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STRONG NMV-ALGEBRAS, COMMUTATIVE BASIC ALGEBRAS AND NABL-ALGEBRAS

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ABSTRACT. The aim of the paper is to investigate the relationship among NMV-algebras, commutative basic algebras and $naBL$-algebras (i.e., non-associative BL-algebras). First, we introduce the notion of strong NMV-algebra and prove that

(1) a strong NMV-algebra is a residuated l-groupoid (i.e., a bounded integral commutative residuated lattice-ordered groupoid);
(2) a residuated l-groupoid is commutative basic algebra if and only if it is a strong NMV-algebra.

Secondly, we introduce the notion of NMV-filter and prove that a residuated l-groupoid is a strong NMV-algebra (commutative basic algebra) if and only if its every filter is an NMV-filter. Finally, we introduce the notion of weak $naBL$-algebra, and show that any strong NMV-algebra (commutative basic algebra) is weak $naBL$-algebra and give some counterexamples.

1. Introduction and preliminaries

Various ordered algebraic structures (for example, MV-algebras, BL-algebras, MTL-algebras or general residuated lattices) play an important role in the research of fuzzy logic, quantum logic and rough set theory (see [18–22, 32–34, 38, 39]). Recently, non-associative fuzzy logic structures have been studied in many

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papers (see ([1, 3–15, 25–27, 35–37]). In particular, Chajda and Kühr [14] introduced the notions of non-associative *MV*-algebra and *NMV*-implication algebra (which is a dual form of non-associative *MV*-algebra). As a non-associative generalization of the Łukasiewicz logic, Botur and Halaš [4] establish fuzzy logic L_{CBA}, and its algebraic counterpart are commutative basic algebras. Moreover, Botur and Halaš [5] investigate non-associative *BL*-logic and non-associative *BL*-algebras (*naBL*-algebras).

In this paper we discuss the relationship among *NMV*-algebras, commutative basic algebras and *naBL*-algebras.

Now, we recall some basic concepts and properties.

Definition 1.1. ([14]) An algebra $(A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called a non-associative *MV*-algebra or an *NMV*-algebra for short if it satisfies the identities

(1) $x \oplus y = y \oplus x$;

(2) $x \oplus 0 = x$;

(3) $\neg\neg x = x$;

(4) $x \oplus 1 = 1$ where $1 = \neg 0$;

(5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(6) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$;

(7) $\neg x \oplus (x \oplus y) = 1$.

Definition 1.2. ([3, 12]) A basic algebra is an algebra $(A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(A1) $x \oplus 0 = x$;

(A2) $\neg\neg x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$ where $1 = \neg 0$;

(A4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$;

(A5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

A basic algebra A is commutative if it satisfies the commutativity identity

$$x \oplus y = y \oplus x.$$

A basic algebra is an *MV*-algebra if it is commutative and associative.

Remark 1.

(1) Chajda and Kolarik [13] proved that (A3) is redundant.

(2) Let $(A; \oplus, \neg, 0)$ be an *NMV*-algebra or a basic algebra. The relation $\leq$ defined by $x \leq y$ iff $\neg x \oplus y = 1$ is a partial order such that 0 and 1 are the least and the greatest element. In case of basic algebras the poset $(A; \leq)$ is a bounded lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ are the

supremum and the infimum of x, y. In case of NMV-algebras, $x \vee y$ ($x \wedge y$) is a common upper (lower) bound of x, y.

Definition 1.3. ([21,35,36]) A residuated l-groupoid (or non-associative residuated lattice, i.e., bounded integral residuated lattice-ordered groupoid) is an algebra $(L; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(i) $(L; \wedge, \vee, 0, 1)$ is a bounded lattice with the top element 1 and the bottom element 0;
(ii) $(L; \otimes, 1)$ is a commutative groupoid with unit 1;
(iii) $x \otimes y \leq z$ iff $y \leq x \rightarrow z$ for all $x, y, z \in L$.

For a residuated l-groupoid (note that it is bounded integral commutative in this paper), we will use the notation $\neg x$ for $x \rightarrow 0$.

Lemma 1.1. ([35,36]) *Let $(L; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated l-groupoid. Then, for all $x, y, z \in L$:*

(1) $x \rightarrow y = \max\{z \in L \mid x \otimes z \leq y\}$;
(2) $x \leq y \rightarrow y \otimes x$;
(3) $x \otimes (x \rightarrow y) \leq y$;
(4) $1 \rightarrow x = x$;
(5) *if* $y \leq z$ *then* $x \otimes y \leq x \otimes z$;
(6) *if* $y \leq z$ *then* $x \rightarrow y \leq x \rightarrow z$;
(7) $x \otimes \neg x = 0$;
(8) $x \leq y \rightarrow z$ *iff* $y \leq x \rightarrow z$;
(9) $x \leq (x \rightarrow y) \rightarrow y$;
(10) $x \leq \neg\neg x$;
(11) *if* $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$;
(12) *if* $x \leq y$ *then* $\neg y \leq \neg x$;
(13) $x \leq y$ *iff* $x \rightarrow y = 1$;
(14) $x \otimes y \leq x \wedge y$;
(15) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$;
(16) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(17) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
(18) $\neg(x \vee y) = \neg x \wedge \neg y$.

Definition 1.4. ([36]) Let L be a residuated l-groupoid. A filter F is a nonempty subset of L, which satisfies:

(1) $1 \in F$;

(2) For any $x, y \in L$, if $x, x \to y \in F$, then $y \in F$;

(3) For any $x, y \in L$, $(x \otimes y) \otimes F = x \otimes (y \otimes F)$, where $a \otimes F = \{a \otimes f \mid f \in F\}$ for any $a \in L$.

Proposition 1.1. ([36]) *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid and $\emptyset \neq F \subseteq L$. Then F is a filter of L if and only if F satisfies:*

(1) *For any $x, y \in F$, $x \otimes y \in F$;*

(2) *For any $x \in F$, if $x \leq y$, then $y \in F$;*

(3) *For any $x, y \in L$, $(x \otimes y) \otimes F = x \otimes (y \otimes F)$.*

Theorem 1.2. ([36]) *Let L be a residuated l-groupoid, F a filter of L. Then*

(1) $\forall x, y, z \in L, \quad x \to (y \to z) \in F \implies x \otimes y \to z \in F$;

(2) $\forall x, y, z \in L, \quad x \otimes y \to z \in F \implies x \to (y \to z) \in F$;

(3) $\forall x, y, z \in L, \quad x \to y \in F \implies (y \to z) \to (x \to z) \in F$;

(4) $\forall x, y, z \in L, \quad x \to y \in F \implies (z \to x) \to (z \to y) \in F$.

Theorem 1.3. ([36]) *Let L be a residuated l-groupoid, F a filter of L. Define the relation $\sim_F$ on L as follows:*

$$x \sim_F y \iff (x \to y \in F \;\;\&\;\; y \to x \in F).$$

Then $\sim_F$ is a congruence relation on L and in the quotient residuated l-groupoid $L/\sim_F$ one has $[x]_F \leq [y]_F$ if and only if $x \to y \in F$.

Definition 1.5. ([5]) Non-associative BL-algebras (more briefly $naBL$-algebras) are the members of the subvariety of the variety of residuated l-groupoids which is generated by its linearly ordered members and which satisfies divisibility axiom

$$x \otimes (x \to y) = x \wedge y.$$

We denote $x \perp y$ if $x \vee y = 1$. For any nonempty subset M of L, let $M^{\perp} = \{x \in L \mid x \perp y \text{ for all } y \in M\}$. Moreover,

$$\alpha_a^b(x) := (a \otimes b) \to a \otimes (b \otimes x), \qquad \beta_a^b(x) := b \to (a \to (a \otimes b) \otimes x).$$

Theorem 1.4. ([5]) *Let V be a subvariety of the variety of residuated l-groupoids. Then the following conditions are equivalent:*

(1) *V is generated by its linearly ordered elements.*

(2) *Subdirectly irreducible algebras in V are linearly ordered.*

(3) *For any $A \in V$ and any $M \subseteq A$ the set $M^{\perp}$ is filter.*

(4) *The quasi-identities $x \perp y \implies x \perp \alpha_a^b(y)$, $x \perp y \implies x \perp \beta_a^b(y)$ hold in V.*

Theorem 1.5. ([5]) *A residuated l-groupoid is an naBL-algebra if and only if it satisfies divisibility and the following identities:*

$$(x \to y) \vee \alpha_a^b(y \to x) = 1 \qquad (\alpha - prelinearity).$$
$$(x \to y) \vee \beta_a^b(y \to x) = 1 \qquad (\beta - prelinearity).$$

Definition 1.6. ([36]) Let L be a residuated l-groupoid. A filter F of L is called a Boolean filter if $x \vee \neg x \in F$ for all $x \in L$.

Obviously, if $F \subseteq G$ are two filters of L and F is Boolean, then G is a Boolean filter. In [34], we prove that, for associative residuated lattice L, F is a Boolean filter of L if and only if quotient residuated lattice $L/{\sim_F}$ is a Boolean algebra.

Theorem 1.6. ([36]) *Let L be a residuated l-groupoid and F a filter of L. Then the following statements are equivalent:*

(1) *F is a Boolean filter;*
(2) *for any $x, y, z \in L$, $x \to (\neg z \to y) \in F$, $y \to z \in F$ implies $x \to z \in F$;*
(3) *for any $x, y \in L$, $(x \to y) \to x \in F$ implies $x \in F$.*

2. Strong *NMV*-algebras

Definition 2.1. An *NMV*-algebra is called strong *NMV*-algebra, if it satisfies:

(SNMV) $x \oplus (y \oplus z) = 1$ if and only if $(x \oplus y) \oplus z = 1$ for all x, y, z.

In an *NMV*-algebra $(A; \oplus, \neg, 0)$, for any $x, y \in A$, denote $x \to y = \neg x \oplus y$. Then, the condition (SNMV) can be written

(SNMV) $\neg x \to (\neg y \to z) = 1$ if and only if $\neg(\neg x \to y) \to z = 1$ for all x, y, z.

It is easy to prove that:

Proposition 2.1. *A strong NMV-algebra is an MV-algebra if and only if it satisfies the identity*

$$(x \to y) \to ((z \to x) \to (z \to y)) = 1.$$

Example 1. Let the operations $\oplus$ and $\neg$ on the set $X = \{0, a, b, c, d, 1\}$ be defined by Table 1 and Table 2. Then $(X; \oplus, \neg, 0)$ is an *NMV*-algebra, but it is not a strong *NMV*-algebra since $a \oplus (d \oplus d) = 1 \neq a = (a \oplus d) \oplus d$.

Example 2. Let the operations $\oplus$ and $\neg$ on the set $X = \{0, a, b, c, d, 1\}$ be defined by Table 3 and Table 4. Then $(X; \oplus, \neg, 0)$ is a strong *NMV*-algebra.

Table 1.

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	a	1
b	b	1	1	b	1	1
c	c	1	b	a	b	1
d	d	a	1	b	a	1
1	1	1	1	1	1	1

Table 2.

x	$\neg x$
0	1
a	c
b	d
c	a
d	b
1	0

Table 3.

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	c	d	1	1
b	b	c	d	1	1	1
c	c	d	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

Table 4.

x	$\neg x$
0	1
a	d
b	c
c	b
d	a
1	0

By [14: Theorem 14] and its proof we can get:

Proposition 2.2. *Let $(A; \oplus, \neg, 0)$ be an NMV-algebra. Define $1 := \neg 0$ and*

$$x \to y := \neg x \oplus y.$$

Then, for any $x, y, z \in A$,

(1) $x \to x = 1$.

(2) $x \to 0 = \neg x$.

(3) $x \to 1 = 1$, $1 \to x = x$ *and* $0 \to x = 1$.

(4) $(x \to y) \to y = (y \to x) \to x$.

(5) $x \to (y \to 0) = y \to (x \to 0)$.

(6) $x \to ((((x \to y) \to y) \to z) \to z) = 1$.

(7) $((x \to y) \to y) \to y = x \to y$.

(8) $x \to (y \to x) = 1$.

Theorem 2.3. *Let $(A; \oplus, \neg, 0)$ be an NMV-algebra. Define $1 := \neg 0$ and for any $x, y \in A$,*

$$x \otimes y := \neg(x \to \neg y),$$
$$x \leq y \iff x \to y = 1,$$
$$x \vee y := (x \to y) \to y, \quad x \wedge y := \neg((\neg x \to \neg y) \to \neg y).$$

Then $(A; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *is a residuated l-groupoid if and only if* $(A; \oplus, \neg, 0)$ *is a strong NMV-algebra.*

P r o o f. It is proved in [14] that in any NMV-algebra $\leq$ is a partial order such that $x \vee y = (x \rightarrow y) \rightarrow y$ is an upper bound of x, y and $x \wedge y$ is a lower bound.

Suppose that $(A; \oplus, \neg, 0)$ is a strong NMV-algebra.

(i) Assume a is an upper bound of x and y, i.e., $x \leq a, y \leq a$. Then, by Proposition 2.2 (4) we have

$$(x \rightarrow y) \rightarrow y \leq (a \rightarrow y) \rightarrow y = (y \rightarrow a) \rightarrow a = 1 \rightarrow a = a.$$

Thus, $(x \rightarrow y) \rightarrow y$ is the supremum of x and y. That is, $x \vee y = (x \rightarrow y) \rightarrow y$.

Similarly,

$$\forall x, y \in A, \quad x \wedge y = \neg((\neg x \rightarrow \neg y) \rightarrow \neg y)$$

is the infimum of x and y.

Hence, $(A; \wedge, \vee, 0, 1)$ is a bounded lattice.

(ii) By the definition of $\otimes$ and Proposition 2.2 (5) we have

$$x \otimes y = y \otimes x.$$

Thus, $(A; \otimes, 1)$ is a commutative groupoid with unit 1.

(iii) By Definition 2.1 (SNMV) we have for all $x, y, z \in A$

$$\begin{aligned} x \leq y \rightarrow z \iff x \leq \neg y \oplus z \iff \neg x \oplus (\neg y \oplus z) = 1 \\ \iff (\neg x \oplus \neg y) \oplus z = 1 \iff \neg(\neg(\neg x \oplus \neg y)) \oplus z = 1 \\ \iff \neg(\neg x \oplus \neg y) \leq z \iff x \otimes y \leq z. \end{aligned}$$

Therefore, by (i), (ii) and (iii), $(A; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a residuated l-groupoid.

Conversely, suppose that $(A; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a residuated l-groupoid. Then the law of residuation holds in A, i.e.,

$$x \otimes y \leq z \iff y \leq x \rightarrow z \qquad \text{for all} \quad x, y, z \in A.$$

Then, for any $x, y, z \in A$,

$$\begin{aligned} \neg x \rightarrow (\neg y \rightarrow z) = 1 \iff \neg x \leq \neg y \rightarrow z \iff (\neg x \otimes \neg y) \leq z \\ \iff \neg(\neg x \rightarrow y) \leq z \iff \neg(\neg x \rightarrow y) \rightarrow z = 1. \end{aligned}$$

By Definition 2.1, $(A; \oplus, \neg, 0)$ is a strong NMV-algebra. □

By [4: Proposition 3] we can get:

Proposition 2.4. *Let* $(A; \oplus, \neg, 0)$ *be a commutative basic algebra. Define* $1 := \neg 0$ *and for any* $x, y \in A$,

$$\begin{aligned} x \rightarrow y &:= \neg x \oplus y, \\ x \otimes y &:= \neg(\neg x \oplus \neg y), \\ x \leq y &\iff \neg x \oplus y = 1. \end{aligned}$$

Then, for any $x, y \in A$,

(1) $\neg x = x \to 0$.

(2) $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

(3) $x \otimes y = y \otimes x$.

(4) $x \otimes 1 = 1 \otimes x = x$.

(5) $x \otimes y \le z \iff y \le x \to z$.

(6) $x \to \neg y = y \to \neg x$.

(7) $(x \to y) \to y = (y \to x) \to x$.

(8) $x \otimes (x \to y) = x \wedge y$.

(9) $(x \to y) \vee (y \to x) = 1$.

Therefore, $(A; \wedge, \vee, \otimes, \to, 0, 1)$ *is a residuated l-groupoid with conditions* $(7), (8)$ *and* (9).

Theorem 2.5. *Let* $(A; \oplus, \neg, 0)$ *be a commutative basic algebra. Then* $(A; \oplus, \neg, 0)$ *is a strong NMV-algebra.*

Proof. For any x, y in A, define

$$x \to y := \neg x \oplus y,$$
$$x \otimes y := \neg(\neg x \oplus \neg y),$$
$$x \le y \iff \neg x \oplus y = 1, \qquad \text{where} \quad 1 := \neg 0.$$

First, we prove that $(A; \oplus, \neg, 0)$ is an NMV-algebra. Comparing Definition 1.1 and 1.2, it suffices to show that the identities (6) and (7) of Definition 1.1 are satisfied. But by Propersition 2.4, $(A; \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated l-groupoid, hence $x \le y \to x$, which is (7). Moreover, by Lemma 1.1 (9), $x \le (x \to y) \to y \le (((x \to y) \to y) \to z) \to z$, which is (6). Thus a commutative basic algebra is an NMV-algebra.

Now, applying Theorem 2.3, $(A; \oplus, \neg, 0)$ is a strong NMV-algebra. □

Theorem 2.6. *Let* $(A; \oplus, \neg, 0)$ *be a strong NMV-algebra. Then* $(A; \oplus, \neg, 0)$ *is a commutative basic algebra.*

Proof.

(1) Obviously, the conditions (A1)–(A3) and (A5) in Definition 1.2 hold in A.

(2) Since $(A; \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated l-groupoid by Theorem 2.3, we have

$$((\neg x \to y) \to y) \to z \le \neg x \to z, \qquad \text{by Lemma 1.1(9) and (11).}$$

Thus, $(((\neg x \to y) \to y) \to z) \to (\neg x \to z) = 1$. That is, $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$. This means that the condition (A4) in Definition 1.2 holds in A. Hence, $(A; \oplus, \neg, 0)$ is a commutative basic algebra. □

Remark 2. From Theorem 2.5 and Theorem 2.6 we know that strong *NMV*-algebra and commutative basic algebra are equivalent. Therefore, above results obtain a characteristic property of commutative basic algebras.

3. *NMV*-filters in non-associative residuated lattices

The notion of *NMV*-implication algebra is introduced in [14], which corresponds to *NMV*-algebra.

Definition 3.1. ([14]) An *NMV-implication algebra* is an algebra $(A; \to, 0, 1)$ of type $(2, 0, 0)$ that satisfies the following identities:

(NI1) $x \to 1 = 1$, $1 \to x = x$ and $0 \to x = 1$.

(NI2) $(x \to y) \to y = (y \to x) \to x$.

(NI3) $x \to (y \to 0) = y \to (x \to 0)$.

(NI4) $x \to ((((x \to y) \to y) \to z) \to z) = 1$.

(NI5) $((x \to y) \to y) \to y = x \to y$.

Theorem 3.1. ([14]) *Let $(A; \oplus, \neg, 0)$ be an NMV-algebra. If we define $x \to y := \neg x \oplus y$, then $(A; \to, 0, 1)$ is an NMV-implication algebra, where $1 := \neg 0$.*

Conversely, if $(A; \to, 0, 1)$ is an NMV-implication algebra and if we put $x \oplus y := (x \to 0) \to y$ and $\neg x := x \to 0$, then $(A; \oplus, \neg, 0)$ is an NMV-algebra.

Remark 3. In what follows, we will not distinguish *NMV*-algebras and *NMV*-implication algebras and we will say that $(A; \to, 0, 1)$ is an *NMV*-algebra.

Lemma 3.1. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. Then $(L; \to, 0, 1)$ is a strong NMV-algebra if and only if it satisfies the identities*

(C1) $(x \to y) \to y = (y \to x) \to x$,

(C2) $(x \to 0) \to y = (y \to 0) \to x$.

Proof. Suppose that L satisfies (C1) and (C2).

For any $x, y \in L$, by (C1), $(x \to 0) \to 0 = x$, $(y \to 0) \to 0 = y$. Thus, by (C2), we have

$$\begin{aligned} x \to (y \to 0) &= ((x \to 0) \to 0) \to (y \to 0) \\ &= ((y \to 0) \to 0) \to (x \to 0) = y \to (x \to 0). \end{aligned}$$

Also, by Lemma 1.1 we have

$$x \leq (x \to y) \to y \leq (((x \to y) \to y) \to z) \to z,$$

That is, $x \to ((((x \to y) \to y) \to z) \to z) = 1$. And, by (C1),

$$((x \to y) \to y) \to y = (y \to (x \to y)) \to (x \to y) = 1 \to (x \to y) = x \to y.$$

Therefore, by Definition 2.1, $(L; \to, 0, 1)$ is a strong NMV-algebra.

Conversely, if $(L; \to, 0, 1)$ is a strong NMV-algebra, then it is easy to verify that (C1) and (C2) hold. $\square$

Lemma 3.2. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. Then $(L; \to, 0, 1)$ is a strong NMV-algebra if and only if it satisfies* (C2) *and the identity*

(C3) $(x \to y) \to y = (((x \to y) \to y) \to x) \to x.$

P r o o f. Suppose that $(L; \to, 0, 1)$ is a strong NMV-algebra. Obviously, condition (C2) holds. Also, by Definition 3.1 (NI2) and (NI5), for any $x, y, z \in L$,

$$\begin{aligned}(((x \to y) \to y) \to x) \to x &= (((y \to x) \to x) \to x) \to x \\ &= (y \to x) \to x = (x \to y) \to y.\end{aligned}$$

That is, condition (C3) holds.

Conversely, suppose that L satisfies the identities (C3) and (C2). For any $x, y \in L$, by Lemma 1.1 (11),

$$y \le (x \to y) \to y \implies (y \to x) \to x \le (((x \to y) \to y) \to x) \to x.$$

Using condition (C3), $(x \to y) \to y = (((x \to y) \to y) \to x) \to x$. It follows that $(y \to x) \to x \le (x \to y) \to y$. Similarly, $(x \to y) \to y \le (y \to x) \to x$. Hence, $(x \to y) \to y = (y \to x) \to x$. This means that L satisfies condition (C1) and (C2). By Lemma 3.1, $(L; \to, 0, 1)$ is a strong NMV-algebra. $\square$

Definition 3.2. Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. A filter F of L is called an NMV-filter if

(NMVF1) $x \to y \in F \implies ((y \to x) \to x) \to y \in F.$

(NMVF2) $x \to (\neg y \to z) \in F \implies x \to (\neg z \to y) \in F.$

Theorem 3.2. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. Then the following statements are equivalent:*

(1) $(L; \to, 0, 1)$ *is a strong NMV-algebra.*

(2) *Every filter of L is an NMV-filter.*

(3) $\{1\}$ *is an NMV-filter of L.*

(4) *For any filter F of L, the quotient algebra L/F is a strong NMV-algebra.*

P r o o f.

(1) $\implies$ (2). Let F be a filter of L. For any x, $y \in L$,

$$x \to y \le ((x \to y) \to y) \to y = ((y \to x) \to x) \to y.$$

By Proposition 1.1 (2) we have $x \to y \in F \implies ((y \to x) \to x) \to y \in F$. That is, (NMVF1) holds.

Also, since $\neg y \to z = \neg z \to y$ holds for any NMV-algebra, so $x \to (\neg y \to z) \in F \implies x \to (\neg z \to y) \in F$. That is, (NMVF2) holds. By Definition 3.2, F is an NMV-filter.

(2) $\implies$ (3). Obviously.

(3) $\implies$ (1). Since $\{1\}$ is a filter of L, then

$$\begin{aligned}\forall x, y \in L, \quad & x \to ((x \to y) \to y) = 1 \in \{1\} \\ & \implies ((((x \to y) \to y) \to x) \to x) \to ((x \to y) \to y) \in \{1\}.\end{aligned}$$

That is,

$$(((x \to y) \to y) \to x) \to x \leq (x \to y) \to y.$$

On the other hand, $(x \to y) \to y \leq (((x \to y) \to y) \to x) \to x$. Thus,

$$(x \to y) \to y = (((x \to y) \to y) \to x) \to x.$$

This means that the condition (C3) in Lemma 3.2 holds for L.

Moreover, $(\neg y \to z) \to (\neg y \to z) = 1 \in \{1\}$, applying (NMVF2) we have $(\neg y \to z) \to (\neg z \to y) \in \{1\}$, that is, $(\neg y \to z) \to (\neg z \to y) = 1$. Similarly, we have $(\neg z \to y) \to (\neg y \to z) = 1$. Hence, $\neg y \to z = \neg z \to y$. This means that the condition (C2) in Lemma 3.2 holds for L.

Therefore, by Lemma 3.2, $(L; \to, 0, 1)$ is a strong NMV-algebra.

(2) $\implies$ (4). We prove that the quotient algebra L/F is an strong NMV-algebra for any NMV-filter F of L.

Let F be an NMV-filter of L, then the quotient algebra L/F is an NMV-algebra. Now, we prove that L/F is strong.

For any $x, y \in L$, since $((x \to y) \to y) \to ((x \to y) \to y) = 1 \in F$, by (NMVF1) of Definition 3.2 we get

$$((((x \to y) \to y) \to x) \to x) \to ((x \to y) \to y) \in F.$$

And, obviously,

$$((x \to y) \to y) \to ((((x \to y) \to y) \to x) \to x) = 1 \in F.$$

Thus, $([x] \to [y]) \to [y] = ((([x] \to [y]) \to [y]) \to [x]) \to [x]$, that is, the identity (C3) holds in L/F.

Moreover, for any $x, y \in L$, since $(\neg x \to y) \to (\neg x \to y) = 1 \in F$, by (NMVF2) of Definition 3.2 we get $(\neg x \to y) \to (\neg y \to x) \in F$. By the same way, we have $(\neg y \to x) \to (\neg x \to y) \in F$. Thus, $([x] \to [0]) \to [y] = ([y] \to [0]) \to [x]$, that is, the identity (C2) holds in L/F.

Therefore, by Lemma 3.2, L/F is a strong NMV-algebra.

(4) $\implies$ (2). We prove that the filter F is an NMV-filter for any strong NMV-algebra L/F.

Let L/F be a strong NMV-algebra, then for any $x, y \in L$,

$$([x] \to [y]) \to [y] = ([y] \to [x]) \to [x],$$
$$([x] \to [0]) \to [y] = ([y] \to [0]) \to [x].$$

That is, for any $x, y \in L$,

(P1) $((x \to y) \to y) \to ((y \to x) \to x) \in F$,

(P2) $(\neg x \to y) \to (\neg y \to x) \in F$.

Now, let $y \to x \in F$, then by Lemma 1.1(9) and Proposition 1.1(2), $((y \to x) \to x) \to x \in F$. And, by (P1) and Theorem 1.2(3), $(((y \to x) \to x) \to x) \to (((x \to y) \to y) \to x) \in F$. Thus, $((x \to y) \to y) \to x \in F$. This means that (NMVF1) holds for F.

Moreover, let $z \to (\neg x \to y) \in F$, then by (P2) and Theorem 1.2(4), $(z \to (\neg x \to y)) \to (z \to (\neg y \to x)) \in F$. Thus, $z \to (\neg y \to x) \in F$.This means that (NMVF2) holds for F.

Therefore, by Definition 3.2, F is an NMV-filter of L. □

Definition 3.3. Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. A Boolean filter F of L is called to be strong, if it satisfies

(NMVF2) $x \to (\neg y \to z) \in F \implies x \to (\neg z \to y) \in F$.

By Theorem 1.2 and 1.6 we have (The proof is similar to that in [32: Proposition 3.9(4)] and it is omitted.)

Lemma 3.3. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid, F Boolean filter of L. Then*

(NMVF1) $x \to y \in F \implies ((y \to x) \to x) \to y \in F$.

Theorem 3.3. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid, F strong Boolean filter of L. Then F is an NMV-filter.*

The inverse of Theorem 3.3 is not true. For example, in Example 2, $\{1\}$ is an NMV-filter but not a Boolean filter.

Theorem 3.4. *Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. Then the following statements are equivalent:*

(1) *$(L; \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra.*

(2) *Every filter of L is a strong Boolean filter of L.*

(3) *$\{1\}$ is a strong Boolean filter of L.*

(4) *For any filter F of L, the quotient algebra L/F is a Boolean algebra.*

Proof.

(1) $\implies$ (2) and (2) $\implies$ (3). Obviously.

(3) $\Longrightarrow$ (1). By Theorem 3.3 and Theorem 3.2, $(L; \to, 0, 1)$ is a strong NMV-algebra. Applying Theorem 2.6, $(L; \oplus, \neg, 0)$ is a commutative basic algebra. By [9: Theorem 3.14], we know that $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice.

By Definition 1.6 and $\{1\}$ is a Boolean filter of L, $x \vee \neg x = 1$ for any $x \in L$. Thus, $x \wedge \neg x = 0$ (applying Lemma 1.1(18) and $\neg\neg x = x$).

Therefore, $(L; \wedge, \vee, \neg, 0, 1)$ is a bounded complemented distributive lattice, that is, it is a Boolean algebra.

(2) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (2). It is similar to the proof of Theorem 3.2. $\square$

Remark 4.

(1) The above results show that a quotient algebra L/F is Boolean algebra iff F is a strong Boolean filter for any residuated l-groupoid L.

(2) Obviously, if a residuated l-groupoid $(L; \wedge, \vee, \otimes, \to, 0, 1)$ satisfy the identity $\neg x \to y = \neg y \to x$ (for examples, all strong NMV-algebras), then its any Boolean filter is strong. By Theorem 3.4, in this kind of residuated l-groupoid, a quotient algebra L/F is Boolean algebra iff F is a Boolean filter. Is it true for all residuated l-groupoid? This is an open problem.

4. Weak non-associative *BL*-algebras

Definition 4.1. Let $(L; \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated l-groupoid. L is called to be a weak non-associative *BL*-algebra (for short, weak *naBL*-algebra), if it satisfies

(W1) $x \otimes (x \to y) = x \wedge y$ (divisibility).

(W2) $(x \to y) \vee (y \to x) = 1$ (prelinearity).

Obviously, every *naBL*-algebra is a weak *naBL*-algebra. The following example shows that there is weak *naBL*-algebra which is not an *naBL*-algebra.

Example 3. Let X be the set $\{0, a, b, c, d, e, 1\}$ with operations defined by Table 5 and Table 6 and order $\leq$ by Figure 1. Then $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is a weak *naBL*-algebra. But $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is not an *naBL*-algebra, since

$$(b \to a) \vee \alpha_a^a(a \to b) = (b \to a) \vee ((a \otimes a) \to a \otimes (a \otimes (a \to b))) = a \neq 1.$$
$$(a \to b) \vee \beta_b^a(b \to a) = (a \to b) \vee (a \to (b \to (b \otimes a) \otimes (b \to a))) = b \neq 1.$$

Example 4. Let X be the set $\{0, a, b, c, d, e, 1\}$ with the order as in Figure 1 and operations defined by Table 7 and Table 8. Then $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is a weak *naBL*-algebra such that $(x \to y) \vee \beta_a^b(y \to x) = 1$. But $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is not an *naBL*-algebra, since $(b \to a) \vee \alpha_c^a(a \to b) = a \neq 1$.

Table 5.

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	b	c	e	1
b	0	a	1	a	d	e	1
c	0	1	1	1	a	e	1
d	0	1	1	1	1	e	1
e	e	1	1	1	1	1	1
1	0	a	b	c	d	e	1

Table 6.

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	c	d	d	e	a
b	0	c	b	c	d	e	b
c	0	d	c	d	d	e	c
d	0	d	d	d	d	e	d
e	0	e	e	e	e	0	e
1	0	a	b	c	d	e	1

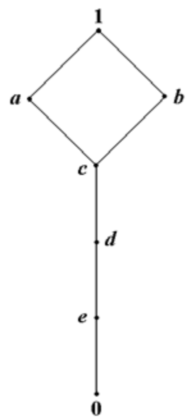

Figure 1.

Table 7.

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	b	d	e	1
b	0	a	1	a	d	e	1
c	0	1	1	1	c	e	1
d	0	1	1	1	1	e	1
e	e	1	1	1	1	1	1
1	0	a	b	c	d	e	1

Table 8.

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	c	c	d	e	a
b	0	c	b	c	d	e	b
c	0	c	c	d	d	e	c
d	0	d	d	d	d	e	d
e	0	e	e	e	e	0	e
1	0	a	b	c	d	e	1

Example 5. Let X be the set $\{0, a, b, c, d, e, 1\}$ with the order as in Figure 1 and operations defined by Table 9 and Table 10. Then $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is an $naBL$-algebra.

Table 9.

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	b	d	e	1
b	0	a	1	a	d	e	1
c	0	1	1	1	d	e	1
d	0	1	1	1	1	d	1
e	e	1	1	1	1	1	1
1	0	a	b	c	d	e	1

Table 10.

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	c	c	d	e	a
b	0	c	b	c	d	e	b
c	0	c	c	c	d	e	c
d	0	d	d	d	e	e	d
e	0	e	e	e	e	0	e
1	0	a	b	c	d	e	1

By Proposition 2.4, Theorem 2.5 and Theorem 2.6 we have:

Theorem 4.1. *Every commutative basic algebra (strong NMV-algebra) is a weak naBL-algebra.*

In Example 5, $(c \to e) \to e = 1 \neq c = (e \to c) \to c$, X is not a commutative basic algebra (strong *NMV*-algebra). This means that the inverse of Theorem 4.1 is not true. Moreover, an *naBL*-algebra with linear order need not be a commutative basic algebra (strong *NMV*-algebra), for example,

Example 6. Let X be the set $\{0, a, b, c, d, 1\}$ with operations defined by Table 11 and Table 12, and the order $0 < a < b < c < d < 1$: Then $(X; \wedge, \vee, \to, \otimes, 0, 1)$ is an *naBL*-algebra, but it is not an *NMV*-algebra since $(b \to a) \to a \neq (a \to b) \to b$.

Table 11.

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	a	1	1	1	1
c	0	a	c	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

Table 12.

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	b	b	b
c	0	a	b	b	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

The connections between *NMV*-algebras, commutative basic algebras and weak *naBL*-algebras can be illustrated by Figure 2.

Remark 5. By [4: Proposition 5] and Theorem 1.4 (or [7: Lemma 5]) we know that α-prelinearity and β-prelinearity conditions are hold in every commutative basic algebra. Therefore, every commutative basic algebra (strong *NMV*-algebra) is a *naBL*-algebra.

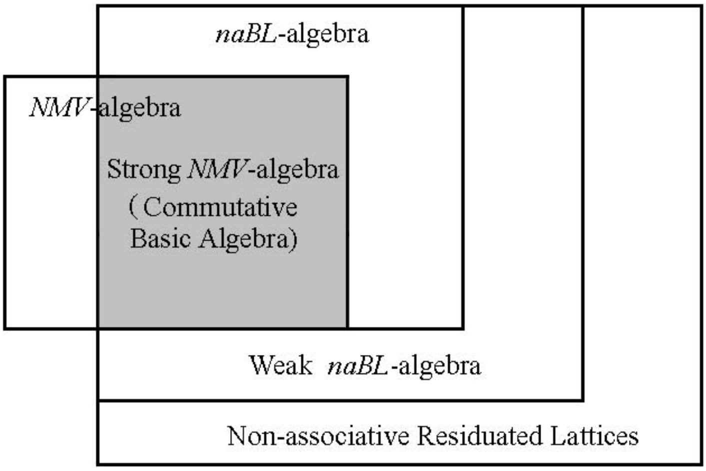

FIGURE 2.

5. Conclusions

In this paper, we investigate the relationship among NMV-algebras, commutative basic algebras and non-associative BL-algebras, some new notions (strong NMV-algebra, NMV-filter and weakly $naBL$-algebra) are introduced, and some important results are obtained, for examples, we present that:

(1) there exists an NMV-algebra which is not a commutative basic algebra (see Example 1 and Theorem 2.6);

(2) an NMV-algebra is a residuated l-groupoid if and only if it is strong (see Theorem 2.3); an NMV-algebra is a commutative basic algebra if and only if it is strong (see Theorem 2.5 and 2.6);

(3) a residuated l-groupoid is a strong NMV-algebra (or commutative basic algebra) if and only if its every filter is an NMV-filter if and only if its every quotient algebra is strong (see Theorem 3.2);

(4) a residuated l-groupoid to be a Boolean algebra if and only if its every filter is a strong Boolean filter if and only if its every quotient algebra is Boolean (see Theorem 3.4);

(5) for residuated l-groupoids, the α-prelinearity and β-prelinearity are independent axioms (see Example 4);

(6) every strong NMV-algebra (or commutative basic algebra) is weak $naBL$-algebra, but the inverse is not true (see Theorem 4.1 and Example 5).

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