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TN-GROUPOIDS

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ABSTRACT. In this paper we introduce the notion of TN-groupoids in an arbitrary groupoid, and discuss some properties in $\mathrm{Bin}(X)$, and obtain several results related with non-negativity of norms.

1. Introduction

J. Neggers and H. S. Kim [5] introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. P. J. Allen et al. [1] developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Recently, some properties of certain real-valued mappings $\triangle$ on binary systems X which satisfy versions of the triangle inequality were investigated by present authors [2]. Especially, they showed that if $\triangle$ is a non-negative, stable and triangular norm of types $(T1) \sim (T4)$ on a d-algebra $(X, *, 0)$, then $(X/\mathrm{Ker}\,\triangle, *, [0]_{\mathrm{Ker}\,\triangle})$ is a d-algebra. In addition, fuzzy versions of these triangular norms and their properties were considered as well.

In this paper we introduce the notion of TN-groupoids in an arbitrary groupoid, and discuss some properties in $\mathrm{Bin}(X)$, and obtain several results related with non-negativity of norms.

2. Preliminaries

The present authors [2] discussed several triangular norms on a groupoid $(X, *)$ as follows: given a groupoid $(X, *)$, we consider mappings $\triangle : X \to \mathbb{R}$ such that one of the following identities holds:

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$(T1)$ $\triangle(x) \leq \triangle(y) + \triangle(x * y)$,

$(T2)$ $\triangle(x * y) \leq \triangle(x) + \triangle(y)$,

$(T3)$ $\triangle(x * z) \leq \triangle(x * y) + \triangle(y * z)$,

$(T4)$ $\triangle((x * z) * (y * z)) + \triangle((z * x) * (z * y)) \leq \triangle(x * y) + \triangle(y * x)$,

for any $x, y, z \in X$. All of $(T1)$, $(T2)$, $(T3)$, $(T4)$ are versions of the "usual" triangle inequality, and thus we shall refer to a mapping $\triangle : X \to \mathbb{R}$ which satisfies an inequality (Ti) above as a *triangular norm of type* (Ti) when $\mathrm{Ker}\,\triangle := \{x \in X \mid \triangle(x) = 0\} \neq \emptyset$ $(i = 1, 2, 3, 4)$. If we don't require that $\mathrm{Ker}\,\triangle \neq \emptyset$, then we shall refer to it as a *norm* of type (Ti) (or (Ti)-*norm*). A mapping $\triangle\colon X \to \mathbb{R}$ is said to be *stable* if $\triangle(x * y) \leq \triangle(x)$ for any $x, y \in X$. A mapping $\triangle\colon X \to \mathbb{R}$ is said to be *non-negative* if $\triangle(x) \geq 0$ for all $x \in X$.

Example 1. ([2]) Let X be the power set of a finite set F, i.e., $x \in X$ means $x \subseteq F$. If $\triangle(x) = |\,x\,|$, the cardinality of x, and $x * y := x - y$, the collection of all elements of x not in y, then $\triangle(\emptyset) = 0$, i.e., $Ker\triangle \neq \emptyset$. Also, $\triangle(x*y) \leq \triangle(x)$, i.e., $\triangle$ is stable, whence certainly $\triangle(x * y) \leq \triangle(x) + \triangle(y)$, i.e., it is a triangular norm of type $(T2)$. Note that $x = (x * y) \cup (x \cap y)$, $\triangle(x) = \triangle(x * y) + \triangle(x \cap y)$ and $\triangle(x \cap y) = \triangle(y * x^C) \leq \triangle(y)$, so that $\triangle(x) \leq \triangle(y) + \triangle(x * y)$ as well, and $\triangle$ is a triangular norm of type $(T1)$. Finally, $x * z \subseteq (x * y) \cup (y * z)$ means that $\triangle(x * z) \leq \triangle(x * y) + \triangle(y * z)$, i.e., $\triangle$ is a triangular norm of type $(T3)$. Since $[(x * z) * (y * z)] \cup [(z * x) * (z * y)] \subseteq (x * y) \cup (y * x)$, we have $\triangle((x*z)*(y*z)) + \triangle((z*x)*(z*y)) \leq \triangle(x*y) + \triangle(y*x)$, i.e., $\triangle\colon X \to \mathbb{R}$ is a stable and triangular norm of types $(T1) \sim (T4)$.

Example 2. ([2]) If $X = \mathbb{R}$ and $x * y := \max\{0, x - y\}$ for all $x, y \in X$, then $\triangle(x) = |\,x\,|$ satisfies $(T2)$. Indeed, if $x \leq y$, then $x * y = 0$ and $\triangle(x * y) = 0 \leq \triangle(x) + \triangle(y)$. If $x > y$, then $\triangle(x * y) = x - y \leq |\,x - y\,| \leq |x| + |y| = \triangle(x) + \triangle(y)$. On the other hand, if $x = -5$, $y = -1$, $x - y = -4$ and $x * y = 0$, $\triangle(x) = 5 \not\leq \triangle(y) + \triangle(x*y)$, so that $(T1)$ fails to hold. Also, if $z = 0$ and if $(T3)$ holds, then $\triangle(x) \leq \triangle(x * y) + \triangle(y)$, which means $(T1)$ holds. Thus $(T3)$ fails, since $(T1)$ fails. Finally, if $x = 5, y = -1, x * y = 6$ and $\triangle(x * y) > \triangle(x)$, i.e., the mapping $\triangle$ is not stable.

Example 3. ([2]) Let $X := \{0, 1, 2, 3\}$ be an algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then it is a d-algebra. If we define a map $\triangle\colon X \to [0, 1]$ by $0 = \triangle(0) < \triangle(1) < \triangle(2) < \triangle(3) \leq 1$, then it is easy to show that $\triangle$ is a fuzzy triangular norm of types $(T1) \sim (T3)$, but not of type $(T4)$, since $\triangle((3*1)*(2*1)) + \triangle((1*3)*(1*2)) > \triangle(3*2) + \triangle(2*3)$.

H. S. Kim and J. Neggers [3] introduced the notion of the semigroup of binary systems as follows: for a given set X, we consider $\mathrm{Bin}(X)$, the collection of all binary systems (groupoids, algebras) defined on X. Given arbitrary groupoids $(X,*)$ and $(X,\bullet)$, define a product $(X,\square) = (X,*)\,\square\,(X,\bullet)$ where, for any $x, y \in X$,

$$x \,\square\, y = (x*y) \bullet (y*x)$$

Theorem 2.1. ([3]) $(\mathrm{Bin}(X), \square)$ *is a semigroup, i.e., the operation* $\square$ *as defined in general is associative. Furthermore, the left zero semigroup is an identity for this operation.*

3. TN-groupoids

Let $(X,*)$ be a groupoid and let $\triangle\colon X \to \mathbb{R}$ be a map satisfying the condition $(T1)$. A system $(X,*,\triangle)$ is said to be a TN-*groupoid* if

(i) $(X,*)$ is a groupoid;

(ii) $\triangle$ satisfies the condition $(T1)$.

Let $\mathrm{Bin}(X)\colon = \{(X,*) \mid *$ is a binary operation on $X\}$. Given $(X,*) \in \mathrm{Bin}(X)$, if we define a map $\triangle_0\colon X \to \mathbb{R}$ by $\triangle_0(x) := 0$ for any $x \in X$, then $(X,*,\triangle_0)$ is a TN-groupoid.

We define some notions as follows:

$$\begin{aligned}
\mathrm{TN}(X,*) &:= \big\{(X,*,\triangle) \mid \triangle \ \text{ satisfies the condition } (T1)\big\};\\
\mathrm{PTN}(X,*) &:= \big\{\triangle \mid (X,*,\triangle) \in \mathrm{TN}(X,*)\big\};\\
\mathrm{Bin}(X,\triangle) &:= \big\{(X,*) \in \mathrm{Bin}(X) \mid (X,*,\triangle) \in \mathrm{TN}(X,*)\big\};\\
\mathrm{St}(X,\triangle) &:= \big\{(X,*) \mid \triangle \ \text{ is stable on } \ (X,*)\big\}.
\end{aligned}$$

Lemma 3.1. *If* $(X,*)$, $(X,\bullet) \in \mathrm{St}(X,\triangle)$, *then* $(X,*)\ \square\ (X,\bullet) = (X,\square) \in \mathrm{St}(X,\triangle)$.

P r o o f. Given $x, y \in X$, we have

$$\triangle(x \,\square\, y) = \triangle((x*y)\bullet(y*x)) \le \triangle(x*y) \le \triangle(x),$$

which shows that $\triangle$ is stable on $(X,\square)$. Hence $(X,\square) \in \mathrm{St}(X,\triangle)$. $\square$

A mapping $\triangle\colon X \to \mathbb{R}$ is said to be *co-stable* if $\triangle(x*y) \ge \triangle(x)$ for any $x, y \in X$. We define a set $\mathrm{CSt}(X,\triangle)\colon = \{(X,*) \mid \triangle$ is co-stable on $(X,*)\}$.

Lemma 3.2. *If* $(X,*), (X,\bullet) \in \mathrm{CSt}(X,\triangle)$, *then* $(X,*)\ \square\ (X,\bullet) = (X,\square) \in \mathrm{CSt}(X,\triangle)$.

P r o o f. The proof is similar to that of Lemma 3.1. $\square$

Proposition 3.1. *Let $\triangle \in \mathrm{PTN}(X, *)$. If $(X, \bullet) \in \mathrm{CSt}(X, \triangle)$ and $(X, \Box) = (X, *) \ \Box \ (X, \bullet)$, then $\triangle \in \mathrm{PTN}(X, \Box)$.*

P r o o f. Since $(X, \bullet) \in \mathrm{CSt}(X, \triangle)$, for any $x, y \in X$,

$$\triangle(x \ \Box \ y) = \triangle((x * y) \bullet (y * x)) \geq \triangle(x * y).$$

Hence $\triangle(x \Box y) + \triangle(y) \geq \triangle(x * y) + \triangle(y) \geq \triangle(x)$, since $\triangle \in \mathrm{PTN}(X, *)$, which proves that $(X, \Box, \triangle)$ is a TN-groupoid. Hence $\triangle \in \mathrm{PTN}(X, \Box)$. $\Box$

Proposition 3.1 means that $\mathrm{Bin}(X, \triangle) \Box \mathrm{CSt}(X, \triangle) \subseteq \mathrm{Bin}(X, \triangle)$.

Proposition 3.2. *If the mapping $\triangle$ is non-negative, then* $\mathrm{CSt}(X, \triangle) \subseteq \mathrm{Bin}(X, \triangle)$.

P r o o f. For any $(X, *) \in \mathrm{CSt}(X, \triangle)$, since $\triangle$ is non-negative, we have

$$\triangle(x * y) \geq \triangle(x) \geq 0$$

for any $x, y \in X$. This proves that $\triangle$ satisfies the condition $(T1)$. Hence $(X, *) \in \mathrm{Bin}(X, \triangle)$. $\Box$

Example 4. Let $X := \mathbb{R}$. Define $x * y := |x| + |y|$ for any $x, y \in X$ and define $\triangle(x) := x$ on X. Then $\triangle(x) = x \leq |x| + |y| = x * y = \triangle(x * y)$ for any $x, y \in X$, proving that $(X, *) \in \mathrm{CSt}(X, \triangle)$. Now, $\triangle(x) = x \leq |x| + |y| + y = \triangle(x * y) + \triangle(y)$, so that $(X, *) \in \mathrm{Bin}(X, \triangle)$. Hence $(X, *) \in \mathrm{CSt}(X, \triangle) \cap \mathrm{Bin}(X, \triangle)$. But $\triangle$ is not non-negative, since $\triangle(x) < 0$ for any $x < 0$.

Question. Does $\mathrm{CSt}(X, \triangle) \subseteq \mathrm{Bin}(X, \triangle)$ imply $\triangle(x) \geq 0$ for all $x \in X$?

If $(X, *) \in \mathrm{St}(X, \triangle) \cap C\,\mathrm{St}(X, \triangle)$, then $\triangle(x) \leq \triangle(x * y) \leq \triangle(x)$ and $\triangle(x * y) = \triangle(x)$, i.e., $\triangle(x * y) = \triangle(x) + 0$ for all $x, y \in X$. With this notion, we may define

$$\mathbb{R}(X, \triangle) := \{(X, *) \,|\, \exists r \in \mathbb{R} \, s.t. \ \triangle(x * y) = \triangle(x) + r, \forall x, y \in X\}$$

Proposition 3.3. $\mathbb{R}(X, \triangle) \ \Box \ \mathrm{Bin}(X, \triangle) \subseteq \mathrm{Bin}(X, \triangle)$.

P r o o f. Let $(X, *) \in \mathbb{R}(X, \triangle)$ and $(X, \bullet) \in \mathrm{Bin}(X, \triangle)$. If we let $(X, \Box) := (X, *) \Box (X, \bullet)$, then for any $x, y \in X$, $\triangle(x * y) \leq \triangle((x * y) \bullet (y * x)) + \triangle(y * x) = \triangle(x \ \Box \ y) + \triangle(y * x)$. This means that $\triangle(x \ \Box \ y) \geq \triangle(x * y) - \triangle(y * x) = (\triangle(x) + r) - (\triangle(y) + r) = \triangle(x) - \triangle(y)$ for some $r \in \mathbb{R}$. Hence $\triangle(x) \leq \triangle(x \ \Box \ y) + \triangle(y)$, proving that $(X, \Box) \in \mathrm{Bin}(X, \triangle)$. $\Box$

Let $(X, *, \triangle_i)$ be a TN-groupoid where $i = 1, 2$. For any real numbers $\alpha, \beta, \gamma \in \mathbb{R}$, and for any $x \in X$, we define $(\alpha \triangle_1 + \beta \triangle_2 + \gamma)(x) := \alpha \cdot \triangle_1(x) + \beta \cdot \triangle_2(x) + \gamma$. Then $(X, *, \alpha \triangle_1 + \beta \triangle_2 + \gamma)$ is a TN-groupoid. With this notion we obtain the following propositions and we omit the proofs.

Proposition 3.4. *If* $(X,*) \in \mathbb{R}(X, \triangle_1) \cap \mathbb{R}(X, \triangle_2)$, *then*

$$(X,*) \in \mathbb{R}(X, \alpha \triangle_1 + \beta \triangle_2)$$

for any $\alpha, \beta \in \mathbb{R}$.

Proposition 3.5. *If* $\alpha, \beta \geq 0$, *then*

$$\mathrm{Bin}(X, \triangle_1) \cap \mathrm{Bin}(X, \triangle_2) \subseteq \mathrm{Bin}(X, \alpha \triangle_1 + \beta \triangle_2).$$

By applying Proposition 3.5, we obtain the following corollary.

Corollary 3.5.1. *Let* $\alpha > 0$. *Then* $(X,*) \in \mathrm{Bin}(X, \triangle)$ *if and only if* $(X,*) \in \mathrm{Bin}(X, \alpha \triangle)$.

4. Homomorphisms and induced mappings

Let $\varphi\colon (X,*) \to (Y,\bullet)$ be a homomorphism of groupoids and let $\triangle \in \mathrm{PTN}(Y,\bullet)$. Then $(Y,\bullet,\triangle)$ is a TN-groupoid. Define a map $\triangle^{\#}_{\varphi}\colon X \to \mathbb{R}$ by $\triangle^{\#}_{\varphi}(x) := \triangle(\varphi(x))$ for any $x \in X$. Then $\triangle^{\#}_{\varphi} \in \mathrm{PTN}(X,*)$. In fact, for any $x, y \in X$, $\triangle^{\#}_{\varphi}(x*y) = \triangle(\varphi(x*y)) = \triangle(\varphi(x) \bullet \varphi(y))$. Since $\triangle \in \mathrm{PTN}(Y,\bullet)$, we obtain

$$\begin{aligned} \triangle^{\#}_{\varphi}(x*y) + \triangle^{\#}_{\varphi}(y) &= \triangle(\varphi(x) \bullet \varphi(y)) + \triangle(\varphi(y)) \\ &\geq \triangle(\varphi(x)) = \triangle^{\#}_{\varphi}(x). \end{aligned}$$

Hence $\varphi^{\#}\colon \mathrm{PTN}(Y,\bullet) \to \mathrm{PTN}(X,*)$ defined by $\varphi^{\#}(\triangle) := \triangle^{\#}_{\varphi}$ is a mapping, called an *induced mapping* from a homomorphism $\varphi\colon (X,*) \to (Y,\bullet)$.

Theorem 4.1. *If* $\triangle_i \in \mathrm{PTN}(Y,\bullet)$, $(i = 1,2)$ *and* $\alpha, \beta \geq 0$, *then*

$$\varphi^{\#}(\alpha \triangle_1 + \beta \triangle_2) = \alpha\varphi^{\#}(\triangle_1) + \beta\varphi^{\#}(\triangle_2).$$

P r o o f. Let $\alpha, \beta \geq 0$ and let $\triangle_i \in \mathrm{PTN}(Y,\bullet)$ $(i = 1,2)$. Then $\alpha \triangle_1 + \beta \triangle_2 \in \mathrm{PTN}(Y,\bullet)$ by Proposition 3.5. Given $x \in X$, we obtain

$$\begin{aligned} \varphi^{\#}(\alpha \triangle_1 + \beta \triangle_2)(x) &= (\alpha \triangle_1 + \beta \triangle_2)^{\#}_{\varphi}(x) \\ &= (\alpha \triangle_1 + \beta \triangle_2)(\varphi(x)) = \alpha \triangle_1(\varphi(x)) + \beta \triangle_2(\varphi(x)) \\ &= \alpha(\triangle_1)^{\#}_{\varphi}(x) + \beta(\triangle_2)^{\#}_{\varphi}(x) = [\alpha(\triangle_1)^{\#}_{\varphi} + \beta(\triangle_2)^{\#}_{\varphi}](x) \\ &= [\alpha\varphi^{\#}(\triangle_1) + \beta\varphi^{\#}(\triangle_2)](x). \end{aligned}$$

$\square$

Proposition 4.1. *If* $\varphi_1\colon (X,*) \to (Y,\bullet)$ *and* $\varphi_2\colon (Y,\bullet) \to (Z,\star)$ *are homomorphisms of groupoids, then* $(\varphi_2 \circ \varphi_1)^{\#} = \varphi_1^{\#} \circ \varphi_2^{\#}$.

Proof. Given $x \in X$, since $\triangle \in \mathrm{PTN}(Z, \star)$, we have $\varphi_2^{\#}(\triangle) = \triangle_{\varphi_2}^{\#} \in \mathrm{PTN}(Y, \bullet)$ and hence $(\varphi_1^{\#} \circ \varphi_2^{\#})(\triangle) = \varphi_1^{\#}(\varphi_2^{\#}(\triangle)) = \varphi_1^{\#}(\triangle_{\varphi_2}^{\#}) = (\triangle_{\varphi_2}^{\#})_{\varphi_1}^{\#} \in \mathrm{PTN}(X, *)$. It follows that $[(\varphi_2 \circ \varphi_1)^{\#}(\triangle)](x) = \triangle((\varphi_2 \circ \varphi_1)(x)) = \triangle_{\varphi_2}^{\#}(\varphi_1(x)) = (\triangle_{\varphi_2}^{\#})_{\varphi_1}^{\#}(x) = [(\varphi_1^{\#} \circ \varphi_2^{\#})(\triangle)](x)$. Hence $(\varphi_2 \circ \varphi_1)^{\#}(\triangle) = (\varphi_1^{\#} \circ \varphi_2^{\#})(\triangle)$, which shows that $(\varphi_2 \circ \varphi_1)^{\#} = \varphi_1^{\#} \circ \varphi_2^{\#}$. □

Proposition 4.2. *Let $\varphi\colon (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids. If $(Y, \bullet) \in \mathbb{R}(Y, \triangle)$, then $(X, *) \in \mathbb{R}(X, \triangle_{\varphi}^{\#})$.*

Proof. Let $\varphi\colon (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $(Y, \bullet) \in \mathbb{R}(Y, \triangle)$. Then there exists $r \in \mathbb{R}$ such that $\triangle(y_1 \bullet y_2) = \triangle(y_1) + r$ for any $y_1, y_2 \in Y$. It follows that $\triangle_{\varphi}^{\#}(x_1 * x_2) = \triangle(\varphi(x_1 * x_2)) = \triangle(\varphi(x_1) \bullet \varphi(x_2)) = \triangle(\varphi(x_1)) + r = \triangle_{\varphi}^{\#}(x_1) + r$, which proves that $(X, *) \in \mathbb{R}(X, \triangle_{\varphi}^{\#})$. □

Proposition 4.3. *Let $\varphi\colon (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids. If $(Y, \bullet) \in \mathrm{CSt}(Y, \triangle)$, then $(X, *) \in \mathrm{CSt}(X, \triangle_{\varphi}^{\#})$.*

Proof. Let $\varphi\colon (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids. If $(Y, \bullet) \in \mathrm{CSt}(Y, \triangle)$, then $\triangle(y_1) \le \triangle(y_1 \bullet y_2)$ for any $y_1, y_2 \in Y$. Since φ is a homomorphism, we have $\triangle(\varphi(x_1)) \le \triangle(\varphi(x_1) \bullet \varphi(x_2)) = \triangle(\varphi(x_1 * x_2))$, which shows that $\triangle_{\varphi}^{\#}(x_1) \le \triangle_{\varphi}^{\#}(x_1 * x_2)$ for any $x_1, x_2 \in X$. This proves that $(X, *) \in \mathrm{CSt}(X, \triangle_{\varphi}^{\#})$. □

5. Non-negativity

Let $(X, *, \triangle)$ be a TN-groupoid and let

$$\beta_X^{\triangle} := \sup\{\triangle(x) \mid x \in X\}$$

A TN-groupoid $(X, *, \triangle)$ is said to be *bounded* if $\beta_X^{\triangle} < \infty$. Similarly, we define

$$\alpha_X^{\triangle} := \inf\{\triangle(x) \mid x \in X\}$$

Example 5. Let $(X, *)$ be a left-zero semigroup. If $\triangle \in \mathrm{PTN}(X, *)$, then $\triangle(x) \le \triangle(x * y) + \triangle(y) = \triangle(x) + \triangle(y)$ for any $x, y \in X$. It follows that $0 \le \triangle(y)$ for any $y \in X$, i.e., $\triangle \in [0, \infty)^X$. Hence $\mathrm{PTN}(X, *) \subseteq [0, \infty)^X$. If $\triangle \in [0, \infty)^X$, then $\triangle(x) \ge 0$ for any $x \in X$. It follows that $\triangle(x) \le \triangle(x) + \triangle(y) = \triangle(x * y) + \triangle(y)$, i.e., $\triangle \in \mathrm{PTN}(X, *)$. Hence $[0, \infty)^X \subseteq \mathrm{PTN}(X, *)$. This means that every norm satisfying $(T1)$ on the left-zero semigroup is non-negative.

Theorem 5.1. *If $(X, *, \triangle)$ is a bounded* TN*-groupoid, then $\alpha_X^{\triangle} \ge 0$.*

Proof. Let $(X, *, \triangle)$ be a bounded TN-groupoid and let $y \in X$ such that $\triangle(y) < 0$. Since $(X, *, \triangle)$ is bounded, if we let $\epsilon := \frac{1}{2}|\triangle(y)|$, then there exists an $x \in X$ such that $\beta_X^{\triangle} - \epsilon < \triangle(x) < \beta_X^{\triangle}$. Since $(X, *, \triangle)$ is a TN-groupoid, $\triangle(x * y) \ge \triangle(x) - \triangle(y) = \triangle(x) + |\triangle(y)| > \beta_X^{\triangle} - \epsilon + 2\epsilon = \beta_X^{\triangle} + \epsilon > \beta_X^{\triangle}$, a contradiction. □

The TN-groupoid in Example 4 is not bounded.

Corollary 5.1.1. *If $(X, *, \triangle)$ is a finite* TN*-groupoid, then $\alpha_X^{\triangle} \geq 0$.*

P r o o f. If $(X, *, \triangle)$ is a finite TN-groupoid, then it is bounded, i.e., $\beta_X^{\triangle} < \infty$. By applying Theorem 5.1, we obtain $\alpha_X^{\triangle} \geq 0$. □

Corollary 5.1.2. *Let $(X, *, \triangle)$ be a* TN*-groupoid. Assume that there exists a finite subgroupoid S_x of $(X, *)$ containing x, for any $x \in X$. Then $\alpha_X^{\triangle} \geq 0$, i.e., $\triangle$ is non-negative.*

P r o o f. Given $x \in X$, by Corollary 5.1.1, we obtain $\alpha_{S_x}^{\triangle} \geq 0$. Since $X = \bigcup_{x \in X} S_x$, we obtain $\alpha_X^{\triangle} = \inf\{\alpha_{S_x}^{\triangle} \mid x \in X\} \geq 0$ and hence $\triangle(x) \geq 0$ for any $x \in X$. □

Thus, if $\triangle(x) < 0$ for the TN-groupoid $(X, *, \triangle)$, then there is no finite subgroupoid $(S, *)$ of $(X, *)$ which contains x. Notice that in Example 4 the only finite subgroupoid of $(\mathbb{R}, *)$ is $(\{0\}, *)$ with $0 * 0 = |0| + |0| = 0$.

Example 6. Let $(X, *)$ be a right-zero semigroup, $x * y = y$ for all $x, y \in X$. If $\triangle \in \mathrm{PTN}(X, *)$, then $\triangle(x) \leq \triangle(x * y) + \triangle(y) = \triangle(y) + \triangle(y) = 2\,\triangle(y)$. Fix $y := x_0$. Then $\triangle(x) \leq 2\,\triangle(x_0)$ for any $x \in X$. It follows that $\beta_X^{\triangle} \leq 2\,\triangle(x_0)$, i.e., $(X, *, \triangle)$ is a bounded TN-groupoid. By Theorem 5.1, we obtain $\alpha_X^{\triangle} \geq 0$.

Example 7. Let $(X, \bullet, e)$ be a group and let $\triangle \in \mathrm{PTN}(X, \bullet)$. Then

$$\triangle(x) \leq \triangle(x \bullet y) + \triangle(y) \tag{1}$$

for all $x, y \in X$. If we let $x := e, y := e$ in (1), then we obtain $\triangle(e) \leq \triangle(e) + \triangle(e)$, i.e., $0 \leq \triangle(e)$. If we let $x := e$ in (1), then $0 \leq \triangle(e) \leq \triangle(e \bullet y) + \triangle(y) = 2\,\triangle(y)$ for any $y \in X$, proving that $\alpha_X^{\triangle} \geq 0$.

Example 8. If we define a binary operation $x * y := x - y$, on $X := \mathbb{R}$ where "$-$" is the usual subtraction. Then $(X, -, 0)$ is a B-algebra. If we define $\triangle(x) := x$ for any $x \in X$, then $\triangle \in \mathrm{PTN}(X, *)$. Since $\triangle(y) = y < 0$ for any $y < 0$, $\alpha_X^{\triangle} < 0$.

These two examples show that groups and B-algebras behave differently and thus they are not "equivalent" ideas, even though they are logically equivalent.

Given $\triangle, \triangle^* \in \mathrm{PTN}(X, *)$, we define $\min\{\triangle, \triangle^*\}(x) := \min\{\triangle(x), \triangle^*(x)\}$ for any $x \in X$. It may not be true that if $\triangle, \triangle^* \in \mathrm{PTN}(X, *)$, then $\min\{\triangle, \triangle^*\} \in \mathrm{PTN}(X, *)$.

Example 9. In Example 8, we know that $\triangle \in \mathrm{PTN}(X, -)$. Define a map $\triangle^*(x) := K(> 0)$ for any $x \in X$. Then $\triangle^*(x) \leq \triangle^*(x - y) + \triangle^*(y)$ for any $x, y \in X$, i.e., $\triangle^* \in \mathrm{PTN}(X, -)$. We claim that $\min\{\triangle, \triangle^*\} \notin \mathrm{PTN}(X, -)$. In fact, if we let $y := -1, x \in X$ such that $x > K$, then $\min\{\triangle, \triangle^*\}(x) = \min\{\triangle, \triangle^*\}(x - y) = K$ and $\min\{\triangle, \triangle^*\}(y) = -1$, i.e., $\min\{\triangle, \triangle^*\}(x) = K > K + (-1) = \min\{\triangle, \triangle^*\}(x - y) + \min\{\triangle, \triangle^*\}(y)$.

Similarly, given $\triangle_1, \triangle_2 \in \mathrm{PTN}(X,*)$, we define $\max\{\triangle_1,\triangle_2\}(x) := \max\{\triangle_1(x),\triangle_2(x)\}$ for any $x \in X$.

Proposition 5.1. *If* $\triangle_1, \triangle_2 \in \mathrm{PTN}(X,*)$, *then* $\max\{\triangle_1,\triangle_2\} \in \mathrm{PTN}(X,*)$.

Proof. Let $\triangle_1, \triangle_2 \in \mathrm{PTN}(X,*)$. Then

$$\triangle_i(x) \le \triangle_i(x*y) + \triangle_i(y) \le \max\{\triangle_1,\triangle_2\}(x*y) + \max\{\triangle_1,\triangle_2\}(y)$$

for any $x, y \in X$ where $i = 1, 2$. Hence $\max\{\triangle_1,\triangle_2\}(x) \le \max\{\triangle_1,\triangle_2\}(x*y) + \max\{\triangle_1,\triangle_2\}(y)$, proving the proposition. □

Proposition 5.1 shows that $(\mathrm{PTN}(X,*), \max)$ is a semi-lattice.

Proposition 5.2. *Let* $X := F$ *be a field and* $a, b, c \in X$ *(fixed). Given* $\triangle \in \mathrm{PTN}(X,*)$, *if we define* $x * y := a + bx + cy$ *for any* $x, y \in X$, *then*

(i) $\triangle(a) \ge 0$;

(ii) *if* $b + c \ne 0$, *then* $\alpha_X^{\triangle} \ge 0$;

(iii) *if* $b - c \ne 0$ *and* $\triangle(x) = \triangle(-x)$ *for any* $x \in X$, *then* $\alpha_X^{\triangle} \ge 0$.

Proof.

(i) Let $\triangle \in \mathrm{PTN}(X,*)$. Then

$$\triangle(x) \le \triangle(x*y) + \triangle(y) = \triangle(a + bx + cy) + \triangle(y) \tag{2}$$

for any $x, y \in X$. If we let $x := 0, y := 0$ in (2), then $\triangle(0) \le \triangle(a) + \triangle(0)$ and so $\triangle(a) \ge 0$.

(ii) If we let $x := z, y := z$ in (2), then $\triangle(z) \le \triangle(a + (b+c)z) + \triangle(z)$ and hence $0 \le \triangle(a + (b+c)z)$ for any $z \in X$. Assume $b + c \ne 0$. Given $x \in X$, if we let $z := \frac{x-a}{b+c}$, then $x = a + (b+c)z$. It follows that $\triangle(x) = \triangle(a + (b+c)z) \ge 0$ for any $x \in X$, proving that $\alpha_X^{\triangle} \ge 0$.

(iii) Assume $b - c \ne 0$ and $\triangle(x) = \triangle(-x)$ for any $x \in X$. If we let $x := z$, $y := -z$ in (2), then $\triangle(z) \le \triangle(a+(b-c)z) + \triangle(-z)$ and hence $\triangle(a+(b-c)z) \ge 0$ for any $z \in X$. Given $x \in X$, if we let $z := \frac{x-a}{b-c}$, then $x = a + (b-c)z$ and hence $\triangle(x) = \triangle(a + (b-c)z) \ge 0$ for any $x \in X$, proving that $\alpha_X^{\triangle} \ge 0$. □

Example 10. Let $(X, *, f)$ be a leftoid, i.e., $x * y := f(x)$ for all $x, y \in X$, where $f\colon X \to X$ is a map and let $\triangle \in \mathrm{PTN}(X,*,f)$. Then $\triangle(x) \le \triangle(x*x) + \triangle(x)$ for any $x \in X$. It follows that $0 \le \triangle(x*x) = \triangle(f(x))$ for any $x \in X$. Obviously, $(f(X), *, f)$ is a subgroupoid of $(X,*,f)$ and $\triangle \in \mathrm{PTN}(f(X),*,f)$. Hence $\alpha_{f(X)}^{\triangle} \ge 0$.

Example 11. Let $(X, *, g)$ be a rightoid, i.e., $x * y := g(y)$ for all $x, y \in X$, where $g\colon X \to X$ be a map and let $\triangle \in \mathrm{PTN}(X,*,g)$. Then $\triangle(x) \le \triangle(x*y) + \triangle(y) = \triangle(g(y)) + \triangle(y)$ for any $x, y \in X$. If we let $y := x$, then $\triangle(x) \le \triangle(g(x)) + \triangle(x)$ for any $x \in X$, proving $0 \le \triangle(g(x))$ for any $x \in X$. Since $(g(X), *, g)$ is a subgroupoid of $(X,*,g)$ and $\triangle \in \mathrm{PTN}(g(X),*,g)$, we have $\alpha_{g(X)}^{\triangle} \ge 0$.

Proposition 5.3. *If $\triangle \in \mathrm{PTN}(X,*)$, then $0 \leq \triangle(x*y)+\triangle(y*x)$ for any $x,y \in X$.*

Proof. Straightforward. $\square$

Proposition 5.4. *Let $(X,*)$ be a groupoid and let $0 \in X$ such that $0*x=0$ for any $x \in X$. Then $\alpha_X^{\triangle} \geq 0$ for any $\triangle \in \mathrm{PTN}(X,*)$.*

Proof. Let $0 \in X$ such that $0*x=0$ for any $x \in X$. Then $\triangle \in \mathrm{PTN}(X,*)$ implies $\triangle(0) \leq \triangle(0*x)+\triangle(x) = \triangle(0)+\triangle(x)$ and hence $\triangle(x) \geq 0$ for all $x \in X$, whence $\alpha_X^{\triangle} \geq 0$. $\square$

Corollary 5.1.3. *If $(X,*,0)$ is a BCK/d-algebra, then $\alpha_X^{\triangle} \geq 0$ for any $\triangle \in \mathrm{PTN}(X,*)$.*

Example 12. Let $\Gamma=(V,E)$ be a digraph with $E \subseteq V \times V$. We define a product on $X=V$, $x*y$, where $x*y=y$ if $(x,y)\in E$ and $x*y=x$ if $(x,y)\notin E$. Hence $(\{x,y\},*)$ is a subgroupoid of $(X,*)$. If $\triangle \in \mathrm{PTN}(X,*)$, then by Corollary 5.1.2, $\alpha_X^{\triangle} \geq 0$.

6. Product-transitive

A groupoid $(X,*)$ is said to be *product-transitive* if for any $u,v \in X$, there exist $x,y \in X$ such that $x*y=u$ and $y*x=v$.

Example 13. If $(X,*)$ is the left-zero semigroup, then $u*v=u$ and $v*u=v$, so that the left-zero semigroup is product-transitive. Similarly, the right-zero semigroup is also product-transitive.

Proposition 6.1. *Let $(X,*)$ be a product-transitive groupoid. Then $\alpha_X^{\triangle} \geq 0$ for any $\triangle \in \mathrm{PTN}(X,*)$.*

Proof. By Proposition 5.4, we obtain $0 \leq \triangle(x*y)+\triangle(y*x)$ for any $x,y \in X$. Since $(X,*)$ is product-transitive, $0 \leq \triangle(u)+\triangle(v)$ for any $u,v \in X$. If we let $v := u$, then $0 \leq 2\,\triangle(u)$ for any $u \in X$, proving that $\alpha_X^{\triangle} \geq 0$. $\square$

Proposition 6.2. *If $(X,*)$ and $(X,\bullet)$ are product-transitive groupoids, then $(X,\Box)=(X,*)\ \Box\ (X,\bullet)$ is also product-transitive.*

Proof. Given $u,v \in X$, there exist $a,b \in X$ such that $u=a\bullet b, v=b\bullet a$, since $(X,\bullet)$ is product-transitive. Since $(X,*)$ is a product-transitive groupoid, there exist $x,y \in X$ such that $a=x*y, b=y*x$. It follows that $x\Box y=(x*y)\bullet(y*x)=a\bullet b=u$. Similarly, $y\Box x=v$. Hence $(X,\Box)$ is a product-transitive groupoid. $\square$

Example 14. Let $X := F$ be a field and $a, b, c \in X$ (fixed) such that $b^2 - c^2 \neq 0$. Given $\triangle \in \mathrm{PTN}(X, *)$, if we define $x * y := a + bx + cy$ for any $x, y \in X$, then it was proved that $\alpha_X^{\triangle} \geq 0$ by Proposition 5.3. We claim that $(X, *)$ is product-transitive. In fact, for any $u, v \in X$, if we let $x := \frac{b(u-a)-c(v-a)}{b^2-c^2}$, $y := \frac{(bv-cu)+a(c-b)}{b^2-c^2}$, then it is easy to prove that $x * y = u, y * x = v$.

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REFERENCES

[1] ALLEN, P. J.—KIM, H. S.—NEGGERS, J.: *Companion d-algebras*, Math. Slovaca **57** (2007), 93–106.
[2] HAN, J. S.—KIM, H. S.—NEGGERS, J.: *Constructions of quotient algebras via triangular norms*, Math. Slovaca, **60** (2010), 279–288.
[3] KIM, H. S.—NEGGERS, J.: *The semigroups of binary systems and some perspectives*, Bull. Korean Math. Soc. **45** (2008), 651–661.
[4] NEGGERS, J.—JUN, Y. B.—KIM H. S.: *On d-ideals in d-algebras*, Math. Slovaca **49** (1999), 243–251.
[5] NEGGERS, J.—Kim, H. S.: *On d-algebras*, Math. Slovaca **49** (1999), 19–26.

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