VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0137-y
Math. Slovaca **63** (2013), No. 4, 829–838

Λ-STRONGLY SUMMABLE SEQUENCE SPACES IN n-NORMED SPACES DEFINED BY IDEAL CONVERGENCE AND AN ORLICZ FUNCTION

Ayhan Esi* — M. Kemal Özdemir**

(*Communicated by Wladyslaw Wilczynski*)

ABSTRACT. In this paper we introduce some certain new sequence spaces via ideal convergence, λ-sequence and an Orlicz function in n-normed spaces and study different properties of these spaces and also establish some inclusion results among them.

1. Introduction

Recall in [2] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M\,(x) + M\,(y)$ then this function is called the modulus function and characterized by Ruckle [5]. An Orlicz function M is said to satisfy Δ_2-condition for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M\,(2)$ for some constant $K > 0$.*

A sequence space X is said to be solid or normal if $(\alpha_k x_k) \in X$, and for all double sequences $\alpha = (\alpha_k)$ of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

2010 Mathematics Subject Classification: Primary 40A99, 40A05.
Keywords: λ-sequence, n-normed space, Orlicz function.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|\cdot, \dots, \cdot\|$ on X satisfying the following four condition:

(i) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,

(iii) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \, \|x_1, x_2, \dots, x_n\|$, $\alpha \in \mathbb{R}$,

(iv) $\|x_1 + x_1^{\imath}, x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1^{\imath}, x_2, \dots, x_n\|$

is called an n-norm on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space [1].

A trivial example of n-normed space is $X = \mathbb{R}$ equipped with the following Euclidean n-norm:

$$\|x_1, x_2, \dots, x_n\|_E = \text{abs}\left(\begin{vmatrix} x_{11} \dots x_{1n} \\ \dots \\ x_{n1} \dots x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

In paper [4], Mursaleen and Noman introduced the notion of λ-convergent and λ-bounded sequences as follows: let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, that is

$$0 < \lambda_o < \lambda_1 < \dots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called a the λ-limit of x, if $\Lambda_n(x) \to L$ as $n \to \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_n |\Lambda_n(x)| < \infty$. It is well known [4] that if $\lim_n x_n = a$ in the ordinary sense of convergence, then

$$\lim_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, |x_k - a| \right) = 0.$$

This implies that

$$\lim_n |\Lambda_n(x) - a| = \lim_n \left| \frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, (x_k - a) \right| = 0$$

which yields that $\lim_n \Lambda_n(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

2. Main Results

Let I be an ideal of $2^{\mathbb{N}}$, M be an Orlicz function, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. Further $w(n-X)$ denotes X-valued sequence space. Now, we define the following sequence spaces:

$$w_\Lambda^I [M, p, \|\cdot, \ldots, \cdot\|]_o = \Bigg\{ x = (x_k) \in w(n-X) :$$

$$\forall \varepsilon > 0, \ \left\{ n \in \mathbb{N} : \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{\Lambda_n(x)}{\rho}, \ z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

$$\text{for some } \ \rho > 0 \text{ and for every } z_1, z_2, \ldots, z_{n-1} \in X \Bigg\},$$

$$w_\Lambda^I [M, p, \|\cdot, \ldots, \cdot\|] = \Bigg\{ x = (x_k) \in w(n-X) :$$

$$\forall \varepsilon > 0, \ \left\{ n \in \mathbb{N} : \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{\Lambda_n(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

$$\text{for some } \ \rho > 0, \ L \in X \text{ and for every } \ z_1, z_2, \ldots, z_{n-1} \in X \Bigg\},$$

$$w_\Lambda^I [M, p, \|\cdot, \ldots, \cdot\|]_\infty = \Bigg\{ x = (x_k) \in w(n-X) :$$

$$\exists K > 0, \ \left\{ n \in \mathbb{N} : \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

$$\text{for some } \ \rho > 0 \text{ and for every } z_1, z_2, \ldots, z_{n-1} \in X \Bigg\}$$

and

$$w_\Lambda [M, p, \|\cdot, \ldots, \cdot\|]_\infty = \Bigg\{ x = (x_k) \in w(n-X) :$$

$$\exists K > 0, \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \leq K, \text{ for some } \rho > 0$$

$$\text{and for every } z_1, z_2, \ldots, z_{n-1} \in X \Bigg\}$$

The following well-known inequality will be used in this study:

If $0 \leq \inf_k p_k = H_o \leq p_k \leq \sup_k = H < \infty$, $D = \max\left(1, 2^{H-1}\right)$, then

$$|x_k + y_k|^{p_k} \leq D\left\{|x_k|^{p_k} + |y_k|^{p_k}\right\}$$

for all $k \in \mathbb{N}$ and $x_k, y_k \in \mathbb{C}$. Also $|x_k|^{p_k} \leq \max\left(1, |x_k|^H\right)$ for all $x_k \in \mathbb{C}$.

Theorem 1. *The sets* $w^I_\Lambda\left[M, p, \|\cdot, \dots, \cdot\|\right]_o$, $w^I_\Lambda\left[M, p, \|\cdot, \dots, \cdot\|\right]$ *and* $w^I_\Lambda\left[M, p, \|\cdot, \dots, \cdot\|\right]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We will prove only for $w^I_\Lambda\left[M, p, \|\cdot, \dots, \cdot\|\right]_o$ and the others can be proved similarly. Let $x, y \in w^I_\Lambda\left[M, p, \|\cdot, \dots, \cdot\|\right]_o$ and $\alpha, \beta \in \mathbb{C}$.

$$\left\{n \in \mathbb{N}:\ \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \in I,$$

for some $\rho_1 > 0$ and

$$\left\{n \in \mathbb{N}:\ \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(y)}{\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \in I,$$

for some $\rho_2 > 0$.

Since $\|\cdot, \dots, \cdot\|$ is a n-norm and M is an Orlicz function, the following inequality holds:

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(\alpha x + \beta y\right)}{|\alpha|\,\rho_1 + |\beta|\,\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\leq \frac{D}{n}\sum_{k=1}^{n}\left[\frac{|\alpha|}{|\alpha|\,\rho_1 + |\beta|\,\rho_2} M\left(\left\|\frac{\Lambda_n(x)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\quad + \frac{D}{n}\sum_{k=1}^{n}\left[\frac{|\beta|}{|\alpha|\,\rho_1 + |\beta|\,\rho_2} M\left(\left\|\frac{\Lambda_n(y)}{\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\leq \frac{DA}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\quad + \frac{DA}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(y)}{\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}
\end{aligned}$$

where

$$A = \max\left\{1, \left(\frac{|\alpha|}{|\alpha|\,\rho_1 + |\beta|\,\rho_2}\right)^H, \left(\frac{|\beta|}{|\alpha|\,\rho_1 + |\beta|\,\rho_2}\right)^H\right\}.$$

From the above inequality we get

$$\left\{n \in \mathbb{N}: \ \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(\alpha x+\beta y\right)}{|\alpha|\,\rho_1+|\beta|\,\rho_2}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subset \ \left\{n \in \mathbb{N}: \ \frac{DA}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}$$

$$\cup\left\{n \in \mathbb{N}: \ \frac{DA}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(y)}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}.$$

Two sets on the right hand side belong to I and this completes the proof. □

It is also easy verify that the space $w_\Lambda\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ is also a linear space.

Theorem 2. *For fixed $n \in \mathbb{N}$, $w_\Lambda\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ paranormed space with respect to the paranorm defined by*

$$h(x)=\inf\left\{\rho^{p_n/H}>0: \ \left(\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1,\right.$$

$$\left.\textit{for some } \ \rho>0 \ \textit{ and for every } \ z_1, z_2, \ldots, z_{n-1} \in X\right\}.$$

Proof. $h(\theta)=0$ and $h(-x)=h(x)$ are easy to prove, so we omit them. Let us take $x, y \in w_\Lambda\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$. Let

$$A(x)=\left\{\rho>0: \ \forall z \in X, \ \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1\right\}$$

and

$$A(y)=\left\{\rho>0: \ \forall z \in X, \ \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(y)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1\right\}.$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. If $\rho=\rho_1+\rho_2$, then we have

$$\begin{aligned}
&\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(x+y\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right] \\
&\quad\le \frac{\rho_1}{\rho_1+\rho_2}\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right] \\
&\qquad + \frac{\rho_2}{\rho_1+\rho_2}\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(y)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right].
\end{aligned}$$

Thus

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(x+y\right)}{\rho_1+\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \le 1$$

and

$$\begin{aligned}
h\left(x+y\right) &= \inf\left\{(\rho_1+\rho_2)^{p_n/H} :\ \rho_1 \in A(x) \text{ and } \rho_2 \in A(y)\right\} \\
&\le \inf\left\{(\rho_1)^{p_n/H} :\ \rho_1 \in A(x)\right\} + \inf\left\{(\rho_2)^{p_n/H} :\ \rho_2 \in A(y)\right\} \\
&= h(x) + h(y).
\end{aligned}$$

Now, let $\lambda_k^u \to \lambda$, where $\lambda_k^u, \lambda \in \mathbb{C}$ and $h\left(x_k^u - x_k\right) \to 0$ as $u \to \infty$. We have to show that $h\left(\lambda_k^u x_k^u - \lambda x_k\right) \to 0$ as $u \to \infty$. Let $\lambda_k \to \alpha$, where $\lambda_k, \lambda \in \mathbb{C}$ and $h\left(x_k^u - x_k\right) \to 0$ as $u \to \infty$. Let

$$\begin{aligned}
A\left(x^u\right) = \Bigg\{&\rho_u > 0 :\ \forall z_1, z_2, \dots, z_{n-1} \in X, \\
&\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(x^u\right)}{\rho_u}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \le 1\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
A\left(x^u - x\right) = \Bigg\{&\rho_u^\imath > 0 :\ \forall z_1, z_2, \dots, z_{n-1} \in X, \\
&\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n\left(x_k^u - x_k\right)}{\rho_u^\imath}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \le 1\Bigg\}.
\end{aligned}$$

If $\rho_u \in A\left(x^u\right)$ and $\rho_u^\imath \in A\left(x^u - x\right)$ then we observe that

$$\begin{aligned}
&M\left(\left\|\frac{\Lambda_n\left(\lambda_k^u x_k^u - \lambda x_k\right)}{\rho_u\left|\lambda_k^u - \lambda\right| + \rho_u^\imath\left|\lambda\right|}, z_1, z_2, \dots, z_{n-1}\right\|\right) \\
&\le M\left(\left\|\frac{\Lambda_n\left(\lambda_k^u x_k^u - \lambda x_k^u\right)}{\rho_u\left|\lambda_k^u - \lambda\right| + \rho_u^\imath\left|\lambda\right|}, z_1, z_2, \dots, z_{n-1}\right\|\right.
\end{aligned}$$

$$+\left\|\frac{\Lambda_n\left(\lambda x_k^u-\lambda x_k\right)}{\rho_u\left|\lambda_k^u-\lambda\right|+\rho_u^\imath\left|\lambda\right|}, z_1, z_2, \ldots, z_{n-1}\right\|\Bigg)$$
$$\leq \frac{\rho_u\left|\lambda_k^u-\lambda\right|}{\rho_u\left|\lambda_k^u-\lambda\right|+\rho_u^\imath\left|\lambda\right|} M\left(\left\|\frac{\Lambda_n\left(x_k^u\right)}{\rho_u}, z_1, z_2, \ldots, z_{n-1}\right\|\right)$$
$$+\frac{\rho_u^\imath\left|\lambda\right|}{\rho_u\left|\lambda_k^u-\lambda\right|+\rho_u^\imath\left|\lambda\right|} M\left(\left\|\frac{\Lambda_n\left(x_k^u-x_k\right)}{\rho_u^\imath}, z_1, z_2, \ldots, z_{n-1}\right\|\right).$$

From this inequality, it follows that

$$\left[M\left(\left\|\frac{\Lambda_n\left(\lambda_k^u x_k^u-\lambda x_k\right)}{\rho_u\left|\lambda_k^u-\lambda\right|+\rho_u^\imath\left|\lambda\right|}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1$$

and consequently

$$\begin{aligned} h\left(\lambda_k^u x_k^u-\lambda x_k\right) = & \inf\left\{\left(\rho_u\left|\lambda_k^u-\lambda\right|+\rho_u^\imath\left|\lambda\right|\right)^{p_n/H} : \rho_u \in A\left(x^u\right), \rho_u^\imath \in A\left(x^u-x\right)\right\} \\ \leq & \left(\left|\lambda_k^u-\lambda\right|\right)^{p_n/H} \inf\left\{\left(\rho_u\right)^{p_n/H} : \rho_u \in A\left(x^u\right)\right\} \\ & + \left(\left|\lambda\right|\right)^{p_n/H} \inf\left\{\left(\rho_u^\imath\right)^{p_n/H} : \rho_u^\imath \in A\left(x^u-x\right)\right\} \\ \leq & \max\left\{\left|\lambda\right|, \left(\left|\lambda\right|\right)^{p_n/H}\right\} h\left(x_k^u-x_k\right). \end{aligned}$$

Hence by our assumption the right hand side tends to zero as $u \to \infty$. This completes the proof. □

Corollary 1. *It can be noted that $g = \inf_{n \in \mathbb{N}} h$ also gives a paranorm on the above sequence spaces. However if one consider the sequence space $w_\Lambda\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ which is larger space than the space $w_\Lambda^I\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ the construction of the paranorm is not clear and we leave it as an open problem. However it should be noted that for a fixed $F \in I$, the space*

$$\begin{aligned} w_\Lambda^F\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty = \Bigg\{ & x = (x_k) \in w\left(n-X\right) : \exists K > 0, \\ & \left\{n \in \mathbb{N} : \sup_{n \in \mathbb{N}/F} \frac{1}{n} \sum_{k=1}^n \left[M\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq K\right\} \in I, \\ & \text{for some } \rho > 0 \text{ and for every } z_1, z_2, \ldots, z_{n-1} \in X \Bigg\} \end{aligned}$$

which is subspace of the space $w_\Lambda^I\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ is a paranormed space with the paranorms h for $n \notin F$ and $g = \inf_{n \in \mathbb{N}/F} h$.

Theorem 3. *Let M, M_1 and M_2 be Orlicz functions. Then we have*

(i) $w_\Lambda^I \left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o \subset w_\Lambda^I \left[MoM_1, p, \|\cdot, \ldots, \cdot\|\right]_o$ *provided that* $p = (p_k)$ *is such that* $H_o > 0$.

(ii) $w_\Lambda^I \left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o \cap w_\Lambda^I \left[M_2, p, \|\cdot, \ldots, \cdot\|\right]_o \subset w_\Lambda^I \left[M_1 + M_2, p, \|\cdot, \ldots, \cdot\|\right]_o$.

Proof.

(i) For given $\varepsilon > 0$, we first choose $\varepsilon_o > 0$ such that $\max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < \varepsilon$. Now using the continuity of M, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M(t) < \varepsilon_o$. Let $x \in w_\Lambda^I \left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o$. Now from the definition of the space $w_\Lambda^I \left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o$, for some $\rho > 0$

$$A(\delta) = \left\{ n \in \mathbb{R} : \ \frac{1}{n} \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta^H \right\} \in I.$$

Thus if $(n, m) \notin A(\delta)$ then

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} < \delta^H \\
\Longrightarrow \ & \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} < n\delta^H, \\
\Longrightarrow \ & \left[M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \delta^H \quad \text{for all } n \in \mathbb{N}, \\
\Longrightarrow \ & M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) < \delta \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Hence from above inequality and using continuity of M, we must have

$$M \left(M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right) < \varepsilon_o \quad \text{for all } n \in \mathbb{N}$$

which consequently implies that

$$\sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right) \right]^{p_k} < n \max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < n\varepsilon,$$

$$\Longrightarrow \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right) \right]^{p_k} < \varepsilon.$$

This shows that

$$\left\{ n \in \mathbb{N} : \ \frac{1}{n} \sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right) \right]^{p_k} \geq \varepsilon \right\} \subset A(\delta)$$

and so belongs to I. This completes the proof.

(ii) Let $x \in w_{\Lambda}^{I}\left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o \cap w_{\Lambda}^{I}\left[M_2, p, \|\cdot, \ldots, \cdot\|\right]_o$. Then the fact that

$$\begin{aligned}
&\frac{1}{n}\left[(M_1 + M_2)\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\le \frac{D}{n}\left[M_1\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\quad + \frac{D}{n}\left[M_2\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}
\end{aligned}$$

gives us the result. □

Theorem 4.

(i) *If* $0 < H_o \le p_k < 1$, *then* $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o \subset w_{\Lambda}^{I}\left[M, \|\cdot, \ldots, \cdot\|\right]_o$.

(ii) *If* $1 \le p_k \le H < \infty$, *then* $w_{\Lambda}^{I}\left[M, \|\cdot, \ldots, \cdot\|\right]_o \subset w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$.

(iii) *If* $0 < p_k < q_k < \infty$ *and* $\frac{q_k}{p_k}$ *is bounded, then* $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$ $\subset w_{\Lambda}^{I}\left[M, q, \|\cdot, \ldots, \cdot\|\right]_o$.

P r o o f. The proof is standard, so we omit it. □

Theorem 5. *The sequence spaces* $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$, $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]$, $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ *and* $w_{\Lambda}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_\infty$ *are solid.*

P r o o f. We give the proof for only $w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$. The others can be proved similarly. Let $x \in w_{\Lambda}^{I}\left[M_1, p, \|\cdot, \ldots, \cdot\|\right]_o$ and $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(\alpha x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \le \varepsilon\right\} \\
&\subset \left\{n \in \mathbb{N} : \frac{T}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Lambda_n(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \le \varepsilon\right\} \in I,
\end{aligned}$$

where $T = \max\limits_k \left\{1, |\alpha_k|^H\right\}$. Hence $\alpha x \in w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$ for all double sequences $\alpha = (\alpha_k)$ with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$ whenever $x \in w_{\Lambda}^{I}\left[M, p, \|\cdot, \ldots, \cdot\|\right]_o$. □

REFERENCES

[1] GUNAWAN, H.: *On n-inner product, n-norms and the Cauchy-Schwarz Inequality*, Sci. Math. Jpn. **5** (2001), 47–54.

[2] KRASNOSELSKI, M. A.—RUTICKII, Y. B.: *Convex Function and Orlicz Spaces*, Groningen, Nederland, 1961.

[3] KOSTYRKO, P.—ŠALÁT, T.—WILCZYNSKI, W.: *I-convergence*, Real Anal. Exchange **26** (2000/2001), 669–686.

[4] MURSALEEN, M.—NOMAN, A. K.: *On the spaces of λ-convergent and bounded sequences*, Thai J. Math. **8** (2010), 311–329.

[5] RUCKLE, W. H.: *FK-spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.

[6] ŠALÁT T.—TRIPATHY, B. C.—ZIMAN, M.: *On I-convergence field*, Ital. J. Pure Appl. Math. **17** (2005), 45–54.

Received 10. 2. 2011
Accepted 20. 8. 2011

* *Adiyaman University*
Science and Arts Faculty
Department of Mathematics
TR–02040 Adiyaman
TURKEY
E-mail: aesi23@hotmail.com

** *Inonu University*
Science and Arts Faculty
Department of Mathematics
TR–44280 Malatya
TURKEY
E-mail: kozdemir73@gmail.com