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PHILLIPS LEMMA ON EFFECT ALGEBRAS OF SETS

A. Aizpuru — S. Moreno-Pulido* — F. Rambla-Barreno*

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ABSTRACT. We prove the classical Phillips Lemma in the setting of measures defined on effect algebras of sets. This leads to several Vitali-Hahn-Saks-type results for these measures.

1. Introduction

In 1982 Schachermayer published his well known memoir [7] on classical properties of measure theory in non-σ-complete Boolean algebras. This memoir has been a milestone in the development of the modern measure theory. He introduced the Nikodym (N), Grothendieck (G) and Vitali-Hahn-Saks (VHS) properties for Boolean algebras. In [2] some of these properties on a more general structure, the so-called effect algebras, are studied and it is shown that under certain conditions an effect algebra has properties (VHS) and (N) for sequences of σ-additive measures. Recently, in [6], [8], [2], [1], [9], interesting measure theory results on effect algebras have been obtained.

In this paper we concentrate on the notion of effect algebra of sets ([2]), which is a weakening of the concept of Dynkin system, inspired in (abstract) effect algebras [5].

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Definition 1.1. Let L be a family of subsets of a set X. We say that L is an effect algebra of sets if L satisfies the following conditions:

(1) $\varnothing \in L$.

(2) If $a, b \in L$ and $a \cap b = \varnothing$, then $a \cup b \in L$.

(3) If $a \in L$, then $a^c \in L$.

We study measures defined on an effect algebra of sets and valued on a Banach space X.

Definition 1.2. Let L be an effect algebra of sets and X a Banach space.

A map $\mu\colon L \to X$ is called measure if for every $a, b \in L$ such that $a \cap b = \varnothing$, we have that $\mu(a \cup b) = \mu(a) + \mu(b)$. If for each sequence of pairwise disjoint sets $(a_i)_i \subseteq L$ we have that $\mu\big(\bigcup a_i\big) = \sum_i \mu(a_i)$ then μ is called countably additive or σ-additive. If for every sequence of pairwise disjoint sets $(a_i)_i \subset L$, we have that $\sum_i \mu(a_i)$ converges, we say that μ is strongly additive. We say that μ is bounded if $\sup\big\{\|\mu(a)\| : \ a \in L\big\} < \infty$.

We say that a sequence of measures $(\mu_i)_i$ is pointwise convergent on L if there exists $\lim_i \mu_i(a)$ for every $a \in L$. We say that a sequence of measures $(\mu_i)_i$ is uniformly strongly additive (USA) if for every sequence of pairwise disjoint sets $(a_j)_j \subseteq L$, then $\lim_j \mu_i(a_j) = 0$ uniformly in $i \in \mathbb{N}$.

We denote by $\mathrm{ba}(L, X)$ the space of bounded measures $\mu\colon L \to X$. This is a Banach space with the norm $\|\mu\| = \sup\big\{\|\mu(a)\| : \ a \in L\big\}$. We denote by $\mathrm{sa}(L, X)$ the subspace of $\mathrm{ba}(L, X)$ of strongly additive measures. We denote by $\mathrm{ca}(L, X)$ the subspace of σ-additive measures.

The following definition is from [2]:

Definition 1.3. Let L be an effect algebra of sets. We will say that L has the Vitali-Hahn-Saks property (VHS) if every sequence of bounded, real valued measures $(\mu_i)_i$ which is pointwise convergent on L must be (USA).

If L satisfies this conditions for sequences of σ-additive measures, we will say that L has the σ-Vitali-Hahn-Saks property (σ-VHS).

Note that if $\mu\colon L \to \mathbb{R}$ is a bounded measure and $(a_i)_i \subseteq L$ is a sequence of pairwise disjoint sets, then $\sum_i |\mu(a_i)| < \infty$ [2].

We define now the concept of natural effect algebra of sets [2].

Definition 1.4. If X is a set, we denote by $\phi(X)$ the family of finite and cofinite subsets of X, and by $\phi_0(X)$ the family of finite subsets of X.

If $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ and $\phi_0(\mathbb{N}) \subseteq \mathcal{F}$, we call $\mathcal{F}$ natural family [1]. We will say that $\mathcal{F}$ is a natural effect algebra if $\mathcal{F}$ is a natural family with structure of effect algebra.

Let L be an effect algebra of sets. We will say that L is subsequentially complete (SC) if for each sequence of pairwise disjoint sets $(a_i)_i$ of L there exists $M \subseteq \mathbb{N}$ infinite such that $\bigcup\limits_{i \in M} a_i \in L$. If L is SC, then L is VHS (see [8]).

The concept of natural family with the S property is defined in [1]. We will say that a natural effect algebra has the S property if for each pair $[(a_i)_i, (b_i)_i]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists $b \in \mathcal{F}$ and $M \subseteq \mathbb{N}$ infinite such that $a_i \subseteq b$ and $b \cap b_i = \varnothing$ if $i \in M$. In [1] there is an example of a natural algebra $\mathcal{F}$ that has the property S but lacks the SC property.

In [10] some interesting results on vector σ-additive measures which are defined on Boolean natural algebras which have the S property have been obtained. In [2] it is shown that the natural effect algebras $\mathcal{F}$ which have the S property, also have the σ-VHS property. Also in [6] some other results of measure theory are obtained on effect algebras.

In this paper we obtain a version of Phillips Lemma for vector measures, not necessarily σ-additive, which are defined on a natural effect algebra that has a certain property we call (S_1) which will be defined in the next section. Finally we also show some applications of the result to vector measure theory on Boolean algebras and we will see how the VHS theorems are deduced from our Phillips Lemma for natural effect algebras.

2. The main result

We will say that a natural effect algebra $\mathcal{F}$ has the (S_1) property if it satisfies

(**a**): for every sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint elements of $\mathcal{F}$, there exist $H \subset \mathbb{N}$ infinite and $B \in \mathcal{F}$ such that $\bigcup\limits_{i \in H} A_i \subseteq B \subseteq \bigcup\limits_{i \in \mathbb{N}} A_i$.

It is easy to prove that a natural effect algebra $\mathcal{F}$ with (S_1) will also satisfy the following

(**b**): If $A \in \mathcal{F}$ and $B \in \phi_0(\mathbb{N})$, then $A \setminus B \in \mathcal{F}$.

The following theorem is proved in [11] for scalar measures defined on the algebra $\mathcal{P}(\mathbb{N})$.

Theorem 2.1. *Let $\mathcal{F}$ be a natural effect algebra (S_1), X a Banach space and $(\mu_i)_i$ a sequence of* $\mathrm{sa}(\mathcal{F}, X)$. *Then we have* $(1) \implies (2) \implies (3)$ *where*

(1) $\lim\limits_n \mu_n(A) = 0$ *for every* $A \in \mathcal{F}$.

(2) $\lim\limits_n \sup\bigl\{\|\mu_n(A)\| :\ A \in \phi_0(\mathbb{N})\bigr\} = 0$.

(3) $\lim\limits_n \sum\limits_j \mu_n(\{j\}) = 0$.

Proof.

$(1) \implies (2)$ We want to see that for every $\varepsilon > 0$ there exists n_0 such that $\|\mu_n(A)\| < \varepsilon$ if $n \geq n_0$ and $A \in \phi_0(\mathbb{N})$. Otherwise, there exists $\varepsilon > 0$ and $A_k \in \phi_0(\mathbb{N})$ such that for every $k \in \mathbb{N}$ there exists $n_k > k$ with $\|\mu_{n_k}(A_k)\| > \varepsilon$.

There exist $n_1 > 1$ and $A_1 \in \phi_0(\mathbb{N})$ such that $\|\mu_{n_1}(A_1)\| > \varepsilon$.

Let $m_1 = \max A_1$.

We have $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{m_1} \|\mu_n(\{j\})\| = 0$, and we can deduce that there exists $l_1 > n_1$ such that if $n \geq l_1$, it is $\sum\limits_{j=1}^{m_1} \|\mu_n(\{j\})\| < \frac{\varepsilon}{5}$.

There exist $n_2 > l_1$ and $A_2' \in \phi_0(\mathbb{N})$ such that $\|\mu_{n_2}(A_2')\| > \varepsilon$, and if $A_2 = A_2' \setminus \{1, \ldots, m_1\}$, we have

$$\|\mu_{n_2}(A_2')\| \leq \|\mu_{n_2}(A_2)\| + \sum_{j \leq m_1} \|\mu_{n_2}(\{j\})\|$$

and $\|\mu_{n_2}(A_2)\| > \varepsilon - \frac{\varepsilon}{5} = \frac{4\varepsilon}{5}$.

Let $m_2 = \max A_2$. We have $m_1 < m_2$ and $A_2 \subseteq (m_1, m_2]$. There exists $l_2 > n_2$ such that if $n \geq n_2$ it is $\sum\limits_{j \leq m_2} \|\mu_n(\{j\})\| < \frac{\varepsilon}{5}$.

There exist $n_3 > l_2$ and $A_3' \in \phi_0(\mathbb{N})$ such that $\|\mu_{n_3}(A_3')\| > \varepsilon$ and if $A_3 = A_3' \setminus \{1, \ldots, m_2\}$, it is $\|\mu_{n_3}(A_3)\| > \frac{4\varepsilon}{5}$. Let $m_3 = \max A_3$, we have $m_2 < m_3$ and $A_3 \subseteq (m_2, m_3]$.

Put $m_0 := 0$. Inductively we obtain sequences $m_1 < m_2 < \cdots < m_k < \ldots$, $n_1 < n_2 < \cdots < n_k < \ldots$ of natural numbers and a sequence $(A_k)_k$ of $\phi_0(\mathbb{N})$ such that:

(1) $A_k \subseteq (m_{k-1}, m_k]$.

(2) $\|\mu_{n_k}(A_k)\| > \frac{4\varepsilon}{5}$.

(3) $\sum\limits_{j \le m_{k-1}} \|\mu_{n_k}(\{j\})\| < \frac{\varepsilon}{5}$.

For every $k \in \mathbb{N}$, we denote $D_k = (m_{k-1}, m_k]$ and let $\mathbb{N} = \bigcup\limits_j L_j$ be an infinite partition of $\mathbb{N}$ in pairwise disjoint, infinite sets. We show that there exists $j_0 \in \mathbb{N}$ such that $\sup\big\{\|\mu_{n_1}(A)\| : \ A \in \mathcal{F}, \ A \subseteq \bigcup\limits_{k \in L_{j_0}} D_k\big\} < \frac{\varepsilon}{5}$. Otherwise, for every $j \in \mathbb{N}$ there exists $E_j \in \mathcal{F}$ with $E_j \subseteq \bigcup\limits_{k \in L_j} D_k$ and $\|\mu_{n_1}(E_j)\| \ge \frac{\varepsilon}{5}$, but $(E_j)_j$ is a sequence of disjoint sets of $\mathcal{F}$ and this contradicts the fact that μ_{n_1} is strongly additive.

We denote $M_1 = L_{j_0}$ and we can suppose that $m_1 < q_1 = \min M_1$. Let $M_1 = \bigcup\limits_j R_j$ be an infinite partition of M_1 in pairwise disjoint, infinite sets, and we deduce as above that there exists R_{j_1} such that $\sup\big\{\|\mu_{n_{q_1}}(A)\| : \ A \in \mathcal{F}, \ A \subseteq \bigcup\limits_{k \in R_{j_1}} D_k\big\} < \frac{\varepsilon}{5}$. Call $M_2 = R_{j_1}$ and we can assume that $q_1 < q_2 = \min M_2$ and $m_{q_1} < q_2$.

Inductively we obtain $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_r \supseteq \dots$, a sequence of infinite sets of $\mathbb{N}$ and $q_1 < q_2 < \cdots < q_r < \dots$, a sequence of $\mathbb{N}$ such that if $r > 1$ then $q_r = \min M_r$ and $m_{q_r} < q_{r+1}$ and $\sup\big\{\|\mu_{n_{q_r}}(A)\| : \ A \in \mathcal{F}, \ A \subset \bigcup\limits_{j \in M_{r+1}} D_j\big\} < \frac{\varepsilon}{5}$.

Consider $M = \{q_1, q_2, \dots, q_r, \dots\}$ and $(A_{q_r})_{r \in \mathbb{N}}$. There exist $H \subset \mathbb{N}$ infinite and $B \in \mathcal{F}$ such that $\bigcup\limits_{r \in H} A_{q_r} \subseteq B \subseteq \bigcup\limits_{r \in \mathbb{N}} A_{q_r}$.

If $r > 1$ and $r \in H$ then $B = (B \cap \{1, \dots, m_{q_r - 1}\}) \cup A_{q_r} \cup (B \setminus \{1, \dots, m_{q_r}\})$ and $\|\mu_{n_{q_r}}(B \cap \{1, \dots, m_{q_r - 1}\})\| \le \sum\limits_{j=1}^{m_{q_r - 1}} \|\mu_{n_{q_r}}(\{j\})\| < \frac{\varepsilon}{5}$.

$B \setminus \{1, \dots, m_{q_r}\} \subset B \cap \big(\bigcup\limits_{j \ge r+1} A_{q_j}\big) \subset \bigcup\limits_{j \in M_{r+1}} D_j$, so we have

$$\|\mu_{n_{q_r}}(B \setminus \{1, \dots, m_{q_r}\})\| < \frac{\varepsilon}{5}$$

and we deduce that $\|\mu_{n_{q_r}}(B)\| > \frac{2\varepsilon}{5}$, which is a contradiction.

(2) $\implies$ (3) If it is false that $\lim\limits_n \sum\limits_j \mu_n(\{j\}) = 0$, we can suppose that there exists $\varepsilon > 0$ such that $\big\|\sum\limits_j \mu_n(\{j\})\big\| > \varepsilon$ if $n \in \mathbb{N}$.

For $n_1 = 1$, there exists $m_1 \in \mathbb{N}$ such that $\left\| \sum_{j=1}^{m_1} \mu_{n_1}(\{j\}) \right\| > \varepsilon$ but from (2) we deduce that there exists $n_2 > n_1$ such that $\|\mu_{n_2}(\{1, \dots, m_1\})\| < \frac{\varepsilon}{2}$ and we have

$$\Big\| \sum_{j=m_1+1}^{\infty} \mu_{n_2}(\{j\}) \Big\| > \frac{\varepsilon}{2}$$

and there exists $m_2 > m_1$ such that

$$\Big\| \sum_{j=m_1+1}^{m_2} \mu_{n_2}(\{j\}) \Big\| > \frac{\varepsilon}{2}.$$

We denote $A_1 = \{1, \dots, m_1\}$, $A_2 = \{m_1 + 1, \dots, m_2\}$ and inductively we obtain a sequence of $\mathbb{N}$, $n_1 < n_2 < \dots < n_k < \dots$ and a sequence of disjoint elements of $\phi_0(\mathbb{N})$, $(A_k)_k$ such that $\|\mu_{n_k}(A_k)\| > \frac{\varepsilon}{2}$ if $k \in \mathbb{N}$ and this contradicts (2). □

Remark 2.2. Note that the previous theorem still works if we require $\mathcal{F}$ to be a natural family (not necessarily an effect algebra of sets) satisfying properties (**a**) and (**b**) mentioned above.

Example 2.3. Let

$$\begin{aligned} O_n &= \{m \in \mathbb{N} : \ m \text{ is odd and } 1 \leq m < 2n\}, \\ E_n &= \{m \in \mathbb{N} : \ m \text{ is even and } 1 < m \leq 2n\}. \end{aligned}$$

Consider the family

$$\mathcal{F} = \big\{A \subset \mathbb{N} : \ (|A \cap E_n| - |A \cap O_n|)_n \notin l_\infty \big\} \cup \phi(\mathbb{N}).$$

It is easy to check that $\mathcal{F}$ is a natural effect algebra with S_1. On the other hand $\mathcal{F}$ lacks the SC property (it is not subsequentially complete), just consider $(F_n)_n = (\{4n - 1, 4n\})_n$. •

Definition 2.4. Let $\mathcal{F}$ be an effect algebra of subsets of a set Ω, we will say that $\mathcal{F}$ is LS_1 if for every sequence $(A_i)_i$ of disjoint elements of $\mathcal{F}$ there exists $H \subset \mathbb{N}$ infinite and another sequence $(B_i)_i$ of disjoint elements of $\mathcal{F}$ such that $B_i = A_i$ if $i \in H$ and $\mathcal{F}[(B_i)_i] = \big\{M \subset \mathbb{N} : \bigcup_{i \in M} B_i \in \mathcal{F}\big\}$ is a natural family with the S_1 property.

THEOREM 2.5. *Let X be a Banach space and $\mathcal{F}$ an effect algebra of subsets of a set Ω with LS_1. If $(\mu_i)_i$ is a sequence of* $\mathrm{sa}(\mathcal{F}, X)$ *such that* $\lim_i \mu_i(A) = 0$ *for every $A \in \mathcal{F}$, then $(\mu_i)_i$ is USA in $\mathcal{F}$. In particular, we have that $\mathcal{F}$ has the VHS property.*

Proof. If $(\mu_i)_i$ is not USA, there exists a sequence $(A_i)_i$ of disjoint elements of $\mathcal{F}$ and there exists $\varepsilon > 0$ such that for certain subsequence of $(\mu_i)_i$, that we denote the same, it is $\|\mu_i(A_i)\| > \varepsilon$. By the assumption, there exists a sequence $(B_i)_i$ of pairwise disjoint elements of $\mathcal{F}$ and $H \subset \mathbb{N}$ infinite such that $B_i = A_i$ if $i \in H$ and $\mathcal{F}_0 = \mathcal{F}[(B_i)_i]$ has the (S_1) property. We define $\overline{\mu}_i$ in $\mathcal{F}_0$ by $\overline{\mu}_i(M) = \mu_i\big(\bigcup_{i \in M} B_i\big)$ if $i \in \mathbb{N}$, $M \in \mathcal{F}_0$.

We have $\lim_i \overline{\mu}_i(M) = 0$ in $\mathcal{F}_0$, but if $i \in H$ then $\|\overline{\mu}_i(\{i\})\| = \|\mu_i(A_i)\| > \varepsilon$, and this contradicts theorem 2.1. □

The classic Vitali-Hahn-Saks theorems for a Boolean algebra $\mathcal{F}$ have been obtained in the following situations:

(1) $\mathcal{F}$ is σ-algebra.

(2) $\mathcal{F}$ has the E property of Schachermeyer [7]: every sequence of disjoint elements has a subsequence such that every further subsequence has supremum.

(3) $\mathcal{F}$ is subsequentially complete of Haydon [4]: every sequence has a subsequence with supremum.

Also some VHS results have been obtained in the case when $\mathcal{F}$ has interpolation properties [3]. But every interpolation property that is known to be enough for a Boolean algebra to have the VHS property, satisfies that the Boolean algebra quotient the ideal of the elements where certain measure is null satisfies some of the properties (1), (2) or (3) [3].

If $(A_i)_i$ is a sequence of disjoint elements of $\mathcal{F}$ we have:

(1) If $\mathcal{F}$ is a σ-algebra, then $\mathcal{F}[(A_i)_i] = \mathcal{P}(\mathbb{N})$.

(2) If $\mathcal{F}$ is E, then $\mathcal{F}[(A_i)_i]$ has the (S_1) property.

(3) If $\mathcal{F}$ is SC, then $\mathcal{F}[(A_i)_i]$ has the (S_1) property.

From the version of the Phillips Lemma for effect algebras that we have shown, the VHS theorems for Boolean algebras until now known have been obtained.

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* *Department of Mathematics*
Faculty of Sciences
University of Cádiz
Apdo. 40–11510
Puerto Real (Cádiz)
SPAIN

E-mail: soledad.moreno@uca.es
fernando.rambla@uca.es