

SOME PARANORMED EULER SEQUENCE SPACES OF DIFFERENCE SEQUENCES OF ORDER m

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ABSTRACT. The main purpose of this work is to extend the sequence spaces which are defined in [KARAKAYA, V.—POLAT, H.: *Some new paranormed sequence spaces defined by Euler and difference operators*, Acta Sci. Math. (Szeged) **76** (2010), 87–100] and [POLAT, H.—BASAR, F.: *Some Euler spaces of difference sequences of order m* , Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 254–266] by using difference operator of order m , and to give their alpha, beta and gamma duals. Furthermore, we characterize some classes of the related matrix transformations.

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1. Introduction

In studies on the sequence spaces, there are some basic approaches which are determination of topologies, matrix mapping and inclusions of sequence spaces (see [18]). These methods are applied to study the matrix domain λ_A of an infinite matrix A defined by $\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}$. Especially, the Euler transformations and the difference operators which are special cases for the matrix A have been studied extensively via the methods mentioned above.

In the literature, some new sequence spaces are defined by using the Euler transformation and the difference operator or by combining both of them. For example, in [11], the difference sequence spaces are first defined by Kızmaz. Further, the authors including Ahmad and Mursaleen [1], Çolak and Et[8], Başar and Altay [6], and the others have defined and studied new sequence spaces by

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considering matrices that represent difference operators. On the other hand, by using Euler transformation, several authors defined some new sequence spaces and studied some properties of these spaces. The articles concerning this work can be found in the list of references [4], [5] and [14].

Recently, by combining the Euler transformation with difference operator or difference operator of order m , various authors have constructed new sequence spaces and studied some properties of these new sequence spaces. Some of them are as follows: Polat and Başar [16] defined and studied some Euler spaces of difference sequences of order m over the normed spaces, Altay and Polat [2] studied the Euler difference sequence spaces. Altay et al [4] defined the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ and examined some properties of these spaces. Also in [10], Karakaya and Polat defined and studied the paranormed Euler spaces of m th order difference sequences.

In this work, our purpose is to introduce new paranormed sequence spaces by combining the Euler transformation and difference operator of order m and also to investigate topological structure, the α -, β -, γ - duals, some inclusion relations and the bases of these sequence spaces. In addition, we characterize some matrix mappings on these spaces. Also these new spaces are the generalization of the sequence spaces defined in [10] and [16].

2. Preliminaries and notations

By w , we denote the set of all sequences $x = (x_k)$. Any subspace of w is called a sequence space. We shall write ℓ_∞ , c , c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs , cs and ℓ_1 , we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $h: X \rightarrow \mathbb{R}$ such that $h(\theta) = 0$, $h(x) = h(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \rightarrow 0$ and $h(x_n - x) \rightarrow 0$ imply $h(\alpha_n x_n - \alpha x) \rightarrow 0$ for all α in $\mathbb{R}$ and all x in X , where θ is the zero vector in linear space X .

Throughout this paper, it is assumed that (p_k) be a bounded sequence over positive real numbers with $\sup p_k = H$ and $M = \max(1, H)$. The linear spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ were defined by Maddox [12], (see also Simons [19]) as follows:

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty \right\};$$

$$c(p) = \left\{ x = (x_k) \in w : |x_k - l|^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } l \in \mathbb{R} \right\};$$

$$c_0(p) = \{x = (x_k) \in w : |x_k|^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty\}.$$

In the sequel of this work, we will need inequalities (see, [13]).

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}) \quad (2.1)$$

$$|a_k + b_k|^{t_k} \leq |a_k|^{t_k} + |b_k|^{t_k} \quad (2.2)$$

where $t_k = \frac{p_k}{M} \leq 1$ and $C = \max(1, 2^{H-1})$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be any infinite matrix of real numbers a_{nk} , where $n, k \in \mathbb{N}$ with $\mathbb{N} = \{0, 1, 2, \dots\}$. Then we say that A defines a matrix mapping from λ into μ by writing $A: \lambda \rightarrow \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = ((Ax)_n)$, the A -transform of x , is in μ , where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \quad (2.3)$$

By (λ, μ) , we denote the class of all matrices A such that $A: \lambda \rightarrow \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right hand side of (2.3) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = ((Ax)_n) \in \mu$ for all $x \in \lambda$. A sequence x is said to be A -summable to a if Ax converges to a which is called as the A -limit of x .

The Euler means of order r is defined by the matrix $E^r = e_{nk}^r$, (see, [17])

$$e_{nk}^r = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

where $0 < r < 1$.

The difference operator of order m is defined by the matrix $\Delta^m = (\Delta_{nk}^m)$, (see [7]).

It is natural to expect that the sequence spaces $e_0^r(\Delta^m)$, $e_c^r(\Delta^m)$ and $e_\infty^r(\Delta^m)$ can be extended to the paranormed sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$ just as c, c_0 and ℓ_∞ were extended $c(p), c_0(p)$ and $\ell_\infty(p)$ respectively.

3. The sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$

We define the following new sequence spaces

$$\begin{aligned} e_0^r(\Delta^m, p) &= \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k \Delta^m x_k \right|^{p_n} = 0 \right\} \\ e_c^r(\Delta^m, p) &= \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k (\Delta^m x_k - l) \right|^{p_n} = 0 \right. \\ &\quad \left. \text{for some } l \in \mathbb{R} \right\} \end{aligned}$$

$$e_{\infty}^r(\Delta^m, p) = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k \Delta^m x_k \right|^{p_n} < \infty \right\}$$

where

$$\Delta^m x = (\Delta^m x)_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k-i} = \sum_{i=\max(0, k-m)}^m (-1)^{k-i} \binom{m}{k-i} x_i.$$

It is trivial that the sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_{\infty}^r(\Delta^m, p)$ may be reduced to some sequence spaces in the special cases of the sequence (p_k) and $m \in \mathbb{N}$. For instance, the sequence spaces $e_0^r(\Delta, p)$, $e_c^r(\Delta, p)$ and $e_{\infty}^r(\Delta, p)$ correspond in the case $p_k = 1$ for all $k \in N$ to the sequence spaces $e_0^r(\Delta^m)$, $e_c^r(\Delta^m)$ and $e_{\infty}^r(\Delta^m)$ defined by Polat and Başar [16]. If taking $m = 1$ and $p_k = 1$ for all $k \in N$, then it can obtain the spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_{\infty}^r(\Delta)$ defined by Altay and Polat [2] from the spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_{\infty}^r(\Delta^m, p)$.

We should note that the new sequence spaces which are defined above can be expressed by

$$\begin{aligned} e_0^r(\Delta^m, p) &= [c_0(p)]_{E^r \Delta^m}, & e_c^r(\Delta^m, p) &= [c(p)]_{E^r \Delta^m}, \\ e_{\infty}^r(\Delta^m, p) &= [\ell_{\infty}(p)]_{E^r \Delta^m} \end{aligned}$$

where $E^r \Delta^m$ denotes the composition Δ^m with E^r . Let us define the sequence $y = (y_k)$ as $E^r \Delta^m$ -transform of the sequence $x = (x_k)$, that is;

$$y_n(r) = (E^r \Delta^m x)_n = \sum_{k=0}^n \sum_{j=k}^n \binom{n}{j} \binom{m}{j-k} (-1)^{j-k} r^j (1-r)^{n-j} x_k \quad (3.1)$$

Throughout this work, we will use $\Delta x_k = x_k - x_{k-1}$.

The following theorem gives inclusion relations among the spaces $e_c^r(\Delta^m, p)$ and $e_{\infty}^r(\Delta^m, p)$. This is routine verification and therefore we omit the proof.

THEOREM 3.1. *The sequence space $e_{\infty}^r(\Delta^m, p)$ includes the sequence space $e_c^r(\Delta^m, p)$.*

After this, we would like to drive addition some inclusion relations concerning the spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_{\infty}^r(\Delta^m, p)$ with respect to r , m and the sequence (p_k) . We shall prove them for the space $e_0^r(\Delta^m, p)$ in Theorems 3.2 and 3.3. The proofs of the spaces $e_c^r(\Delta^m, p)$ and $e_{\infty}^r(\Delta^m, p)$ can be proved in the similar manner.

THEOREM 3.2. *Let $0 < s < r < 1$. Then the inclusions $e_0^r(\Delta^m, p) \subset e_0^s(\Delta^m, p)$, $e_c^r(\Delta^m, p) \subset e_c^s(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p) \subset e_\infty^s(\Delta^m, p)$ hold.*

Proof. Let $x \in e_0^r(\Delta^m, p)$. We consider the equality

$$\begin{aligned} |v_n|^{p_n} &= \left| \sum_{k=0}^n \left[\sum_{j=k}^n \binom{n}{j} \binom{m}{j-k} (-1)^{j-k} s^j (1-s)^{n-j} \right] x_k \right|^{p_n} \\ &= \left| \sum_{k=0}^n e_{nk}^{\frac{s}{r}} y_k \right|^{p_n}. \end{aligned}$$

Hence we get $(v_n) \in c_0(p)$ whenever $(y_n) \in c_0(p)$ which means that $e_0^r(\Delta^m, p) \subset e_0^s(\Delta^m, p)$. $\square$

THEOREM 3.3. *The inclusions $e_0^r(\Delta^m, p) \subset e_0^r(\Delta^{m+1}, p)$, $e_c^r(\Delta^m, p) \subset e_c^r(\Delta^{m+1}, p)$ and $e_\infty^r(\Delta^m, p) \subset e_\infty^r(\Delta^{m+1}, p)$ hold.*

Proof. Let $x \in e_0^r(\Delta^m, p)$. We consider (2.1) and the following inequality

$$\begin{aligned} & |(E^r \Delta^{m+1} x)_n|^{p_n} \\ &= |(E^r \Delta^m (\Delta x))_n|^{p_n} \\ &= |(E^r \Delta^m x)_n - (E^r \Delta^m x)_{n-1}|^{p_n} \\ &\leq C (|(E^r \Delta^m x)_n|^{p_n} + |(E^r \Delta^m x)_{n-1}|^{p_n}) \end{aligned} \tag{3.2}$$

It is easy to show that (3.2) tends to zero as $n \rightarrow \infty$. So we obtain $x \in e_0^r(\Delta^{m+1}, p)$. This shows that the inclusion $e_0^r(\Delta^m, p) \subset e_0^r(\Delta^{m+1}, p)$. $\square$

THEOREM 3.4. *The sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$ are complete linear metric space with paranorm defined by*

$$h(x) = \sup_{n \in N} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k \Delta^m x_k \right|^{\frac{p_n}{M}}. \tag{3.3}$$

Proof. We shall give the proof for only the space $e_0^r(\Delta^m, p)$. The others can be proved similarly.

The linearity of $e_0^r(\Delta^m, p)$ with respect to the coordinatewise addition and scalar multiplication follows from the inequality (2.2). Because for any $\lambda \in \mathbb{R}$, we have

$$|\lambda|^{p_n} \leq \max \left(1, |\lambda|^M \right).$$

It is clear that $h(\theta) = 0$, $h(x) = h(-x)$ for all $x \in e_0^r(\Delta^m, p)$. Also it can be seen that for $\lambda \in \mathbb{R}$

$$h(\lambda x) \leq \max\{1, |\lambda|\}h(x).$$

Let (x^j) be any sequence of the points of the space $e_0^r(\Delta^m, p)$ such that $h(x^j - x) \rightarrow 0$ and (λ_j) also be any sequence of scalars such that $\lambda_j \rightarrow \lambda$. Then since the inequality $h(x) \leq h(x^j - x) + h(x^j)$ holds and h is subadditive, the sequence $(h(x^j))_{j \in \mathbb{N}}$ is bounded and we have

$$\begin{aligned} h(\lambda_j x^j - \lambda x) &= \sup_n \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k (\lambda_j \Delta x_k^j - \lambda \Delta x_k) \right|^{\frac{pn}{M}} \\ &\leq |\lambda_j - \lambda| h(x^j) + |\lambda| h(x^j - x). \end{aligned}$$

The last expression tends to zero as $j \rightarrow \infty$, that is, the scalar multiplication is continuous. Hence h is a paranorm on the space $e_0^r(\Delta^m, p)$.

Now let (x^i) be any Cauchy sequence in the space $e_0^r(\Delta^m, p)$ where $x = (x_k^i) = \{x_0^i, x_1^i, x_2^i, \dots\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $N_0(\varepsilon)$ such that

$$h(x^i - x^j) < \frac{\varepsilon}{2}$$

for every $i, j \geq N_0(\varepsilon)$. By using the definition of h , for each fixed $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} &|(E^r \Delta^m x^i)_n - (E^r \Delta^m x^j)_n|^{\frac{pn}{M}} \\ &\leq \sup_n |(E^r \Delta^m x^i)_n - (E^r \Delta^m x^j)_n|^{\frac{pn}{M}} < \frac{\varepsilon}{2} \end{aligned} \quad (3.4)$$

for every $i, j \geq N_0(\varepsilon)$ which leads us to the fact that $\{(E^r \Delta^m x^0)_n, (E^r \Delta^m x^1)_n, \dots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete,

$$(E^r \Delta^m x^j)_n \rightarrow (E^r \Delta^m x)_n$$

as $j \rightarrow \infty$. From (3.4) with $j \rightarrow \infty$, we have

$$|(E^r \Delta x^i)_n - (E^r \Delta x)_n|^{\frac{pn}{M}} \leq \frac{\varepsilon}{2} \quad (3.5)$$

for every $i \geq N_0(\varepsilon)$. Since $(x^i) = (x_k^i) \in e_0^r(\Delta^m, p)$ for each $i \in \mathbb{N}$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|(E^r \Delta^m x^i)_n|^{\frac{pn}{M}} \leq \frac{\varepsilon}{2}$ for every $n_0(\varepsilon) \in \mathbb{N}$ and for each fixed $i \in \mathbb{N}$. Therefore taking a fixed $i \geq N_0(\varepsilon)$, we obtain by (3.5) that

$$|(E^r \Delta^m x)_n|^{\frac{pn}{M}} \leq |(E^r \Delta^m x^i)_n - (E^r \Delta^m x)_n|^{\frac{pn}{M}} + |(E^r \Delta^m x^i)_n|^{\frac{pn}{M}} \leq \varepsilon$$

for every $n \geq n_0(\varepsilon)$. This shows that $x \in e_0^r(\Delta^m, p)$. Since (x^i) is an arbitrary Cauchy sequence, the space $e_0^r(\Delta^m, p)$ is complete. $\square$

THEOREM 3.5. *The sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$ are linearly isomorphic to the spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively, where $0 < p_k < H < \infty$ for all $k \in \mathbb{N}$.*

Proof. We only prove for the space $e_0^r(\Delta, p)$. Since the isomorphism of the spaces $e_c^r(\Delta, p)$ and $e_\infty^r(\Delta, p)$ can be proved by the same way, so we omit them. Firstly, we should show the existence of a linear bijection between the spaces $e_0^r(\Delta^m, p)$ and $c_0(p)$. According to the notation (3.1), we define the transformation T from $e_0^r(\Delta^m, p)$ to $c_0(p)$ such that $x \mapsto y = Tx$. Since, for every $x, z \in e_0^r(\Delta^m, p)$ and for all $\alpha \in \mathbb{R}$, $T(\alpha x + z) = \alpha Tx + Tz$, the transformation T is linear. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and T is injective.

Let $y \in c_0(p)$ and define the sequence $x = (x_k)$ by

$$x_n(m, r) = \sum_{k=0}^n \left[\sum_{i=k}^n \binom{m+n-i-1}{n-i} \binom{i}{k} (r-1)^{i-k} r^{-i} \right] y_k \quad (n \in \mathbb{N}). \quad (3.6)$$

Then, from (3.6), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} |(E^r \Delta^m x)_n|^{p_n} &= \lim_{n \rightarrow \infty} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k \Delta^m x_k \right|^{p_n} \\ &= \lim_{n \rightarrow \infty} |y_n(m, r)|^{p_n} = 0 \end{aligned}$$

Thus, we obtain that $x \in e_0^r(\Delta^m, p)$. Therefore, it is obtained that T is surjective. Hence since T is linear bijection, the spaces $e_0^r(\Delta^m, p)$ and $c_0(p)$ are linearly isomorphic. $\square$

We now give a theorem about the Schauder bases of the sequence spaces $e_0^r(\Delta^m, p)$ and $e_c^r(\Delta^m, p)$. If a sequence space λ paranormed by h contains a sequence (b_n) with the property that for every $x \in \lambda$, there is a unique sequence of scalar (α_n) such that

$$\lim_{n \rightarrow \infty} h \left(x - \sum_{k=0}^n \alpha_k b_k \right) = 0,$$

then (b_n) is called a Schauder basis for λ .

THEOREM 3.6. *Define the sequence $b^{(k)}(m, r) = \{b_n^k(m, r)\}_{k \in \mathbb{N}}$ of the elements of the space $e_0^r(\Delta^m, p)$ by*

$$b_n^k(m, r) = \begin{cases} 0 & \text{if } 0 < k \\ \sum_{i=k}^n \binom{m+n-i-1}{n-i} \binom{i}{k} (r-1)^{i-k} r^{-i} & \text{if } k \geq n \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then,

- (a) The sequence $\{b^k(m, r)\}_{k \in \mathbb{N}}$ is a basis for the space $e_0^r(\Delta^m, p)$ and any $x \in e_0^r(\Delta^m, p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(m, r) b^k(m, r) \quad (3.7)$$

where $\lambda_k(m, r) = (E^r \Delta^m x)_k$ for all $k \in \mathbb{N}$.

- (b) The set $\{e, b^k(m, r)\}$ is a basis for the space $e_c^r(\Delta^m, p)$ and any $x \in e_c^r(\Delta^m, p)$ has a unique representation of the form

$$x = le + \sum_k [\lambda_k(m, r) - l] b^k(m, r); \quad (3.8)$$

where

$$l = \lim_{k \rightarrow \infty} ((E^r \Delta^m x)_k). \quad (3.9)$$

Proof.

(a) It is clear that $\{b_n^k(m, r)\} \subset e_0^r(\Delta^m, p)$, since $E^r \Delta^m b^{(k)}(m, r) = e^{(k)} \in c_0(p)$ where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

Let $x \in c_0(p)$ be given. For every non-negative integer s , we put

$$x^{[s]} = \sum_{k=0}^s \lambda_k(m, r) b^k(m, r). \quad (3.10)$$

Then, by applying $(E^r \Delta^m)$ to (3.10) with (3.7), we get that

$$\begin{aligned} E^r \Delta^m x^{[s]} &= \sum_{k=0}^s \lambda_k(m, r) E^r \Delta^m b^k(m, r) \\ &= \sum_{k=0}^s \lambda_k(m, r) e^{(k)} = \sum_{k=0}^s (E^r \Delta^m x)_k e^{(k)} \end{aligned}$$

and for $i, s \in \mathbb{N}$.

$$\{E^r \Delta^m (x - x^{[s]})\}_i = \begin{cases} 0 & \text{if } 0 \leq i \leq s \\ (E^r \Delta^m x)_i & \text{if } i > s \end{cases}.$$

Given $\varepsilon > 0$, then there is an integer s_0 such that $|(E^r \Delta^m x)_s|^{\frac{pn}{M}} < \frac{\varepsilon}{2}$ for all $s \geq s_0$. Hence, we have

$$h(x - x^{[s]}) = \sup_{n \geq s} |(E^r \Delta^m x)_n|^{\frac{pn}{M}} \leq \sup_{n \geq s_0} |(E^r \Delta^m x)_n|^{\frac{pn}{M}} < \varepsilon$$

for all $s \geq s_0$. So we get that $x \in e_0^r(\Delta^m, p)$ is represented as in (3.7).

Let us show the uniqueness of the representation for $x \in e_0^r(\Delta^m, p)$ given by (3.7). On the contrary, suppose that there exists a representation $x = \sum_{k=0}^s \mu_k(m, r) b^k(m, r)$. Since the linear transformation T , from $e_0^r(\Delta^m, p)$ to $c_0(p)$, used in the proof of Theorem 3.5 is continuous we have

$$\begin{aligned} (E^r \Delta^m x)_n &= \sum_k \mu_k(m, r) (E^r \Delta^m b^k(m, r))_n \\ &= \sum_k \mu_k(m, r) e_n^{(k)} = \mu_n(m, r) \end{aligned}$$

which contradicts the fact that $(E^r \Delta^m x)_n = \lambda_n(m, r)$ for all $n \in \mathbb{N}$. Hence the representation (3.7) of $x \in e_0^r(\Delta^m, p)$ is unique.

(b) Since $b_n^k(m, r) \in e_0^r(\Delta^m, p)$ and $e \in c$, the inclusion $\{e, b^k(m, r)\} \subset e_c^r(\Delta^m, p)$ trivially holds. Let us take $x \in e_c^r(\Delta^m, p)$. Then, there uniquely exists an l satisfying (3.9). We thus have the fact that $u \in e_0^r(\Delta^m, p)$ whenever we set $u = x - le$. Therefore, we deduce by the Part (a) of the present theorem that the representation of u is unique. This leads us to the fact that the representation of x given by (3.8) is unique and this step concludes the proof. $\square$

4. The α -, β -, γ -duals of the spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$

In this section, we prove the theorems determining the α -, β - and γ -duals of the sequence spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$. For the sequence spaces λ and μ , define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \bigcap_{x \in \lambda} x^{-1} * \mu = \{a = (a_k) \in w : ax \in \mu \text{ for all } x \in \lambda\} \quad (4.1)$$

is called the multiplier space of λ and μ . With the notation of (4.1), α -, β - and γ -duals of a sequence space λ , which are respectively denote by λ^α , λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs), \quad \lambda^\gamma = S(\lambda, bs).$$

We now give some lemmas which are needed for the proofs of the theorems given in sequel.

LEMMA 4.1. ([9: Theorem 5.1.1]) *$A \in (c_0(p), \ell(q))$ if and only if*

$$\lim_{M \rightarrow \infty} \sup_K \sum_n \left| \sum_{k \in K} a_{nk} M^{-\frac{1}{p_k}} \right|^{q_n} = 0. \quad (4.2)$$

LEMMA 4.2. ([9: Theorem 5.1.9]) $A \in (c_0(p), c(q))$ if and only if, for $(\alpha_k) \subset \mathbb{R}$,

$$\lim_{M \rightarrow \infty} \sup_n \sum_k |a_{nk}| M^{-\frac{1}{p_k}} = 0; \quad (4.3)$$

$$\lim_{M \rightarrow \infty} \sup_n \left(\sum_k |a_{nk} - \alpha_k| M^{-\frac{1}{p_k}} \right)^{q_n} = 0; \quad (4.4)$$

$$\lim_{n \rightarrow \infty} |a_{nk} - \alpha_k|^{q_n} = 0 \quad \text{for all } k. \quad (4.5)$$

LEMMA 4.3. ([9: Theorem 5.1.13]) $A \in (c_0(p), \ell_\infty(q))$ if and only if

$$\sup_n \left(\sum_k |a_{nk}| M^{-\frac{1}{p_k}} \right)^{q_n} < \infty. \quad (4.6)$$

In the following theorems, by K , we denote the finite subset of $\mathbb{N}$ and also M is a positive integer.

THEOREM 4.1. Let us define the following sets for α -duals of the spaces $e_0^r(\Delta^m, p)$, $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$.

$$\begin{aligned} d_1^r(p) &= \left\{ a = (a_k) \in w : \lim_{M \rightarrow \infty} \sup_K \left| \sum_{k \in K} d_{nk} M^{-\frac{1}{p_k}} \right| = 0 \right\} \\ d_2^r &= \left\{ a = (a_k) \in w : \sum_n \left| \sum_k d_{nk} \right| < \infty \right\} \\ d_3^r(p) &= \left\{ a = (a_k) \in w : \sup_K \sum_n \left| \sum_{k \in K} d_{nk} M^{\frac{1}{p_k}} \right| < \infty \text{ for every } M \in \mathbb{N} \right\} \end{aligned}$$

where

$$d_{nk} = \begin{cases} \sum_{i=k}^n \binom{m+n-i-1}{n-i} \binom{i}{k} (r-1)^{i-k} r^{-i} a_n & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases} \quad (4.7)$$

Thus, $[e_0^r(\Delta, p)]^\alpha = d_1^r(p)$, $[e_c^r(\Delta, p)]^\alpha = [e_0^r(\Delta, p)]^\alpha \cap d_2^r$, $[e_\infty^r(\Delta, p)]^\alpha = d_3^r(p)$.

Proof. We only give the proof for the space $e_0^r(\Delta^m, p)$. Since the proof may be obtained by the same way for the spaces $e_c^r(\Delta^m, p)$ and $e_\infty^r(\Delta^m, p)$, so we omit them.

Let us define the matrix D whose rows are the product of the rows of the matrix $(E^r \Delta^m)^{-1}$ and the sequence $a = (a_n)$. By (3.1), we derive that

$$\begin{aligned} a_n x_n &= \sum_{k=0}^n \left[\sum_{i=k}^n \binom{m+n-i-1}{n-i} \binom{i}{k} (r-1)^{i-k} r^{-i} a_n \right] y_k \\ &= \sum_{k=0}^n d_{nk} y_k = (Dy)_n \end{aligned}$$

Also since $y \in c_0(p)$, there exists $M > 1$ such that $|y_k|^{p_k} < \frac{1}{M}$. Therefore we observe that $(a_n x_n) \in \ell_1$ whenever $x \in e_0^r(\Delta^m, p)$ if and only if $Dy \in \ell_1$ whenever $y \in c_0(p)$. By using Lemma 4.1 with $q_n = 1$ for every $n \in \mathbb{N}$ we get

$$\lim_{M \rightarrow \infty} \sup_K \sum_n \left| \sum_{k \in K} d_{nk} M^{-\frac{1}{p_k}} \right| = 0.$$

Consequently, we write that $[e_0^r(\Delta^m, p)]^\alpha = d_1^r(p)$. This completes the proof. $\square$

THEOREM 4.2. *Suppose that the matrix $D = (d_{nk})$ given with (4.7) and define the following sets:*

$$\begin{aligned} d_4^r(p) &= \left\{ a = (a_k) \in w : \lim_{M \rightarrow \infty} \sup_n \sum_k |d_{nk}| M^{-\frac{1}{p_k}} = 0 \right\}; \\ d_5^r(p) &= \left\{ a = (a_k) \in w : \lim_{M \rightarrow \infty} \sup_n \sum_k |d_{nk} - \alpha_k| M^{-\frac{1}{p_k}} = 0 \right\}; \\ d_6^r &= \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} |d_{nk} - \alpha_k| = 0 \text{ for all } k \in \mathbb{N} \right\}; \\ d_7^r &= \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} \sum_k |d_{nk} - \alpha_k| = 0 \text{ for all } k \in \mathbb{N} \right\}; \\ d_8^r(p) &= \left\{ a = (a_k) \in w : \sup_n \sum_k |d_{nk}| M^{\frac{1}{p_k}} < \infty \text{ for every } M \in \mathbb{N} \right\}; \\ d_9^r(p) &= \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} \sum_k |d_{nk} - \alpha_k| M^{\frac{1}{p_k}} = 0 \text{ for every } M \in \mathbb{N} \right\}; \\ d_{10}^r(p) &= \left\{ a = (a_k) \in w : \sup_n \sum_k |d_{nk}| M^{-\frac{1}{p_k}} < \infty \text{ for some } M \in \mathbb{N} \right\}; \\ d_{11}^r &= \left\{ a = (a_k) \in w : \sup_n \left| \sum_k d_{nk} \right| < \infty \right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} [e_0^r(\Delta^m, p)]^\beta &= d_4^r(p) \cap d_5^r(p) \cap d_6; & [e_0^r(\Delta^m, p)]^\gamma &= d_{10}^r(p); \\ [e_c^r(\Delta^m, p)]^\beta &= d_4^r(p) \cap d_5^r(p) \cap d_6 \cap d_7^r; & [e_c^r(\Delta^m, p)]^\gamma &= d_{10}^r(p) \cap d_{11}^r; \\ [e_\infty^r(\Delta^m, p)]^\beta &= d_8^r(p) \cap d_9^r(p); & [e_\infty^r(\Delta^m, p)]^\gamma &= d_8^r(p). \end{aligned}$$

Proof. We only give the proof for the space $e_0^r(\Delta^m, p)$. We consider the following equation

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^n \left[\sum_{j=0}^k \sum_{i=j}^k \binom{m+k-i-1}{k-i} \binom{i}{j} (r-1)^{i-j} r^{-i} y_j \right] a_k$$

$$\begin{aligned}
 &= \sum_{k=0}^n \left[\sum_{j=k}^n \sum_{i=k}^j \binom{m+j-i-1}{j-i} \binom{i}{k} (r-1)^{i-k} r^{-i} a_j \right] y_k \\
 &= \sum_{k=0}^n d_{nk} y_k = (Dy)_n
 \end{aligned}$$

where $D = (d_{nk})$ defined by (4.7). Therefore it is easy to see from Lemma 4.2. with $q_n = 1$ that $ax = (a_n x_n) \in cs$ whenever $x \in e_0^r(\Delta^m, p)$ if and only if $Dy \in c$ whenever $y \in c_0(p)$. This shows that

$$[e_0^r(\Delta^m, p)]^\beta = d_4^r(p) \cap d_5^r(p) \cap d_6^r.$$

By the same way, we can show that $ax = (a_n x_n) \in bs$ whenever $x \in e_0^r(\Delta^m, p)$ if and only if $Dy \in \ell_\infty$ whenever $y \in c_0(p)$. By Lemma 4.3 with $q_n = 1$ for all $n \in \mathbb{N}$, we obtain that

$$[e_0^r(\Delta^m, p)]^\gamma = d_{10}^r(p).$$

This completes the proof. $\square$

5. Some matrix mappings on the sequence space $e_0^r(\Delta, p)$

In this section, our demand is to characterize the matrix transformations from the sequence space $e_0^r(\Delta^m, p)$ into any given sequence space. Define the infinite matrix $B = (b_{nk})$ via an infinite matrix $A = (a_{nk})$ by $B = A(E^r \Delta^m)^{-1}$, i.e.,

$$b_{nk}(m, r) = \sum_{j=k}^{\infty} \binom{m+n-j-1}{n-j} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj} \quad (n, k \in \mathbb{N}) \quad (5.1)$$

and $(E^r \Delta^m)^{-1}$ is the inverse of products of the matrices Δ^m and E^r .

We now record our basic theorem for this section.

THEOREM 5.1. *Let μ be any given sequence space and the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation (5.1). Then $A \in (e_0^r(\Delta^m, p); \mu)$ if and only if $B \in (c_0(p); \mu)$.*

Proof. Suppose that (5.1) holds and μ be any given sequence space. Let $A \in (e_0^r(\Delta^m, p); \mu)$ and take any $y \in c_0(p)$. Then since $(a_{nk})_{k \in \mathbb{N}} \in [e_0^r(\Delta^m, p)]^\beta$, A -transform of $x \in e_0^r(\Delta^m, p)$ exists. Also since for every $x \in e_0^r(\Delta^m, p)$, $x_k(m, r) = ((\Delta^m)^{-1} E^{\frac{1}{r}})_k y_k(m, r)$, we have that $Ax = A((\Delta^m)^{-1} E^{\frac{1}{r}})y = By$, which leads us to the consequence $B \in (c_0(p); \mu)$.

Conversely, let $B \in (c_0(p); \mu)$ and (5.1) holds. Since for every $y \in c_0(p)$, $y_k(r, m) = (E^r \Delta^m x)_k$ whenever $x \in e_0^r(\Delta^m, p)$ and $(b_{nk})_{k \in \mathbb{N}} \in \ell_1$, B -transform

of y exists. Then we have that $By = B(E^r \Delta^m)x = Ax$ and this shows that $A \in (e_0^r(\Delta, p); \mu)$. This completes the proof. $\square$

We give some corollaries by combining Theorem 5.1 with Lemmas 4.1–4.3.

COROLLARY 5.1.1. $A \in (e_0^r(\Delta, p); \ell_\infty(q))$ if and only if (4.6) holds with b_{nk} in place of a_{nk} for all $n, k \in \mathbb{N}$.

COROLLARY 5.1.2. $A \in (e_0^r(\Delta, p); c(q))$ if and only if (4.3)–(4.5) hold with b_{nk} in place of a_{nk} for all $n, k \in \mathbb{N}$.

COROLLARY 5.1.3. $A \in (e_0^r(\Delta, p); c_0(q))$ if and only if (4.4), (4.5) hold with $\alpha_k = 0$ and b_{nk} in place of a_{nk} for all $n, k \in \mathbb{N}$.

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