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EXTREMAL SOLUTIONS OF CAUCHY PROBLEMS FOR ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the extremal solutions of Cauchy problems for abstract fractional differential equations. Some definitions such as L^1-Lipschitz-like, L^1-Carathéodory-like and L^1-Chandrabhan-like are introduced. By virtue of the singular integral inequalities with several nonlinearities due to Medveď, the properties of solutions are given. By using a hybrid fixed point theorem due to Dhage, existence results for extremal solutions are established. Finally, we present an example to illustrate our main results.

1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ will be a Banach spaces, and $T > 0$, $J := [0, T]$. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, X)$.

We consider the following Cauchy problems of fractional differential equation

$$\begin{cases} {}^cD^\alpha u(t) = \overline{F}(t, u(t)), & \text{a.e. } t \in J, \\ u(0) = u_0, \end{cases} \tag{1}$$

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where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $\overline{F}(t,u(t)) = f(t,u(t)) + g(t,u(t)) + h(t,u(t))$, $f\colon J \times X \to X$, $g\colon J \times X \to X$, $h\colon J \times X \to X$ are given functions satisfying some assumptions that will be specified latter.

Fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of biomedical sciences, engineering, physics and economics. For more details, one can see the monographs of Diethelm [8], Kilbas et al. [12], Lakshmikantham et al. [13], Miller and Ross [18], Podlubny [24] and Tarasov [25]. Moreover, fractional differential equations (inclusions) and optimal controls in Banach spaces are studied by Balachandran et al. [2,3], Benchohra et al. [4,5], N'Guérékata [21,22], Mophou and N'Guérékata [19], Wang et al. [26,27] and Zhou et al. [28–30] and etc.

On the one hand, the existence of extremal solutions for integer differential equations have been investigated by Aizicovici and Papageorgiou [1], Dhage [6], Nieto and Rodriguez-López [20] and references therein. On the other hand, the existence results of extremal solutions for fractional differential equations with deviating arguments involving Riemann-Liouville derivative has been reported by Jankowski [11]. To deal with the problem of the existence of extremal solutions, Jankowski applied the well known monotone iterative technique. Recently, Zhou and Jiao [28] discussed the existence of extremal solutions for discontinuous fractional functional differential equations involving Caputo derivative and applied the hybrid fixed point methods to study such problems. However, both the results obtained in [11] and [28] hold only in finite dimensional spaces.

To our knowledge, the problem of the extremal solutions for abstract fractional differential equations has not been studied extensively. In the present paper, we study the existence of the extremal solutions for system (1) in a Banach space X. Utilizing fractional calculus, the singular integral inequalities with several nonlinearities (Lemma 6) and fixed point method, existence results for the system (1) are presented. Compared with the earlier results obtained in [28], many more general definitions such as L^1-Lipschitz-like, L^1-Carathéodory-like and L^1-Chandrabhan-like are introduced. We firstly give an important result which display the equivalent relationship between the solution (in the Carathéodory sense) for the system (1) and a Volterra fractional integral equation. Secondly, we study the properties of solutions for the system (1). Then, the existence results for extremal solutions are proved by applying the Dhage hybrid fixed point theorem again.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, some definitions of solutions such as lower solution, supper solution, maximal solution, minimal solution and two important lemmas are given. In Section 4, the existence results for extremal solutions are proved. Finally, we give an example to illustrate the usefulness of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us recall the following known definitions. For more details see [12].

Definition 1. The fractional integral of order γ with the lower limit zero for a function $f\colon [0,\infty) \to \mathbb{R}$ is defined as

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} \, \mathrm{d}s, \qquad t > 0, \ \ \gamma > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f\colon [0,\infty) \to \mathbb{R}$ can be written as

$$^{L}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} \, \mathrm{d}s, \qquad t > 0, \ \ n-1 < \gamma < n.$$

Definition 3. The Caputo derivative of order γ for a function $f\colon [0,\infty) \to \mathbb{R}$ can be written as

$$^{c}D^{\gamma} f(t) = {}^{L}D^{\gamma} \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \qquad t > 0, \ \ n-1 < \gamma < n.$$

Remark 1. (i) If $f(t) \in C^n[0,\infty)$, then

$$^{c}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} \, \mathrm{d}s = I^{n-\gamma} f^{(n)}(t), \qquad t > 0, \ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 1 and 2 are taken in Bochner's sense.

Assume that $1 \leq p \leq \infty$. For measurable functions $m\colon J \to \mathbb{R}$, define the norm

$$\|m\|_{L^pJ} = \begin{cases} \Big(\int\limits_J |m(t)|^p \,\mathrm{d}t\Big)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \Big\{ \sup\limits_{t \in J - \bar{J}} |m(t)| \Big\}, & p = \infty \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J,\mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m\colon J \to \mathbb{R}$ with $\|m\|_{L^pJ} < \infty$.

Lemma 1 (**Hölder inequality**). *Assume that $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi \in L^p(J,\mathbb{R})$, $\varphi \in L^q(J,\mathbb{R})$, then for $1 \leq q \leq \infty$, $\phi\varphi \in L^1(J,\mathbb{R})$ and*

$$\|\phi\varphi\|_{L^1J} \leq \|\phi\|_{L^pJ}\|\varphi\|_{L^qJ}.$$

Lemma 2 (**Bochner theorem**). *A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 3 (**Mazur lemma**). *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{\mathrm{conv}}\mathcal{K}$ is compact.*

Lemma 4 (**Ascoli-Arzela theorem**). *Let $\mathcal{W} = \{s(t)\}$ be a function family of continuous mappings $s\colon J \to X$. If $\mathcal{W}$ is uniformly bounded and equicontinuous, and for any $t^* \in J$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1, 2, \ldots, t \in J)$ in $\mathcal{W}$.*

Definition 4. An operator $S\colon X \to X$ is called compact if $\overline{S(X)}$ is a compact subset of X. $S\colon X \to X$ is called totally bounded if S maps the bounded subsets of X into the relatively compact subsets of X. Finally, $S\colon X \to X$ is called a completely continuous operator, if it is a continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on the bounded subsets of X.

Definition 5. A nonempty closed set K in a Banach space X is called a cone if satisfies the following conditions

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for $\lambda \in R, \lambda \geq 0$ and
(iii) $\{-K\} \cap K = \{0\}$, where 0 is the zero element of X.

We introduce an order relation "$\leq$" in X as follows. Let $z, y \in X$. Then $z \leq y$ if and only if $y - z \in K$.

Definition 6. A cone K is called *normal* if the norm $\|\cdot\|$ is semimonotone increasing on K, that is, there is a constant $N > 0$ such that $\|z\| \leq N\|y\|$ for all $z, y \in K$ with $z \leq y$.

It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. Similarly, the cone K in X is called regular if every monotone increasing (resp. decreasing) order bounded sequence in X converges in norm. The details of cones and their properties appear in Heikkilä and Lakshmikantham [9].

For any $a, b \in X$, $a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{z \in X : \ a \leq z \leq b\}.$$

Definition 7. Let X and Y be two ordered Banach spaces. A mapping $S\colon X \to Y$ is said to be nondecreasing or monotone increasing if $z \leq y$ implies $Sz \leq Sy$ for all $z, y \in [a, b]$.

We use the following hybrid fixed point theorem of Dhage [7].

Lemma 5 (Hybrid fixed point theorem). *Let X be a Banach space and let $A, B, C\colon X \to X$ be three monotone increasing operators such that*

(i) *A is a contraction with contraction constant $\ell < 1$,*
(ii) *B is completely continuous,*
(iii) *C is totally bounded, and*
(iv) *there exist elements a and b in X such that*

$$a \leq Aa + Ba + Ca \quad \text{and} \quad b \geq Ab + Bb + Cb \qquad \text{with} \quad a \leq b.$$

Further if the cone K in X is normal, then the operator equation $Az + Bz + Cz = z$ has a least and a greatest solution in the order interval $[a, b]$.

To end this section, we recalled the following singular integral inequalities with several nonlinearities which appeared in Medveď [17].

Lemma 6. *We make the following assumptions:*

(C1): *the functions $w_i\colon \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are continuous and positive on $(0, \infty)$ such that $w_{i_1} \propto w_{i_2} \propto \cdots \propto w_{i_n}$ for some $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}$ simultaneously different, where $w_{i_k} \propto w_{i_{k+1}}$ denotes $\frac{w_{i_k}}{w_{i_{k+1}}}$ is nondecreasing on $(0, \infty)$, $k = 1, 2, \ldots, n$.*

(C2): *the functions v, $\lambda_i\colon [0, \overline{T}] \to (0, \infty)$, $i = 1, 2, \ldots, N$, $\infty > \overline{T} > 0$, are continuous, nonnegative, and $\bar{a}$ is a positive constant.*

(C3): *$0 < \beta_i < 1$, $i = 1, 2, \ldots, n$, $\beta_i \neq \beta_j$ for $i \neq j$.*

If $\varepsilon > 0$, $k \geq 1$ and the function $v(t)$ satisfies the following inequality

$$v(t)^k \leq \bar{a} + \sum_{i=1}^{N} \int_0^t (t-s)^{\beta_i - 1} \lambda_i(s) w_i(v(s))\,\mathrm{d}s + \sum_{j=N+1}^{n} \int_0^t \lambda_j(s) w_j(v(s))\,\mathrm{d}s, \qquad t \in [0, \overline{T}],$$

then

$$v(t) \leq \left(W_{i_n}^{-1} \left[W_{i_n}(c_{i_n - 1}) + \int_0^t \widehat{\lambda}_{i_n}(s)\,\mathrm{d}s \right] \right)^{\frac{1}{kr}}, \qquad t \in J = [0, T],$$

where

$$W_{i_m}(v) = \int_{v_m}^{v} \frac{\mathrm{d}\sigma}{[w_{i_m}(\sigma^{\frac{1}{kr}})]^r}, \qquad v \geq v_m = W_{i_{m-1}}^{-1}(v_{i_{m-1}}), \quad v_0 > 0$$

$$r = r(\varepsilon) := q_1 q_2, \ldots, q_N, \qquad q_i = \frac{1}{\beta_i} + \varepsilon, \quad \varepsilon > 0,$$

$$\beta_i = \frac{1}{1 + z_i}, \qquad z_i > 0, \quad i = 1, 2, \ldots, N,$$

$W_{i_m}^{-1}$ *is the inverse function of* W_{i_m}, $c_0 = (n+1)^{r-1} \bar{a}^r$,

$$c_{i_m} = W_{i_m}^{-1} \left[W_{i_m}(c_{i_{m-1}}) + \int_0^{\overline{T}} \widehat{\lambda}_{i_m}(s)\,\mathrm{d}s \right], \qquad m = 1, 2, \ldots, n$$

$$\widehat{\lambda}_i(t) = \overline{T}^{\widehat{q}_i - 1}(n+1)^{r-1} d_i^r \mathrm{e}^{-rt} \lambda_i(t)^r, \qquad \widehat{q}_i = q_1 q_2 \ldots q_{i-1} q_{i+1} \cdots q_N,$$

$$d_i = c_i^{\frac{1}{p_i}} \mathrm{e}^{\overline{T}}, \quad i = 1, 2, \ldots, N, \qquad \widehat{\lambda}_j(t) = \overline{T}^{r-1}(n+1)^{r-1} \lambda_j(s)^r,$$

$$j = N+1, \ldots, n, \qquad p_i = \frac{1 + z_i + \varepsilon}{z_i + \varepsilon}, \qquad c := c_i = c_i(\alpha_i, p_i) > 0$$

satisfies the following inequality

$$\int_0^t (t-s)^{-\alpha_i p_i} \mathrm{e}^{\beta_i s}\,\mathrm{d}s \leq C \mathrm{e}^{\beta_i t}, \qquad t \geq 0, \quad C = C(\alpha_i p_i, \quad \beta_i) > 0,$$

with $\alpha_i = 1 - \beta_i$ *and the number* $T \in [0, \overline{T}]$ *is the largest number such that*

$$\int_0^T \widehat{\lambda}_{i_m}(s)\,\mathrm{d}s \leq \int_{c_{m-1}}^{\infty} \frac{\mathrm{d}\sigma}{[w_{i_m}(\sigma^{\frac{1}{kr}})]^r}, \qquad m = 1, 2, \ldots, n.$$

Remark 2. This lemma is a generalization of the Henry-Gronwall lemma (see [10]) and the Pinto integral inequality (see [23]). Its proof is based on a desingularization method developed in the paper [14] (for applications of this method see [15, 16]). We remark that $C = C(\alpha_i p_i, \beta_i) = \frac{\Gamma(1-\alpha_i p_i)}{p_i^{1-\alpha_i p_i}}$. Since $\alpha_i p_i = 1$ for $\varepsilon = 0$, it is necessary to assume $\varepsilon > 0$. For any $\varepsilon > 0$ the argument $1 - \alpha_i p_i$ of the Gamma function is positive and thus $C = C(\alpha_i p_i, \beta_i)$ is a positive number.

3. Some definitions and important lemmas

We give the following definitions in the sequel.

Definition 8 (L^1-*Lipschitz-like*)**.** A mapping $f\colon J \times X \to X$ is called L^1-Lipschitz-like if

(i) $f(\cdot, u)$ is Lebesgue measurable for all $u \in X$,

(ii) there exist functions l_1 *and* w_1, $l_1\colon J \to \mathbb{R}_+$ is continuous, nonnegative, $w_1\colon [0, \infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that

$$\|f(t,u) - f(t,v)\| \leq l_1(t) w_1(\|u - v\|), \qquad t \in J$$

for all $u, v \in X$, where $w_1(\|u - v\|) = \|u - v\|$.

Definition 9 (L^1-*Carathéodory-like*)**.** A mapping $g\colon J \times X \to X$ is said to be Carathéodory if

(i) $g(\cdot, u)$ is Lebesgue measurable for all $u \in X$,

(ii) $g(t, u)$ is continuous with respect to u for any $u \in X$ and almost all $t \in J$.

Furthermore, a Carathéodory function $g(t, u)$ is called L^1-Carathéodory-like if

(iii) there exist functions l_2 and w_2, $l_2\colon J \to \mathbb{R}_+$ are continuous and nonnegative, $w_2\colon [0, \infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that

$$\|g(t,u)\| \leq l_2(t) w_2(\|u\|), \qquad t \in J$$

for all $u \in X$.

Definition 10 (L^1-*Chandrabhan-like*)**.** A mapping $h\colon J \times X \to X$ is said to be Chandrabhan if

(i) $h(\cdot, u)$ is Lebesgue measurable for all $u \in X$,

(ii) $h(t, u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.

Furthermore, a Chandrabhan function $h(t, u)$ is called L^1-Chandrabhan-like if

(iii) there exist functions l_3 and w_3, $l_3\colon J \to \mathbb{R}_+$ are continuous and nonnegative, and $w_3\colon [0,\infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that

$$\|h(t,u)\| \le l_3(t) w_3(\|u\|), \qquad t \in J,$$

for all $u \in X$.

Definition 11. A function $u \in C(J,X)$ is called a solution of system (1) on J if

(i) the function $u(t)$ is absolutely continuous on J,
(ii) $u(0) = u_0$, and
(iii) u satisfies the equation in (1).

We need the following hypotheses in the sequel.

(H_1) f, g, $h\colon J \times X \to X$,
(f_1) f is L^1-Lipschitz-like, and there exists a $\eta \in [0,\alpha)$ such that $\|f(t,0)\| \in L^{\frac{1}{\eta}}(J,\mathbb{R}_+)$,
(g_1) g is L^1-Carathéodory-like,
(h_1) h is L^1-Chandrabhan-like.

For any positive constant $\rho > 0$, let $\mathcal{B}_\rho = \{u \in C(J,X) : \|u\|_C \le \rho\}$. For brevity, set

$$L_i = \max_{t\in J}\{l_i(t)\}, \qquad i = 1,2,3,$$
$$q_1 = \frac{\alpha-1}{1-\eta} \in (-1,0), \qquad F = \|f(t,0)\|_{L^{\frac{1}{\eta}} J}.$$

Lemma 7. *Assume that the hypotheses (H_1), (f_1), (g_1) and (h_1) hold. A function $u \in C(J,X)$ is a solution of the fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s,u(s)) + g(s,u(s)) + h(s,u(s))] \,\mathrm{d}s, \tag{2}$$

if and only if u is a solution of the system (1).

Proof. For any constants $\rho > 0$, set $u \in \mathcal{B}_\rho$. According to (f_1), (g_1), (h_1), $f(t,u(t))$, $g(t,u(t))$, $h(t,u(t))$ are measurable functions for $t \in J$. Further, we have

$$\begin{aligned}
\int_0^t \|(t-s)^{\alpha-1} f(s,u(s))\| \,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1}(l_1(s)\|u(s)\| + \|f(s,0)\|)\,\mathrm{d}s \\
&\le \frac{L_1 \rho T^\alpha}{\alpha} + \frac{F T^{(1+q_1)(1-\eta)}}{(1+q_1)^{1-\eta}}, \\
\int_0^t \|(t-s)^{\alpha-1} g(s,u(s))\| \,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1} l_2(s) w_2(\|u(s)\|)\,\mathrm{d}s \\
&\le \frac{L_2 w_2(\rho) T^\alpha}{\alpha}, \\
\int_0^t \|(t-s)^{\alpha-1} h(s,u(s))\| \,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1} l_3(s) w_3(\|u(s)\|)\,\mathrm{d}s \\
&\le \frac{L_3 w_3(\rho) T^\alpha}{\alpha}.
\end{aligned}$$

It comes from the well known Bochner theorem that $(t-s)^{\alpha-1} f(s,u(s))$, $(t-s)^{\alpha-1} g(s,u(s))$ and $(t-s)^{\alpha-1} h(s,u(s))$ are Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $u \in \mathcal{B}_\rho$.

As a result, $(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $u \in \mathcal{B}_\rho$.

Let $G(\tau,s) = (\cdot - \tau)^{-\alpha}|\tau - s|^{\alpha-1}$. Since $G(\tau,s)$ is a nonnegative, measurable function on $D = [0,t] \times [0,t]$ for $t \in J$, we have

$$\int_0^t \left[\int_0^t G(\tau,s)\,\mathrm{d}s\right]\mathrm{d}\tau = \int_D G(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau = \int_0^t \left[\int_0^t G(\tau,s)\,\mathrm{d}\tau\right]\mathrm{d}s$$

and

$$\begin{aligned}
\int_D G(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau &= \int_0^t \left[\int_0^t G(\tau,s)\,\mathrm{d}s\right]\mathrm{d}\tau = \int_0^t (t-\tau)^{-\alpha}\left[\int_0^t |\tau-s|^{\alpha-1}\,\mathrm{d}s\right]\mathrm{d}\tau \\
&= \int_0^t (t-\tau)^{-\alpha}\left[\int_0^\tau (\tau-s)^{\alpha-1}\,\mathrm{d}s\right]\mathrm{d}\tau + \int_0^t (t-\tau)^{-\alpha}\left[\int_\tau^t (s-\tau)^{\alpha-1}\,\mathrm{d}s\right]\mathrm{d}\tau \\
&\le \frac{2T^\alpha}{\alpha(1-\alpha)}.
\end{aligned}$$

Since $G_1(\tau,s) = (\cdot - \tau)^{-\alpha}(\tau - s)^{\alpha-1}g(s,u(s))$ is a Lebesgue integrable function on $D = [0,t] \times [0,t]$, we have

$$\int_0^t \mathrm{d}\tau \int_0^\tau G_1(\tau,s)\,\mathrm{d}s = \int_0^t \mathrm{d}s \int_s^t G_1(\tau,s)\,\mathrm{d}\tau.$$

Similarly, $G_2(\tau,s) = (\cdot - \tau)^{-\alpha}(\tau - s)^{\alpha-1}h(s,u(s))$ is a Lebesgue integrable function on $D = [0,t] \times [0,t]$, we also have

$$\int_0^t \mathrm{d}\tau \int_0^\tau G_2(\tau,s)\,\mathrm{d}s = \int_0^t \mathrm{d}s \int_s^t G_2(\tau,s)\,\mathrm{d}\tau.$$

We now prove that

$$\begin{aligned} {}^{L}D^{\alpha}(I^{\alpha}[f(t,u(t)) + g(t,u(t)) + h(t,u(t))]) \\ = [f(t,u(t)) + g(t,u(t)) + h(t,u(t))], \end{aligned} \tag{3}$$

for $t \in (0,T]$, where ${}^{L}D^{\alpha}$ is Riemann-Liouville fractional derivative.

First, one can verify that

$$\begin{aligned} &\int_s^t (t-\tau)^{-\alpha}(\tau-s)^{\alpha-1}\,\mathrm{d}\tau = \int_s^t (\tau-s)^{\alpha-1}(t-\tau)^{-\alpha}\,\mathrm{d}\tau \\ &= -\int_s^t (\tau-s)^{\alpha-1}(t-\tau)^{-\alpha}\,\mathrm{d}t-\tau = -\int_{t-s}^0 (\tau-s)^{\alpha-1}\eta^{-\alpha}\,\mathrm{d}\eta \qquad (\text{let } \eta = t-\tau) \\ &= \int_0^{t-s} (t-s-\eta)^{\alpha-1}\eta^{-\alpha}\,\mathrm{d}\eta = \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(\alpha-\alpha+1)}(t-s)^{\alpha-\alpha} = \Gamma(1-\alpha)\Gamma(\alpha). \end{aligned}$$

Second, note that $G_1(\tau,s) = (\cdot - \tau)^{-\alpha}(\tau - s)^{\alpha-1}g(s,u(s))$ and one can derive the following equality

$$\begin{aligned} {}^{L}D^{\alpha}(I^{\alpha}g(t,u(t)) &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\alpha}\left[\int_0^\tau (\tau-s)^{q-1}g(s,u(s))\,\mathrm{d}s\right]\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{d}\tau \int_0^\tau G_1(\tau,s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{d}s \int_s^t G_1(\tau, s)\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(s, u(s))\,\mathrm{d}s \bigg(\int_s^t (t-\tau)^{-\alpha} (\tau - s)^{\alpha-1}\,\mathrm{d}\tau \bigg) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(s, u(s))\,\mathrm{d}s \\
&= g(t, u(t)).
\end{aligned}
$$

Similarly, we can get ${}^L D^\alpha (I^\alpha f(t, u(t)) = f(t, u(t))$, ${}^L D^\alpha (I^\alpha h(t, u(t)) = h(t, u(t))$, for $t \in (0, T]$ which implies that (3) holds.

If u satisfies the relation (2), then we get that $u(t)$ is absolutely continuous on J. In fact, for any disjoint family of open intervals $\{(a_i, b_i)\}_{1 \le i \le n}$ on J with $\sum_{i=1}^{n} (b_i - a_i) \to 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n} \|u(b_i) - u(a_i)\| \\
\le\ & \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \bigg\| \int_{a_i}^{b_i} (b_i - s)^{\alpha-1} [f(s, u(s)) + g(s, u(s)) + h(s, u(s))]\,\mathrm{d}s \bigg\| \\
&+ \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \bigg\| \int_0^{a_i} (b_i - s)^{\alpha-1} [f(s, u(s)) + g(s, u(s)) + h(s, u(s))]\,\mathrm{d}s \\
&- \int_0^{a_i} (a_i - s)^{\alpha-1} [f(s, u(s)) + g(s, u(s)) + h(s, u(s))]\,\mathrm{d}s \bigg\| \\
\le\ & \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \int_{a_i}^{b_i} (b_i - s)^{\alpha-1} [l_1(s)\|u(s)\| + \|f(s, 0)\| \\
&+ l_2(s) w_2(\|u(s)\|) + l_3(s) w_3(\|u(s)\|)]\,\mathrm{d}s \\
&+ \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \int_0^{a_i} ((a_i - s)^{\alpha-1} - (b_i - s)^{\alpha-1}) [l_1(s)\|u(s)\| + \|f(s, 0)\| \\
&+ l_2(s) w_2(\|u(s)\|) + l_3(s) w_3(\|u(s)\|)]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\left(\frac{L_1\rho(b_i-a_i)^\alpha}{\Gamma(\alpha+1)}+\frac{F(b_i-a_i)^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\right)\\
&\quad+\sum_{i=1}^{n}\left(\frac{L_2w_2(\rho)(b_i-a_i)^\alpha}{\Gamma(\alpha+1)}+\frac{L_3w_3(\rho)(b_i-a_i)^\alpha}{\Gamma(\alpha+1)}\right)\\
&\quad+\sum_{i=1}^{n}\left(\frac{L_1\rho(a_i^\alpha-b_i^\alpha+(b_i-a_i)^\alpha)}{\Gamma(\alpha+1)}+\frac{F(a_i^{1+q_1}-b_i^{1+q_1}+(b_i-a_i)^{(1+q_1)(1-\eta)})}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\right)\\
&\quad+\sum_{i=1}^{n}\left(\frac{L_2w_2(\rho)(a_i^\alpha-b_i^\alpha+(b_i-a_i)^\alpha)}{\Gamma(\alpha+1)}+\frac{L_3w_3(\rho)(a_i^\alpha-b_i^\alpha+(b_i-a_i)^\alpha)}{\Gamma(\alpha+1)}\right)\\
&\leq 2\sum_{i=1}^{n}\frac{L_1\rho(b_i-a_i)^\alpha}{\Gamma(\alpha+1)}+2\sum_{i=1}^{n}\frac{F(b_i-a_i)^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}\\
&\quad+2\sum_{i=1}^{n}\frac{L_2w_2(\rho)(b_i-a_i)^\alpha}{\Gamma(\alpha+1)}+2\sum_{i=1}^{n}\frac{L_3w_3(\rho)(b_i-a_i)^\alpha}{\Gamma(\alpha+1)} \ \to\ 0.
\end{aligned}
$$

Therefore, $u(t)$ is absolutely continuous on J which implies that $u(t)$ is differentiable for almost all $t\in J$.

According to the argument above and Remark 1, for almost all $t\in(0,T]$, we have

$$
\begin{aligned}
{}^cD^\alpha u(t) &= {}^cD^\alpha\left[u_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]\,\mathrm{d}s\right]\\
&= {}^cD^\alpha\left[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]\,\mathrm{d}s\right]\\
&= {}^cD^\alpha(I^\alpha[f(t,u(t))+g(t,u(t))+h(t,u(t))]\\
&= {}^LD^\alpha(I^\alpha[f(t,u(t))+g(t,u(t))+h(t,u(t))])\\
&\qquad-(I^\alpha[f(t,u(t))+g(t,u(t))+h(t,u(t))])_{t=0}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

We need to prove

$$
(I^\alpha[f(s,u(s))+g(s,u(s))+h(s,u(s))])_{t=0}=0. \tag{4}
$$

In other word, we need to prove

$$
\left\|\int_0^t(t-s)^{\alpha-1}f(s,u(s))\,\mathrm{d}s\right\|\to 0, \qquad \text{as}\quad t\to 0. \tag{5}
$$

$$\left\| \int_0^t (t-s)^{\alpha-1} g(s,u(s))\,\mathrm{d}s \right\| \to 0, \qquad \text{as} \quad t \to 0. \tag{6}$$

$$\left\| \int_0^t (t-s)^{\alpha-1} h(s,u(s))\,\mathrm{d}s \right\| \to 0, \qquad \text{as} \quad t \to 0. \tag{7}$$

Using our assumptions (f_1), (g_1), (h_1), we have

$$\begin{aligned} 0 \le \int_0^t \|(t-s)^{\alpha-1} f(s,u(s))\|\,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1}(l_1(s)\|u(s)\| + \|f(s,0)\|)\,\mathrm{d}s \\ &\le \frac{L_1 \rho t^\alpha}{\alpha} + \frac{F t^{(1+q_1)(1-\eta)}}{(1+q_1)^{1-\eta}} \to 0, \quad \text{as } t \to 0. \end{aligned}$$

$$\begin{aligned} 0 \le \int_0^t \|(t-s)^{\alpha-1} g(s,u(s))\|\,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1} l_2(s) w_2(\|u(s)\|)\,\mathrm{d}s \\ &\le \frac{L_2 w_2(\rho) t^\alpha}{\alpha} \to 0, \qquad \text{as} \quad t \to 0. \end{aligned}$$

$$\begin{aligned} 0 \le \int_0^t \|(t-s)^{\alpha-1} h(s,u(s))\|\,\mathrm{d}s &\le \int_0^t (t-s)^{\alpha-1} l_3(s) w_3(\|u(s)\|)\,\mathrm{d}s \\ &\le \frac{L_3 w_3(\rho) t^\alpha}{\alpha} \to 0, \qquad \text{as} \quad t \to 0. \end{aligned}$$

Thus, (5)–(7) hold. This yields that (4) holds.

Since $(t-s)^{\alpha-1}[f(s,u(s))+g(s,u(s))+h(s,u(s))]$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$, we known that $(I^\alpha[f(s,u(s)) + g(s,u(s)) + h(s,u(s))])_{t=0} = 0$ which implies that

$${}^cD^\alpha u(t) = f(t,u(t)) + g(t,u(t)) + h(t,u(t)), \text{ a.e. } t \in J.$$

Moreover, $u(0) = u_0$. Thus, $u \in C(J,X)$ is a solution of system (1). On the other hand, if $u \in C(J,X)$ is a solution of system (1), then u satisfies the integral equation (2). □

We apply Lemma 6 to study the properties of solutions for the system (1).

Lemma 8. *Let the functions* $\omega_1, \omega_2, \omega_3 \colon [0,\infty) \to \mathbb{R}$ *be as in Lemma 6 for* $n = 3$. *Assume the hypothesis* (H_1), (f_1), (g_1) *and* (h_1) *hold and there exists an* $\varepsilon > 0$ *such that*

$$\int_0^\infty \frac{s^{r-1}}{w_i(s)^r}\,\mathrm{d}s = \infty, \qquad i = 1,2,3,$$

where $r = r(\varepsilon) := q_1 q_2 q_3$, $q_i = \frac{1}{\alpha} + \varepsilon$, $0 < \alpha < 1$, $i = 1, 2, 3$. *Then*

$$\lim_{t \to \overline{T}^-} \sup \|u(t)\| < \infty,$$

for any solution $u\colon [0, \overline{T}) \to X$ *of the system* (1) *with* $0 < \overline{T} < \infty$, *that is, there exists a constant* $\rho > 0$ *and* $T \in (0, \overline{T})$ *such that*

$$\|u(t)\| \le \rho, \ t \in [0, T].$$

P r o o f. Assume that $u\colon [0, \overline{T}) \to X$ is a continuous solution of the system (1), where $0 < \overline{T} < \infty$ with $\lim_{t \to \overline{T}^-} \sup \|u(t)\| = \infty$. By Lemma 7, the solution u of the system (1) is given by the following fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, u(s)) + g(s, u(s)) + h(s, u(s))] \, \mathrm{d}s.$$

The hypothesis (f_1), (g_1) and (h_1) yield that

$$\begin{aligned}
\|u(t)\| &\le \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, u(s)) + g(s, u(s)) + h(s, u(s))\| \, \mathrm{d}s \\
&\le \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[l_1(s) w_1(\|u(s)\|) + w_1(\|f(s, 0)\|) \right] \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[l_2(s) w_2(\|u(s)\|) + l_3(s) w_3(\|u(s)\|) \right] \mathrm{d}s \\
&\le \|u_0\| + \frac{F \overline{T}^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{3} \int_0^t (t-s)^{\alpha-1} l_i(s) w_i(\|u(s)\|) \, \mathrm{d}s.
\end{aligned}$$

Applying Lemma 6 we obtain

$$W_{i_3}(\|u(t)\|^r) \le W_{i_3}(\kappa_{i_2}) + \int_0^t \widehat{l}_{i_3}(s) \, \mathrm{d}s,$$

where

$$\kappa_0 = 4^{r-1}\eta^r, \qquad r = r(\varepsilon) := \left(\frac{1}{\alpha} + \varepsilon\right)^3,$$

$$\eta = \|u_0\| + \frac{F\overline{T}^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}}, \qquad \kappa_{i_m} = W_{i_m}^{-1}\left[W_{i_m}(\kappa_{i_m-1}) + \int_0^{\overline{T}} \widehat{l}_{i_m}(s)\,\mathrm{d}s\right],$$

$$W_{i_m}(\|v\|) = \int_{\|v_m\|}^{\|v\|} \frac{1}{[w_{i_m}(\sigma^{\frac{1}{r}})]^r}\,\mathrm{d}\sigma, \qquad \|v\| \geq \|v_m\| = W_{i_{m-1}}^{-1}(\|v_{m-1}\|),$$

$$\widehat{l}_{i_m}(t) = \overline{T}^{\widehat{q}_{i_m}-1}4^{r-1}d_{i_m}^r l_{i_m}(t), \qquad \widehat{q}_1 = q_1q_2,\ \widehat{q}_2 = q_1q_3,$$

$$\widehat{q}_3 = q_1q_2, \qquad d_{i_m} = c_{i_m}^{\frac{1}{p_{i_m}}} e^{\overline{T}},$$

c_{i_m}, p_{i_m} are constants from Lemma 6 with $m = 1, 2, 3$ and the functions W_{i_m} are also as Lemma 6 with $m = 1, 2, 3$. This yields that

$$\int_{\|u(0)\|^r}^{\|u(t)\|^r} \frac{\mathrm{d}\sigma}{[w_{i_3}(\sigma^{\frac{1}{r}})]^r} \leq W_{i_3}(\kappa_{i_2}) + \int_0^{\overline{T}} \widehat{l}_{i_3}(s)\,\mathrm{d}s < \infty. \tag{8}$$

However,

$$\lim_{t\to\overline{T}^-}\sup \int_{\|u(0)\|^r}^{\|u(t)\|^r} \frac{\mathrm{d}\sigma}{[w_{i_3}(\sigma^{\frac{1}{r}})]^r} = r\int_0^{\infty} \frac{s^{r-1}}{w_{i_3}(s)^r}\,\mathrm{d}s = \infty$$

and this is a contradiction. The proof is completed. □

4. Existence results of extremal solutions

Define the order relation "≤" by the cone K in $C(J,X)$, given by

$$K = \left\{z \in C(J,X) :\ z(t) \geq 0 \ \text{for all}\ t \in J\right\}.$$

Clearly, the cone K is normal in $C(J,X)$.

Definition 12. A function $a \in C(J,X)$ is called a lower solution of system (1) on J if the function $a(t)$ is absolutely continuous on J, and

$$\begin{cases} {}^cD^\alpha a(t) \leq \overline{F}(t, a(t)), & \text{a.e.}\quad t \in J \\ a(0) \leq u_0. \end{cases}$$

Definition 13. A function $b \in C(J, X)$ is called a upper solution of system (1) on J if the function $b(t)$ is absolutely continuous on J, and

$$\begin{cases} {}^cD^\alpha b(t) \geq \overline{F}(t, b(t)), & \text{a.e.} \quad t \in J \\ b(0) \geq u_0. \end{cases}$$

Definition 14. A function $u \in C(J, X)$ is a solution of system (1) on J if it is a lower as well as a upper solution of system (1) on J.

Definition 15. A solution $u_{\max}$ of system (1) is said to be maximal if for any other solution u to system (1), one has $u(t) \leq u_{\max}(t)$ for all $t \in J$.

Definition 16. A solution $u_{\min}$ of system (1) is said to be minimal if if for any other solution u to system (1), one has $u_{\min}(t) \leq u(t)$ for all $t \in J$.

In addition to the hypotheses in Section 3, we introduce the following hypotheses.

- (H_2) system (1) has a lower solution a and an upper solution b with $a \leq b$.
- (H_3) $\liminf\limits_{\rho \to \infty} \dfrac{w_i(\rho)}{\rho} = \delta_i < \infty, \ \ i = 2, 3.$
- (f_2) $f(t, u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.
- (g_2) $g(t, u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$.
- (g_3) for every $t \in J$, the set $\mathbb{S}_g = \big\{(t-s)^{\alpha-1} g(s, u(s)) : \ u \in C(J, X), \ \ s \in [0, t]\big\}$ is relatively compact.
- (h_2) for every $t \in J$, the set $\mathbb{S}_h = \big\{(t-s)^{\alpha-1} h(s, u(s)) : \ u \in C(J, X), \ \ s \in [0, t]\big\}$ is relatively compact.

Theorem 1. *Suppose all conditions in Lemma 8 are satisfied and the hypotheses (H_2), (H_3), (f_2), (g_2), (g_3), (h_2) hold. Then system* (1) *has a minimal and a maximal solution in the order interval* $[a, b]$ *provided that*

$$\frac{L_1 T^\alpha}{\Gamma(\alpha + 1)} + \frac{L_2 T^\alpha \delta_2}{\Gamma(\alpha + 1)} + \frac{L_3 T^\alpha \delta_3}{\Gamma(\alpha + 1)} < 1. \tag{9}$$

P r o o f. By Lemma 7, system (1) is equivalent to the fractional integral equation (2). Consider the order interval $[a, b]$ in $C(J, X)$ which is well defined in view of hypothesis (H_2). Define three operators A, B and C on $C(J,X)$ as follows

$$\begin{cases} (Au)(t) = u_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s,u(s))\,\mathrm{d}s, & \\ (Bu)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} g(s,u(s))\,\mathrm{d}s, & \text{for} \quad t \in J, \\ (Cu)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s,u(s))\,\mathrm{d}s, & \end{cases}$$

where $u \in X$. Clearly the operators A, B, C are well defined on $[a,b]$ in view of hypotheses (f_1), (g_1) and (h_1). Then fractional integral equation (2) is equivalent to the operator equation $Au(t) + Bu(t) + Cu(t) = u(t)$, $t \in J$. We shall show that A, B and C satisfy the conditions of Lemma 5 on $[a,b]$.

The proof is divided into several steps.

Step 1. $Au + Bu + Cu \in \mathcal{B}_\rho$ for every $u \in \mathcal{B}_\rho$.

For any $u \in \mathcal{B}_\rho$ and $\delta > 0$, by using Hölder inequality, we get

$$\begin{aligned} &\|(Au)(t+\delta) - (Au)(t)\| \\ &\qquad \le \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s)^{\alpha-1} - (t+\delta-s)^{\alpha-1}) \|f(s,u(s))\|\,\mathrm{d}s \\ &\qquad\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}) \|f(s,u(s))\|\,\mathrm{d}s \\ &\qquad \le \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s)^{\alpha-1} - (t+\delta-s)^{\alpha-1}) (l_1(s)\|u(s)\| + \|f(s,0)\|)\,\mathrm{d}s \\ &\qquad\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} (l_1(s)\|u(s)\| + \|f(s,0)\|)\,\mathrm{d}s \\ &\qquad \le \frac{L_1\rho}{\Gamma(\alpha+1)} (t^\alpha - (t+\delta)^\alpha + \delta^\alpha) \\ &\qquad\quad + \frac{F}{\Gamma(\alpha)(1+q_1)^{1-\eta}} \left(t^{1+q_1} - (t+\delta)^{1+q_1} + \delta^{1+q_1}\right)^{1-\eta} \\ &\qquad\quad + \frac{L_1\rho}{\Gamma(\alpha+1)} \delta^\alpha + \frac{F}{\Gamma(\alpha)(1+q_1)^{1-\eta}} \delta^{(1+q_1)(1-\eta)} \\ &\qquad \le \frac{2L_1\rho}{\Gamma(\alpha+1)} \delta^\alpha + \frac{2F}{\Gamma(\alpha)(1+q_1)^{1-\eta}} \delta^{(1+q_1)(1-\eta)}. \end{aligned}$$

It is easy to see that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore $Au \in C(J,X)$. Using the similar argument, we can get that $Bu \in C(J,X)$, $Cu \in C(J,X)$. Therefore $Au + Bu + Cu \in C(J,X)$.

Moreover, we claim that $(A+B+C)(\mathcal{B}_\rho) \subset \mathcal{B}_\rho$. If it is not true, then for each $\rho > 0$, there would exist $u_\rho \in \mathcal{B}_\rho$ and $t_\rho \in J$ such that $\|(Au_\rho + Bu_\rho + Cu_\rho)(t_\rho)\| > \rho$. Thus,

$$\begin{aligned}
\rho &< \|Au_\rho + Bu_\rho + Cu_\rho\| \\
&\leq \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^{t_\rho} (t_\rho - s)^{\alpha-1} \|f(s,u(s)) + g(s,u(s)) + h(s,u(s))\| \, \mathrm{d}s \\
&\leq \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^{t_\rho} (t_\rho - s)^{\alpha-1} [l_1(s)\|u(s)\| + \|f(s,0)\| \\
&\quad + l_2(s)w_2(\|u(s)\|) + l_3(s)w_3(\|u(s)\|)] \, \mathrm{d}s \\
&\leq \|u_0\| + \frac{L_1 \rho T^\alpha}{\Gamma(\alpha+1)} + \frac{F T^{(1+q_1)(1-\eta)}}{\Gamma(\alpha)(1+q_1)^{1-\eta}} + \frac{L_2 w_2(\rho) T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 w_3(\rho) T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Dividing both sides by ρ and taking the lower limit as $\rho \to \infty$, we obtain

$$1 \leq \frac{L_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 T^\alpha}{\Gamma(\alpha+1)} \delta_2 + \frac{L_3 T^\alpha}{\Gamma(\alpha+1)} \delta_3,$$

which is a contradictions with (9). Thus, $Au + Bu + Cu \in \mathcal{B}_\rho$.

From Lemma 7, we get that system (1) is equivalent to the operator equation $(Au)(t) + (Bu)(t) + (Cu)(t) = u(t)$ for $t \in J$. Now we show that the operator equation $Au + Bu + Cu = u$ has a least and a greatest solution in $[a,b]$.

Step 2. A is a contraction in $\mathcal{B}_\rho$.

For any $u, v \in \mathcal{B}_\rho$ and $t \in J$, according to (f_1), we have

$$\begin{aligned}
\|(Au)(t) - (Av)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s)) - f(s,v(s))\| \, \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_1(s) \|u(s) - v(s)\| \, \mathrm{d}s \\
&\leq \frac{L_1 T^\alpha}{\Gamma(\alpha+1)} \|u - v\|_C,
\end{aligned}$$

which implies that

$$\|Ax - Ay\|_C \leq \frac{L_1 T^\alpha}{\Gamma(\alpha+1)} \|u - v\|_C.$$

Therefore, A is a contraction in $\mathcal{B}_\rho$ according to the condition (9).

Step 3. B is a completely continuous operator and C is a totally bounded operator.

For any $u \in \mathcal{B}_\rho$, Let $\{u_n\}$ be a sequence of $\mathcal{B}_\rho$ such that $u_n \to u$ in $\mathcal{B}_\rho$. Then, $g(s, u_n(s)) \to g(s, u(s))$ as $n \to \infty$ due to the hypotheses (g_1). Moreover, for all $t \in J$, we have $\|g(s, u_n(s)) - g(s, u(s))\| \leq 2l_2(s)w_2(\rho)$. Note that the functions $s \to (t-s)^{\alpha-1}2l_2(s)w_2(\rho)$ is integrable on J, and $\|u_n(s) - u(s)\| \to 0$, $\|g(s, u_n(s)) - g(s, u(s))\| \to 0$ a.e. $s \in J$ as $n \to \infty$. By means of the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}\|(Bu_n)(t) - (Bu)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(s, u_n(s)) - g(s, u(s))\| \,\mathrm{d}s \\ &\to 0.\end{aligned}$$

Therefore, $Bu_n \to Bu$ as $n \to \infty$ which implies that B is continuous.

Now we only need to check that $\{Bu : u \in \mathcal{B}_\rho\}$ is relatively compact. For any $u \in \mathcal{B}_\rho$ and $t \in J$, we have

$$\begin{aligned}\|(Bu)(t)\| &\leq \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(s, u(s))\| \,\mathrm{d}s \\ &\leq \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_2(s) w_2(\|u(s)\|) \,\mathrm{d}s \\ &\leq \|u_0\| + \frac{L_2 w_2(\rho) T^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Thus $\{Bu : u \in \mathcal{B}_\rho\}$ is uniformly bounded.

In the following, we will show that $\{Bu : u \in \mathcal{B}_\rho\}$ is a family of equicontinuous functions.

For any $u \in \mathcal{B}_\rho$ and $0 \leq t_1 < t_2 \leq T$, we get

$$\begin{aligned}\|(Bu)(t_2) - (Bu)(t_1)\| \leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) g(s, u(s)) \,\mathrm{d}s \right\| \\ + \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(s, u(s)) \,\mathrm{d}s \right\|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \|((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) g(s, u(s))\| \, \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \|(t_2 - s)^{\alpha-1} g(s, u(s))\| \, \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) l_2(s) w_2(\|u(s)\|) \, \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} l_2(s) w_2(\|u(s)\|) \, \mathrm{d}s \\
&\leq \frac{L_2 w_2(\rho)}{\Gamma(\alpha+1)} (t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha) + \frac{L_2 w_2(\rho)}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha \\
&\leq \frac{2 L_2 w_2(\rho)}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha.
\end{aligned}$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $u \in \mathcal{B}_\rho$. We get that $\{Bu : u \in \mathcal{B}_\rho\}$ is a family of equicontinuous functions.

In view of the condition (g_3) and the Lemma 3, we know that $\overline{\mathrm{conv}}\, \mathbb{S}_g$ is compact.

For any $t^* \in J$,

$$\begin{aligned}
(Bu_n)(t^*) &= \frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^* - s)^{\alpha-1} g(s, u_n(s)) \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)} \lim_{k\to\infty} \sum_{i=1}^{k} \frac{t^*}{k} (t^* - \frac{it^*}{k})^{\alpha-1} g(\frac{it^*}{k}, u_n(\frac{it^*}{k})) \\
&= \frac{t^*}{\Gamma(\alpha)} \zeta_n,
\end{aligned}$$

where

$$\zeta_n = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{k} (t^* - \frac{it^*}{k})^{\alpha-1} g(\frac{it^*}{k}, u_n(\frac{it^*}{k})).$$

Since $\overline{\mathrm{conv}}\, \mathbb{S}_g$ is convex and compact, we know that $\zeta_n \in \overline{\mathrm{conv}}\, \mathbb{S}_g$. Hence, for any $t^* \in J$, the set $\{Bu_n\}$, $(n = 1, 2, \ldots)$ is relatively compact.

From Ascoli-Arzela theorem every $\{Bu_n(t)\}$ contains a uniformly convergent subsequence $\{Bu_{n_k}(t)\}$, $(k = 1, 2, \dots)$ on J. Thus, the set $\{Bu : \ u \in \mathcal{B}_\rho\}$ is relatively compact.

Step 4. C is a totally bounded operator.

Since C is a continuous operator, using the similar argument in Step 3, we can get that $\{Cu : \ u \in \mathcal{B}_\rho\}$ is also relatively compact, which means that C is totally bounded. Therefore, C is a totally bounded operator.

Step 5. A, B and C are three monotone increasing operators.

Since $u, v \in C(J, X)$ with $u \leq v$ for $t \in J$, according to (f_2), we have

$$\begin{aligned}(Au)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\,\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s))\,\mathrm{d}s \ = \ (Av)(t).\end{aligned}$$

Hence A is a monotone increasing operator. Similarly, we can conclude that B and C are also monotone increasing operators according to (g_2) and (h_1).

Clearly, K is a normal cone. From (H_2) and Definition 12, we have that $a \leq Aa + Ba + Ca$ and $b \geq Ab + Bb + Cb$ with $a \leq b$. Thus the operators A, B and C satisfy all the conditions of Lemma 5 and hence the operator equation $Au + Bu + Cu = u$ has a least and a greatest solution in $[a, b]$. Therefore, system (1) has a minimal and a maximal solution on J. This completes the proof. □

5. Example

In this section we give an example to illustrate the usefulness of our main results.

Consider the following the model governed by fractional partial differential equations:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(t,y) = \mu(t, u(t,y)), & \text{a.e.} \quad t \in (0, T], \\ u(t,0) = u(t,1) = 0, & t > 0, \\ u(0,y) = u_0(0,y), & y \in [0,1], \end{cases} \tag{10}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes fractional partial derivative of order $\alpha \in (0,1)$, $\mu(t, u(t,y)) = \mu_1(t, u(t,y)) + \mu_2(t, u(t,y)) + \mu_3(t, u(t,y))$.

Take $X = L^2[0,1]$. Let $u(t)(y) = u(t,y)$, ${}^cD^\alpha u(t)(y) = \frac{\partial^\alpha}{\partial t^\alpha} u(t,y)$. Suppose that $\overline{F}(t, u(t))(y) = \mu(t, u(t,y))$ with $f(t, u(t))(y) = \mu_1(t, u(t,y))$, $g(t, u(t))(y)$

$= \mu_2(t,u(t,y))$, $h(t,u(t))(y) = \mu_3(t,u(t,y))$. Then, the system (1) is the abstract formulation of the model (10).

We make the following assumptions:

(A_0) the model (10) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ with $\underline{u} \leq \overline{u}$.

(A_1) $\mu_1 \colon J \times X \to X$ satisfies the following conditions:
- (i) μ_1 is Lebesgue measurable with respect to t for any $(t,u) \in J \times X$,
- (ii) there exist functions l_1 and w_1, $l_1 \colon J \to R_+$ is continuous, nonnegative, $w_1 \colon [0,\infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that $\|\mu_1(t,u) - \mu_1(t,v)\| \leq l_1(t)w_1(\|u-v\|)$, $t \in J$ for all $u,v \in X$, where $w_1(\|u-v\|) = \|u-v\|$,
- (iii) there exists a $\eta \in [0,\alpha)$ such that $\|\mu_1(t,0)\| \in L^{\frac{1}{\eta}}(J,\mathbb{R}_+)$.
- (iv) $\mu_1(t,u)$ is nondecreasing with respect to u.

(A_2) $\mu_2 \colon J \times X \to X$ satisfies the following conditions:
- (i) $\mu_2(t,u)$ is Lebesgue measurable with respect to t for any $(t,u) \in J \times X$,
- (ii) $\mu_2(t,u)$ is continuous with respect to u for any $u \in X$ and almost all $t \in J$,
- (iii) there exist functions l_2 and w_2, $l_2 \colon J \to \mathbb{R}_+$ are continuous and nonnegative, $w_2 \colon [0,\infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that $\|\mu_2(t,u)\| \leq l_2(t)w_2(\|u\|)$, $t \in J$ for all $u \in X$.
- (iv) $\mu_2(t,u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$,
- (v) for every $t \in J$, the set $\mathbb{S}_{\mu_2} = \{(t-s)^{\alpha-1}\mu_2(s,u(s)) : u \in C(J,X), s \in [0,t]\}$ is relatively compact.

(A_3) $\mu_3 \colon J \times X \to X$ satisfies the following conditions:
- (i) $\mu_3(t,u)$ is Lebesgue measurable with respect to t for any $(t,u) \in J \times X$,
- (ii) $\mu_3(t,u)$ is nondecreasing with respect to u for any $u \in X$ and almost all $t \in J$,
- (iii) there exist functions l_3 and w_3, $l_3 \colon J \to \mathbb{R}_+$ are continuous and nonnegative, and $w_3 \colon [0,\infty) \to \mathbb{R}$ is continuous, positive and nondecreasing such that $\|h(t,u)\| \leq l_3(t)w_3(\|u\|)$, $t \in J$ for all $u \in X$,
- (iv) for every $t \in J$, the set $\mathbb{S}_{\mu_3} = \big\{(t-s)^{\alpha-1}\mu_3(s,u(s)) : u \in C(J,X), s \in [0,t]\big\}$ is relatively compact.

(A_4) for some α, there exists an $\varepsilon > 0$ and $r = r(\varepsilon) := \big(\frac{1}{\alpha} + \varepsilon\big)^3$ such that $\int_0^\infty \frac{s^{r-1}}{w_i(s)^r}\,\mathrm{d}s = \infty$, $i = 1,2,3$. Moreover, $\liminf_{\rho\to\infty} \frac{w_2(\rho)}{\rho} = \delta_2 < \infty$ and $\liminf_{\rho\to\infty} \frac{w_3(\rho)}{\rho} = \delta_3 < \infty$.

Obviously, if $\frac{L_1T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2T^\alpha\delta_2}{\Gamma(\alpha+1)} + \frac{L_3T^\alpha\delta_3}{\Gamma(\alpha+1)} < 1$, then all assumptions given in Theorem 1 are satisfied, our results can be applied to the model (10).

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