VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0131-4
Math. Slovaca **63** (2013), No. 4, 725–732

ON RAMANUJAN SUMS OVER THE GAUSSIAN INTEGERS

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(*Communicated by Jörg Brüdern*)

ABSTRACT. This note deals with Ramanujan sums $c_{\mathbf{m}}(\mathbf{n})$ over the ring $\mathbb{Z}[i]$, in particular with asymptotics for sums of $c_{\mathbf{m}}(\mathbf{n})$ taken over both variables $\mathbf{m}$, $\mathbf{n}$.

1. Introduction. Classical Ramanujan sums

For positive integers m and n, the Ramanujan sum $c_m(n)$ is well known as

$$c_m(n) = \sum_{\substack{1\le j\le m \\ \gcd(j,m)=1}} e\left(j\,\frac{n}{m}\right) = \sum_{d\,|\,\gcd(m,n)} d\,\mu\left(\frac{m}{d}\right), \tag{1.1}$$

where $e(z) := \mathrm{e}^{2\pi\mathrm{i}z}$ and μ denotes the Möbius function. For a convenient basic reference, see e.g., E. Krätzel [7: p. 52; p. 129, f]. The problem of the asymptotic behavior of averages of $c_m(n)$, taken over both variables, has been addressed only quite recently, by T. H. Chan and A. V. Kumchev [1]. Their result (Theorem 1.1) tells us that, for large reals X and $Y \ge X$,

$$\begin{aligned} S_1(X,Y) &:= \sum_{1\le m\le X}\ \sum_{1\le n\le Y} c_m(n) \\ &= Y - \frac{3X^2}{2\pi^2} + O(XY^{1/3}\log X) + O(X^3Y^{-1}), \end{aligned} \tag{1.2}$$

with slight improvements possible in the first O-term, by a more elaborate treatment of certain exponential sums involved.

2010 Mathematics Subject Classification: Primary 11N37, 11L05, 11L07, 11R11.
Keywords: Ramanujan sums, Gaussian integers, quadratic fields, exponential sums.
The author gratefully acknowledges support from the Austrian Science Fund (FWF) under project Nr. P20847-N18.

We remark parenthetically that Chan and Kumchev also investigated the second moment

$$S_2(X,Y) := \sum_{1 \le n \le Y} \left(\sum_{1 \le m \le X} c_m(n) \right)^2 , \tag{1.3}$$

which requires a completely different and very sophisticated method based on complex integration. The asymptotics obtained (Theorem 1.2) is very deep and interesting, but too involved to be cited here in detail.

2. Ramanujan sums over the Gaussian integers

In order to obtain unique prime factorization, we actually consider the set $\mathcal{I}_{\mathbb{Z}[i]}$ of integer ideals over $\mathbb{Z}[i]$, whose elements will be denoted throughout by bold face lower case letters $\mathbf{m}, \mathbf{n}, \dots$. The definition of the Möbius function on $\mathcal{I}_{\mathbb{Z}[i]}$ is natural and classical: If there exists a prime ideal $\mathbf{p} \in \mathcal{I}_{\mathbb{Z}[i]}$ such that $\mathbf{p}^2$ divides $\mathbf{m} \in \mathcal{I}_{\mathbb{Z}[i]}$, then $\mu(\mathbf{m}) = 0$. If $\mathbf{m}$ is the product of r distinct prime ideals, then $\mu(\mathbf{m}) = (-1)^r$. Further, if $m \in \mathbb{Z}[i]$ generates $\mathbf{m}$ (i.e., $\mathbf{m} = m\mathbb{Z}[i]$), the norm of $\mathbf{m}$ is defined by $N(\mathbf{m}) = |m|^2$.

In view of the second representation in (1.1), it is straightforward to generalize as well the concept of Ramanujan sums to $\mathcal{I}_{\mathbb{Z}[i]}$: For any nonzero $\mathbf{m}, \mathbf{n} \in \mathcal{I}_{\mathbb{Z}[i]}$,

$$c_{\mathbf{m}}(\mathbf{n}) := \sum_{\mathbf{d} \colon \mathbf{d} \mid \mathbf{m},\, \mathbf{d} \mid \mathbf{n}} N(\mathbf{d}) \mu\left(\frac{\mathbf{m}}{\mathbf{d}}\right) . \tag{2.1}$$

In fact, the notion of Ramanujan sums has been generalized much further, namely to arbitrary arithmetical semigroups: See, e.g., A. Grytczuk [2], and the monograph by J. Knopfmacher [6: p. 185].

Returning to $\mathbb{Z}[i]$, the objective of this note will be to seek for an asymptotics for the sum

$$S(X,Y) := \sum_{1 \le N(\mathbf{m}) \le X} \; \sum_{1 \le N(\mathbf{n}) \le Y} c_{\mathbf{m}}(\mathbf{n}) . \tag{2.2}$$

In what follows, we shall write (as usual) $r(n)$ for the numbers of ways to write the positive integer n as a sum of two squares of integers, and

$$P(w) := \sum_{1 \le n \le w} r(n) - \pi w$$

for the so-called lattice point discrepancy of an origin-centered circular disc of radius $\sqrt{w}$, w a positive real. By an obvious approach,

$$\begin{aligned} S(X,Y) &= \sum_{1\le N(\mathbf{n})\le Y}\left(\sum_{\mathbf{d},\mathbf{k}:\,1\le N(\mathbf{dk})\le X,\,\mathbf{d}\,|\,\mathbf{n}} N(\mathbf{d})\mu(\mathbf{k})\right) \\ &= \sum_{\mathbf{d},\mathbf{k}:\,1\le N(\mathbf{dk})\le X} N(\mathbf{d})\mu(\mathbf{k})\left(\frac{\pi}{4}\frac{Y}{N(\mathbf{d})}+\frac{1}{4}P\left(\frac{Y}{N(\mathbf{d})}\right)\right) \\ &= \frac{\pi}{4}Y+\frac{1}{4}\sum_{\mathbf{d},\mathbf{k}:\,1\le N(\mathbf{dk})\le X} N(\mathbf{d})\mu(\mathbf{k})\,P\left(\frac{Y}{N(\mathbf{d})}\right). \end{aligned} \tag{2.3}$$

Let $\mathcal{R}=\mathcal{R}(X,Y)$ denote the last sum here, then readily

$$\mathcal{R} \ll \sum_{1\le\ell\le X} r(\ell)\left|\sum_{1\le n\le X/\ell} r(n)\,n\,P\left(\frac{Y}{n}\right)\right|. \tag{2.4}$$

Using the present "record" in the Gaussian circle problem, namely M. Huxley's [3] bound $P(w)\ll w^{131/416+\varepsilon}$, we immediately infer that

$$\mathcal{R} \ll X^{701/416}\,Y^{131/416+\varepsilon}. \tag{2.5}$$

This implies that, as $Y\to\infty$,

$$S(X,Y)\sim\frac{\pi}{4}Y \tag{2.6}$$

provided that $Y\gg X^{\theta}$ for some

$$\theta>\frac{701}{285}=2.4596\ldots.$$

The aim of the present paper is to establish the asymptotics (2.6) for a larger range of θ, by an appropriate application of the method of exponential sums.

Theorem 1. *The asymptotic formula* (2.6) *is true provided that* $Y\gg X^{\theta}$ *for some*

$$\theta>\frac{29}{12}=2.41\dot{6}\,.$$

More precisely, for $Y>X$ *and arbitrary fixed* $\varepsilon>0$,

$$S(X,Y)=\frac{\pi}{4}Y+O\big(X^{29/20}Y^{2/5+\varepsilon}\big)+O\big(X^{37/20}Y^{1/5}\big).$$

We will state some comments concerning possible refinements and generalizations at the end of this note.

3. Two classical auxiliary results

Lemma 1 (**The truncated Hardy identity**). *For large real parameters x, y, and any $\varepsilon > 0$, it follows that*

$$P(x) := \sum_{1 \le n \le x} r(n) - \pi x$$
$$= \frac{1}{\pi}\, x^{1/4} \sum_{1 \le k \le y} \frac{r(k)}{k^{3/4}} \cos\bigl(2\pi\sqrt{kx} - 3\pi/4\bigr) + O\bigl(x^{1/2+\varepsilon}\, y^{-1/2}\bigr) + O(y^{\varepsilon})\,.$$

P r o o f. This is contained in [5: (1.9)] of A. Ivić. (The analogue for the number-of-divisors function $d(n)$ instead of $r(n)$ is well known and can be found, e.g., in the book of E. C. Titchmarsh [9: p. 319].) □

Lemma 2 (**Third derivative test for exponential sums**). *Let F be a real function with three continuous derivatives on the closure of a bounded interval I. Suppose that $L \ge 1$ and $\Delta_3 > 0$ are real parameters such that* $\operatorname{length}(I) \le L$ *and, throughout I, $\Delta_3 \le |F'''(u)| \le C\,\Delta_3$, where C is a positive constant. Then it follows that*

$$\Bigl|\sum_{n \in I} e(F(n))\Bigr| \le C' \left(L\,\Delta_3^{1/6} + L^{1/2}\,\Delta_3^{-1/6}\right)\,,$$

where the constant C' depends only on C.

P r o o f. This classical Van der Corput estimate is contained in [8: p. 34, Theorem 2.6], E. Krätzel's monograph. □

4. Proof of the Theorem

We start from (2.4) and estimate the inner sum which we spilt up by a dyadic sequence. To this end, for ℓ fixed for the moment, let W run through $(2^{-r}X/\ell)$, $r = 1, \dots, [\log(X/\ell)/\log 2]$. Then, by Lemma 1, with a certain parameter $U > 0$ at our disposal,

$$\sum_{W<n\le 2W} r(n)\,n\,P(Y/n) \ll \Bigl|Y^{1/4} \sum_{W<n\le 2W} r(n)\,n^{3/4} \sum_{1\le j\le U} \frac{r(j)}{j^{3/4}}\, e\left(\sqrt{jY/n}\right)\Bigr| + Y^{1/2+\varepsilon}U^{-1/2} \sum_{W<n\le 2W} r(n)\,n^{1/2} + U^{\varepsilon}\,W^2\,. \tag{4.1}$$

Interchanging the order of summation, this is

$$\ll Y^{1/4} \sum_{1\le j\le U} \frac{r(j)}{j^{3/4}} \, |E_{W,j}| + Y^{1/2+\varepsilon} U^{-1/2} W^{3/2} + U^{\varepsilon} W^2 \,, \tag{4.2}$$

where

$$E_{W,j} := \sum_{W<n\le 2W} r(n)\, n^{3/4} e\left(Z/\sqrt{n}\right) \,, \qquad Z := \sqrt{jY}\,.$$

In order to write $E_{W,j}$ as a double sum, let

$$\mathcal{D} = \left\{(u_1,u_2)\in\mathbb{R}^2 :\ W < u_1^2+u_2^2 \le 2W\,,\ \ u_2>u_1>0\right\}.$$

Then, by symmetry,

$$E_{W,j} = 8 \sum_{(m_1,m_2)\in\mathcal{D}} (m_1^2+m_2^2)^{3/4}\, e\left(Z/\sqrt{m_1^2+m_2^2}\right) + O(W^{5/4})\,. \tag{4.3}$$

Here the O-term takes care of the terms corresponding to $|m_1| = |m_2|$, or to $m_1 m_2 = 0$. Throughout, summation extends over all integers, resp., integer points, of the set indicated. We split up the domain $\mathcal{D}$ into subsets

$$\mathcal{D}' = \left\{(u_1,u_2)\in\mathcal{D} :\ u_2 > 2u_1\right\}, \qquad \mathcal{D}'' = \left\{(u_1,u_2)\in\mathcal{D} :\ u_2 \le 2u_1\right\}.$$

Let $f(u_1,u_2) = Z(u_1^2+u_2^2)^{-1/2}$, and write

$$E_{W,j}^{(1)} = \sum_{m_1\in\mathcal{D}_1'} \ \sum_{m_2 :\, (m_1,m_2)\in\mathcal{D}'} (m_1^2+m_2^2)^{3/4} e(f(m_1,m_2))\,,$$

where $\mathcal{D}_1'$ denotes the projection of $\mathcal{D}'$ onto the horizontal axis. Applying summation by parts to the last inner sum, we conclude that

$$E_{W,j}^{(1)} \ll \sum_{m_1\in\mathcal{D}_1'} W^{3/4} \sup_I \Big| \sum_{m_2\in I} e(f(m_1,m_2)) \Big| \,,$$

where, for every fixed m_1, I ranges over all subintervals of

$$\left\{u_2\in\mathbb{R} :\ (m_1,u_2)\in\mathcal{D}'\right\}.$$

Now

$$\frac{\partial^3 f}{\partial u_2^3} = \frac{3Zu_2(3m_1^2-2u_2^2)}{(m_1^2+u_2^2)^{7/2}} \asymp ZW^{-2}$$

on every such interval I. Thus, using Lemma 2 with $L=\sqrt{W}$, $\Delta_3 \asymp ZW^{-2}$, we get

$$\begin{aligned} E_{W,j}^{(1)} &\ll W^{5/4}\left(W^{1/2}(ZW^{-2})^{1/6} + W^{1/4}(ZW^{-2})^{-1/6}\right) \\ &\ll W^{17/12} Z^{1/6} + W^{11/6} Z^{-1/6}\,. \end{aligned}$$

Similarly, let

$$E_{W,j}^{(2)} = \sum_{m_2\in\mathcal{D}_2''} \ \sum_{m_1 :\, (m_1,m_2)\in\mathcal{D}''} (m_1^2+m_2^2)^{3/4} e(f(m_1,m_2))\,,$$

with $\mathcal{D}_2''$ denoting the projection of $\mathcal{D}''$ onto the vertical axis. Since now

$$\frac{\partial^3 f}{\partial u_1^3} = \frac{3Zu_1(3m_2^2 - 2u_1^2)}{(u_1^2+u_2^2)^{7/2}} \asymp ZW^{-2}$$

on $\left\{u_1 \in \mathbb{R} : \ (u_1, m_2) \in \mathcal{D}''\right\}$, $m_2 \in \mathcal{D}_2''$, we conclude again that

$$E_{W,j}^{(2)} \ll W^{17/12} Z^{1/6} + W^{11/6} Z^{-1/6}\,.$$

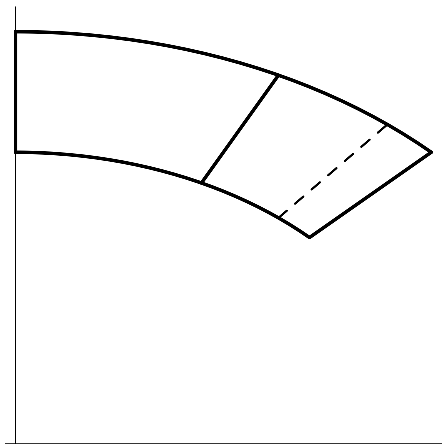

FIGURE 1. The domains $\mathcal{D}'$ (left) and $\mathcal{D}''$ (right). On the dashed line, $\partial^3 f/\partial u_2^3$ vanishes.

Adding up and going back to (4.3), we arrive at

$$E_{W,j} \ll W^{17/12} Z^{1/6} + W^{11/6} Z^{-1/6} + W^{5/4}\,. \tag{4.4}$$

Using this in (4.2), we obtain

$$\begin{aligned}
&\sum_{W<n\le 2W} r(n)\, n\, P(Y/n) \\
&\ll Y^{1/4} \sum_{1\le j\le U} \frac{r(j)}{j^{3/4}} \left(W^{17/12}(jY)^{1/12} + W^{11/6}(jY)^{-1/12} + W^{5/4}\right) \\
&\qquad + Y^{1/2+\varepsilon} U^{-1/2} W^{3/2} + U^{\varepsilon} W^2 \\
&\ll (YU)^{1/3} W^{17/12} + (YU)^{1/6} W^{11/6} + (YU)^{1/4} W^{5/4} \\
&\qquad + Y^{1/2+\varepsilon} U^{-1/2} W^{3/2} + U^{\varepsilon} W^2\,.
\end{aligned}$$

To optimize the estimate, we balance the first against the next-to-last term (apart from the ε) to get $U \asymp Y^{1/5} W^{1/10}$, and thus

$$\sum_{W<n\le 2W} r(n)\, n\, P(Y/n) \ll Y^{2/5+\varepsilon} W^{29/20} + Y^{1/5} W^{37/20} + Y^{\varepsilon} W^2\,.$$

Summation over $W = 2^{-r}X/\ell$, $r = 1, \dots, [\log(X/\ell)/\log 2]$, replaces just W by X/ℓ at the right hand side. Finally, going back to (2.4) and summing over ℓ, we arrive at

$$\mathcal{R} \ll X^{29/20}Y^{2/5+\varepsilon} + X^{37/20}Y^{1/5} + X^2Y^{\varepsilon}\,.$$

For $Y > X$ the last term is less than the next-to-last one, hence the proof of our Theorem is complete.

5. Concluding remark

Seeking for improvements of the estimate obtained, one option is to treat the exponential sum by M. Huxley's deep "Discrete Hardy-Littlewood Method". In its strongest and most recent form [4], this amounts to replacing the bound in Lemma 2 essentially by $O(M^{397/410+\varepsilon}\Delta_3^{32/205})$ which yields a similar slight improvement in the final result. This however is achieved at the cost of considerable technical complications (like conditions on the second and fourth derivative) and the dependance on a very deep and difficult theory. Therefore, in this note we preferred the present exposition which uses only the classical Van der Corput method.

Acknowledgement. The author expresses his gratitude to Professor László Tóth, presently visiting professor at BOKU Wien, for directing his attention to the papers of Chan and Kumchev [1] and of Grytczuk [2], and for enlightening discussions on the subject. Special thanks are also due to Professor Angel Kumchev for the permission to quote the unpublished paper [1] and for sending to the author a revised version of that article.

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Received 24. 1. 2011
Accepted 17. 2. 2011

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