_p) \cong \mathbb{Z}_p y_2$ by Lemma 2.1 (i). Since $p|m$, this differential is zero. It follows that the spectral sequence collapses and we get $H^*(W_{n,k;m}; \mathbb{Z}_p) \cong H^*(\mathbb{S}^1; \mathbb{Z}_p) \otimes H^*(PW_{n,k}; \mathbb{Z}_p)$. Note that $y_1^2 = 0$ as p is odd. The rest of the statement follows

from the description of the $\mathbb{Z}_p$-cohomology of $PW_{n,k}$ due to Astey, Gitler, Micha and Pastor [2].

(iii) We proceed as in (ii) and obtain that $H^*(W_{n,k;m};\mathbb{Z}_2) \cong H^*(\mathbb{S}^1;\mathbb{Z}_2) \otimes H^*(PW_{n,k};\mathbb{Z}_2)$ as an $H^*(PW_{n,k};\mathbb{Z}_2)$-module. Denoting by ν the real line bundle associated to the double cover $f\colon W_{n,k;l} \longrightarrow W_{n,k;m}$ where $m = 2l$ we have $w_1(\nu) =: y_1$ is the generator of $H^1(W_{n,k;m};\mathbb{Z}_2)$. Also, $\nu \otimes_{\mathbb{R}} \mathbb{C}$ is evidently an element of order 2 in $\mathrm{Pic}(W_{n,k;m})$ and hence $\nu \otimes_{\mathbb{R}} \mathbb{C} \cong \xi_{n,k;m}^l$ by Lemma 2.2. It follows that $y_1^2 = c_1(\nu \otimes \mathbb{C}) = lc_1(\xi_{n,k;m}) = ly_2$. Hence $y_1^2 = 0$ if l is even, and $y_1^2 = y_2$ if l is odd.

Finally, write $d = \dim W_{n,k;m}$. Then

$$H^{d-1}(PW_{n,k};\mathbb{Z}_2) \otimes \mathbb{Z}_2 y_1 \cong H^d(W_{n,k;m};\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Therefore $y_1 y_2^{N'-1} y_{2n-2k+1} \ldots \hat{y}_{N-1} \ldots y_{2n-1}$ generates $H^d(W_{n,k;m};\mathbb{Z}_2) \cong \mathbb{Z}_2$. Using this, and the property that square-free monomials in $y_{2n-2k+1}, \ldots, \hat{y}_{N-1}$, $\ldots, y_{2n-1}$ are linearly independent, it follows that the same property holds for $y_1, y_{2n-2k+1}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1}$. This completes the proof. $\square$

We now turn to the integral cohomology of $W_{n,k;m}$. It is easily seen that $H^2(W_{n,k;m};\mathbb{Z}) \cong \mathbb{Z}_m$ generated by $y_2 = c_1(\xi_{n,k;m})$. We are mainly interested in the height of y_2. Recall that the *height* of $0 \neq y \in H^q(X;R)$ is the largest positive integer h such that $y^h \neq 0$. In view of the fact that the complex Stiefel manifold $W_{n,k}$ is $2(n-k)$-connected, we see that the $2(n-k)$-skeleton of $W_{n,k;m}$ with respect to any CW-structure may be regarded as the $2(n-k)$-skeleton of the infinite lens space $L^\infty(m)$ with fundamental group $\mathbb{Z}_m$. So $H^q(W_{n,k;m};\mathbb{Z}) \cong H^q(L^\infty(m);\mathbb{Z}) \cong H^q(\mathbb{Z}_m;\mathbb{Z})$ for $q < 2(n-k)$. It is well-known that $H^*(L^\infty(\mathbb{Z}_m);\mathbb{Z}) \cong \mathbb{Z}[y_2]/\langle my_2 \rangle$; see [6]. It follows that $H^q(W_{n,k;m};\mathbb{Z}) \cong \mathbb{Z}_m y^r$ where $q = 2r < 2n-2k$. However, the following theorem gives the precise value of the height.

DEFINITION 3.2. Fix integers n, k, m such that $m > 1$ and $1 \leq k < n$. We define $m_r := m$ if $r \leq n - k$ and $m_r := \gcd\{m, \binom{n}{j} \mid n - k < j \leq r\}$ if $n - k < r \leq n$.

The integral cohomology ring of lens spaces is well-known. We shall now establish the following

THEOREM 3.3. *With the above notations, the (additive) order of $y_2^r \in H^{2r}(W_{n,k;m};\mathbb{Z})$ is m_r for $1 \leq r \leq n$. In particular the height of $y_2 \in H^2(W_{n,k;m};\mathbb{Z}) \cong \mathbb{Z}_m$ is the largest integer h, $n - k < h \leq n$, such that $m_h > 1$.*

P r o o f. By our observation above, we need only consider the case $r > n - k$.

Let E be a contractible CW complex on which Γ_m acts freely so that the quotient $E/\Gamma_m = K(\Gamma_m, 1)$ has the same homotopy type as the infinite lens

space $L^\infty(m)$. Then the fibre product $W' := E \times_{\Gamma_m} W_{n,k}$ fibres over $W_{n,k;m}$ with fibre E. In particular, W' has the same homotopy type as $W_{n,k;m}$. Also one has a fibre bundle with fibre $W_{n,k}$ with projection $W' \longrightarrow K(\Gamma_m, 1)$. We choose E conveniently so that it is easier to determine the differential in the Serre spectral sequence associated to the $W_{n,k}$-bundle over $K(\Gamma_m, 1)$.

Let $E := W_{\infty,n} = \bigcup_{r>n} W_{r,n}$ be the space of all unitary n-frames in $\mathbb{C}^\infty = \bigcup_{r>1} \mathbb{C}^r$. An element of $\mathbb{C}^\infty$ is viewed as a column vector whose entries are eventually 0. The space E is contractible since the inclusion $W_{r,n} \subset W_{r+1,n}$ is $2(r-n)$-connected for all $r > n$.

The group Γ_m acts on $\mathbb{C}^\infty$ via scalar multiplication and hence one has the diagonal action of Γ_m on $W_{\infty,n}$. The quotient $W_{\infty,n}/\Gamma_m =: L$ has the homotopy type of the infinite lens space $L^\infty(m)$. The image of $\mathbf{v} = (v_1, \ldots, v_n) \in W_{\infty,n}$ in L under the quotient map $q: W' \longrightarrow L$ will be denoted $[v_1, \ldots, v_n]_m$ or $[\mathbf{v}]$. The n-dimensional complex vector space $\mathbb{C}v_1 + \cdots + \mathbb{C}v_n$ will be denoted $\langle \mathbf{v} \rangle$. Denote by $W_k(V)$ the space of all unitary k-frames in the complex vector space $V \subset \mathbb{C}^\infty$.

Let $W' := \left\{ \left([\mathbf{v}]_m; u_1, \ldots, u_k\right) \mid \mathbf{v} \in W_{\infty,n}, \ (u_1, \ldots, u_k) \in W_k\left(\langle v \rangle\right) \right\}$. The space W' is just the $W_{n,k}$-bundle over L associated to the n-plane bundle $n\gamma$ where γ is the complex line bundle associated to the character $\pi_1(L) = \Gamma_m \subset \mathbb{S}^1$. Let $\widetilde{f}: W' \longrightarrow E/U(n-k) = BU(n-k)$ be defined as $\widetilde{f}([\mathbf{v}]_m; u_1, \ldots, u_k) = \left(\langle \mathbf{v} \rangle; u_1, \ldots, u_k\right)$. We let $f: L \longrightarrow E/U(n) = BU(n)$ be the map $[\mathbf{v}]_m \mapsto \langle \mathbf{v} \rangle$. One has the projection $\pi: BU(n-k) \longrightarrow BU(n)$, defined as $(\langle \mathbf{u} \rangle; u_1, \ldots, u_k) \mapsto \langle \mathbf{u} \rangle$, of a fibre bundle with fibre $W_{n,k}$. This is just the projection of the $W_{n,k}$-bundle associated to the universal n-plane bundle $\gamma_{\infty,n}$. Clearly $\pi \circ \widetilde{f} = f \circ q$ and $f^*(\gamma_{\infty,n}) \cong n\gamma$. Thus the following diagram commutes:

$$
\begin{array}{ccc}
W' & \xrightarrow{\ \widetilde{f}\ } & BU(n-k) \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\
L & \xrightarrow{\ f\ } & BU(n).
\end{array}
$$

The $W_{n,k}$-bundle $W' \longrightarrow L$ is the pull-back of the bundle $BU(n-k) \longrightarrow BU(n)$. In particular the former bundle is $\mathbb{Z}$-orientable. We consider the Serre spectral sequence of the $W_{n,k}$-bundle $q: W' \longrightarrow L$ which converges to $H^*(W'; \mathbb{Z}) \cong H^*(W_{n,k;m}; \mathbb{Z})$. We have

$$E_2^{p,q} = H^p(L; H^q(W_{n,k}; \mathbb{Z})) = H^p(L^\infty(m); \mathbb{Z}) \otimes H^q(W_{n,k}; \mathbb{Z})$$

since $H^*(W_{n,k}; \mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_{2n-2k+1}, \ldots, y_{2n-1})$ is free abelian. It is well-known that $H^*(L^\infty(m); \mathbb{Z}) \cong \mathbb{Z}[y_2]/\langle my_2 \rangle$. By comparing the Serre spectral sequence of

$\pi\colon BU(n-k) \longrightarrow BU(n)$, we see that the cohomology classes $y_{2n-2k+2j-1}$, $1 \leq j \leq k$, are transgressive. Indeed $\tau(y_{2n-2k+2j-1}) = c_{2n-2k+2j}(n\gamma) \in H^*(L;\mathbb{Z})$, $1 \leq j \leq k$. That is $\tau(y_{2n-2k+2j-1}) = \binom{n}{k-j}y_2^{n-k+j}$. It follows that $\binom{n}{j}y_2^r = 0$ in $H^{2r}(W';\mathbb{Z})$ for $n-k < j \leq r$ and so, since $my_2 = 0$, we see that the order of $y_2^r \in H^{2n-2k+2j}(W';\mathbb{Z})$ equals m_r. In particular, the height h of y_2 is as stated in the theorem. $\qquad\square$

A complete description of the ring structure of $H^*(W_{n,k;m};\mathbb{Z})$ appears to be more intricate. However, it is clear from the above proof that the torsion subgroup in $H^*(W_{n,k;m};\mathbb{Z})$ is generated by the y_2^j, $1 \leq j < h$. Also, it can be seen readily that there exist classes $v_{2n-2k+2j-1} \in H^*(W_{n,k;m};\mathbb{Z})$, $1 \leq j \leq k$, which generate a free abelian group of rank k. Furthermore, their reduction mod any prime p not dividing m are the generators of $H^*(W_{n,k;m};\mathbb{Z}_p)$ given in Theorem 3.1(i). They arise from the generators of the kernel of the transgression in the spectral sequence in the above proof.

As an application we have the following theorem. We write $p(M)$ (resp. $w(M)$) for the total Pontrjagin class (resp. total Stiefel-Whitney class) of a differentiable manifold M. (See [16].)

PROPOSITION 3.4. *Let* $2 \leq k \leq n-2$ *and let* $m \geq 2$. *One has* $p(W_{n,k;m}) = (1+y_2^2)^{nk}$ *for all* $r \geq 1$. *The total Stiefel-Whitney class* $w(W_{n,k;m}) = (1+y_1^2)^{nk}$, *where it is understood that* $y_1 = 0$ *when* m *is odd.*

P r o o f. Consider the complexified tangent bundle $\tau_\mathbb{C} := \tau W_{n,k;m} \otimes_\mathbb{R} \varepsilon_\mathbb{C}$. From (4), $\tau_\mathbb{C}$ is stably equivalent to the complex vector bundle $nk(\xi_{n,k;m} \oplus \xi_{n,k;m}^\vee)$. Therefore $c(\tau_\mathbb{C}) = (1+y_2)^{nk}(1-y_2)^{nk} = (1-y_2^2)^{nk}$. It follows that the jth Pontrjagin class $p_j(W_{n,k;m}) = \binom{nk}{j}y_2^{2j}$.

Using (4) we get $w(W_{n,k;m}) = w(\xi_{n,k;m})^{nk} = (1+y_1^2)^{nk}$. $\qquad\square$

Recall from Theorem 2.4 that $W_{n,n-1;m}$ is parallelizable for all m. The rest of the $W_{n,k;m}$ are *not* stably parallelizable for most values of m.

THEOREM 3.5. *Let* $1 < k \leq n-2$ *and* $m \geq 2$. *If there exists an* $r \geq 1$ *such that* $\binom{nk}{r}$ *is not divisible by* m_{2r}, *then* $W_{n,k;m}$ *is not stably parallelizable. In particular, if* $W_{n,k;m}$ *is stably parallelizable, then* m *divides* nk.

P r o o f. If $\binom{nk}{r}$ is not divisible by m_{2r}, then $m_{2r} > 1$ and so $h > 2r$. Therefore $p_r(W_{n,k;m}) = \binom{nk}{r}y^{2r} \neq 0$ by Proposition 3.4. It follows that $W_{n,k;m}$ is not stably parallelizable (cf. [16: Lemma 15.2]).

As for the second assertion, since $h > n-k \geq 2$, one has $y_2^2 \neq 0$. If $W_{n,k;m}$ is stably parallelizable, then $p_1(W_{n,k;m}) = nky_2^2 = 0$ and hence $m_2 = m$ divides nk. $\qquad\square$

Remark 3.6. The above theorem does not settle completely the question of stable parallelizability of $W_{n,k;m}$. Suppose that n, k are powers of a prime p and $m = p$. Then $h = n$ and $m_r = p$, for all $r < n$. In this case, $p_j(W_{n,k;m}) = 0$, $w_j(W_{n,k;m}) = 0$ for all $j > 0$. We remark that in the case of lens spaces $L^n(p)$ where p is an odd prime, Kambe [9] has obtained non-immersion results using K-theory calculations. Combined with the work of Adams [1] on the order of the Hopf bundle $\xi_{n,1;2}$, one obtains that for a fixed n, for all but finitely many $m > 1$, the lens spaces are not stably parallelizable.

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