Mathematica Slovaca

DOI: 10.2478/s12175-013-0121-6
Math. Slovaca **63** (2013), No. 3, 595–610

A GENERALIZATION OF A 4-DIMENSIONAL EINSTEIN MANIFOLD

Yunhee Euh* — Jeong Hyeong Park*ᶜ — Kouei Sekigawa**

(*Communicated by Július Korbaš*)

ABSTRACT. A weakly Einstein manifold is a natural generalization of a 4-dimensional Einstein manifold. In this paper, we shall give a characterization of a weakly Einstein manifold in terms of so-called generalized Singer-Thorpe bases. As an application, we prove a generalization of the Hitchin inequality for compact weakly Einstein 4-manifolds. Examples are provided to illustrate the theorems.

1. Introduction

Berger [2] derived a curvature identity on a 4-dimensional compact oriented Riemannian manifold from the generalized Gauss-Bonnet formula. Further, Labbi [11] extended the curvature identity to the higher dimensional cases. In the previous work [5], we gave a direct proof of the fact that the curvature identity holds on any 4-dimensional Riemannian manifold which is not necessarily compact. Consequently, we proved that the following curvature identity holds on any 4-dimensional Riemannian manifold $M = (M, g)$:

$$\check{R} - 2\check{\rho} - L\rho + \tau\rho - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)g = 0. \tag{1.1}$$

Here,

$$\check{R} : \ \check{R}_{ij} = R_{abci}R^{abc}{}_j, \qquad \check{\rho} : \ \check{\rho}_{ij} = \rho_{ai}\rho^a_j, \qquad L : \ (L\rho)_{ij} = 2R_{iabj}\rho^{ab},$$

2010 Mathematics Subject Classification: Primary 53B20, 53C20.
Keywords: Einstein manifold, Singer-Thorpe basis, Hitchin inequality.
Research of Y. Euh was supported by the National Research Foundation of Korea Grant funded by the Korean Government [NRF-2010-355-C00007].
Research of J. H. Park was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2010-0028097).

where R, ρ and τ are the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. Here, we remark that the curvature tensor of Berger [2] is different from ours by a sign.

An n-dimensional Einstein manifold $M = (M, g)$ is called super-Einstein if the following curvature identity is satisfied

$$\check{R} = \frac{1}{n}|R|^2 g \tag{1.2}$$

(see [6]). We note that the constancy of $|R|^2$ follows from (1.2) automatically in dimensions $n > 4$ (see [3: Lemma 3.3]), however the constancy of $|R|^2$ is not automatically satisfied but it is usually required (see [6]) for a 4-dimensional super-Einstein manifold.

From the curvature identity (1.1), we may easily check that any 4-dimensional Einstein manifold $M = (M, g)$ satisfies the curvature equation (1.2) with $n = 4$, namely

$$\check{R} = \frac{1}{4}|R|^2 g. \tag{1.3}$$

However, the converse of the above is not necessarily valid (see [5]). In [5], we defined a weakly Einstein manifold based on this situation as a 4-dimensional Riemannian manifold satisfying the condition (1.3) (with $|R|^2$ not necessarily constant). By the definition, a weakly Einstein manifold is a generalization of a 4-dimensional Einstein manifold. Therefore, it is natural to ask how large the difference between Einsteinness and weakly Einsteinness. As a characterization of a 4-dimensional Einstein manifold, the following theorem is well-known.

Theorem A. ([12]) *A 4-dimensional Riemannian manifold $M = (M, g)$ is Einstein if and only if there exists a Singer-Thorpe basis of T_pM at each point $p \in M$.*

The main purpose of the present paper is to give a generalization of Theorem A, which is a characterization of a weakly Einstein manifold. Namely, we shall prove the following:

Theorem B. *A 4-dimensional Riemannian manifold $M = (M, g)$ is weakly Einstein if and only if there exists a generalized Singer-Thorpe basis of T_pM at each point $p \in M$.*

Now, we can easily see that Theorem A plays an important role in the proof of the Hitchin inequality [7]. As an application of Theorem B, we establish a Hitchin-type inequality for a compact oriented weakly Einstein manifold (§4, Proposition 4.1), which is a generalization of the Hitchin inequality. Further, we may derive another Hitchin-type inequality from the inequality in Proposition 4.1. Namely, we shall prove the following:

Theorem C. *Let $M = (M, g)$ be a compact weakly Einstein manifold. Then, the following inequality holds on M:*

$$2\chi(M) \pm p_1(M) \geqq -\frac{1}{16\pi^2} \int_M |R|^2 \, \mathrm{d}v_g, \tag{1.4}$$

where $\chi(M)$ and $p_1(M)$ are denoted the Euler number and the first Pontrjagin number of M, respectively.

In §2, we shall prepare some fundamental terminologies and notational conventions for the forthcoming arguments. In §3, we shall give a proof of Theorem B. In §4, we shall give proofs of Proposition 4.1 and Theorem C, and further provide an example to illustrate the inequalities.

The authors would like to express their thanks to Professor O. Kowalski for his helpful comments and valuable suggestions. The authors also wish to thank to the referee for the kind comments that helped to refine the manuscript.

2. Preliminaries

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and $\mathfrak{X}(M)$ be the Lie algebra of all smooth vector fields on M. We denote the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of M by ∇, R, ρ and τ, respectively. We assume that the curvature tensor R is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for X, Y, $Z \in \mathfrak{X}(M)$. Further, we denote the Ricci transformation by Q given by $\rho(X,Y) = g(QX,Y)$ for X, $Y \in \mathfrak{X}(M)$. Then, we may easily check that Q is symmetric with respect to the metric g, namely, $g(QX,Y) = g(X,QY)$ for X, $Y \in \mathfrak{X}(M)$. Now, we may rewrite the curvature identity (1.1) as follows:

$$\begin{aligned}\sum_{a,b,c} R_{abci}R_{abcj} - 2\sum_{a} \rho_{ai}\rho_{aj} - 2\sum_{a,b} \rho_{ab}R_{iabj} \\ +\tau\rho_{ij} - \frac{1}{4}\big(|R|^2 - 4|\rho|^2 + \tau^2\big)\delta_{ij} = 0,\end{aligned} \tag{2.1}$$

with respect to an orthonormal basis $\{e_i\}$ ($1 \leqq i \leqq 4$) of T_pM at any point $p \in M$, where $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$ and $\rho_{ij} = \rho(e_i,e_j)$.

We here introduce some special kinds of orthonormal basis of T_pM at any point $p \in M$ and explain their intermediate relationships. We assume that an orthonormal basis $\{e_i\}$ ($1 \leqq i \leqq 4$) of T_pM is simultaneously a Ricci eigenbasis and Chern basis [4, 9, 10] satisfying

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1314} = R_{1323} = 0. \tag{2.2}$$

Then, we have further

$$R_{2434} = R_{2334} = R_{1434} = R_{1334} = R_{2324} = R_{1424} = 0. \tag{2.3}$$

Thus, from (2.2) and (2.3), we have

$$R_{ijjk} = 0 \qquad (i \neq k, \ \ 1 \leqq i, j, k \leqq 4). \tag{2.4}$$

Conversely, if (2.4) holds with respect to an orthonormal basis $\{e_i\}$ of T_pM, then we see that the basis $\{e_i\}$ is a Ricci eigenbasis and Chern basis at the same time.

The following example shows that a Ricci eigenbasis is not necessarily always a Chern basis.

Example 2.1. Let $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$ be a 4-dimensional real Lie algebra equipped with the following Lie bracket operation:

$$\begin{aligned} &[e_1, e_2] = 2e_2, && [e_1, e_3] = -e_3, && [e_1, e_4] = 2e_3 - e_4, \\ &[e_2, e_3] = 0, && [e_2, e_4] = 0, && [e_3, e_4] = 0, \end{aligned} \tag{2.5}$$

and $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{g}$ given by $\langle e_i, e_j \rangle = \delta_{ij}$. Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$ of G and g the G-invariant Riemannian metric on G determined by $\langle \cdot, \cdot \rangle$. We set $\nabla_{e_i} e_j = \sum_{k=1}^{4} \Gamma_{ijk} e_k$ $(1 \leqq i, j \leqq 4)$. Then, we get

$$\Gamma_{ijk} = -\Gamma_{ikj} \tag{2.6}$$

and further, from (2.5), we obtain

$$\begin{aligned} &\Gamma_{134} = -1, && \Gamma_{212} = -2, && \Gamma_{313} = 1, \\ &\Gamma_{314} = -1, && \Gamma_{413} = -1, && \Gamma_{414} = 1, \end{aligned} \tag{2.7}$$

and otherwise being zero up to sign. From (2.6) and (2.7), by direct calculations, we have

$$\begin{aligned} &R_{1212} = 4, && R_{1414} = 4, && R_{2323} = -2, \\ &R_{2424} = -2, && R_{1314} = -2, && R_{2324} = 2, \end{aligned} \tag{2.8}$$

and otherwise being zero up to sign. Thus, we may easily check that the orthonormal basis $\{e_i\}$ is a Ricci eigenbasis satisfying $Qe_i = \lambda_i e_i$ $(1 \leqq i \leqq 4)$, where $\lambda_1 = -8$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = -2$. However, from (2.8), we see that the basis $\{e_i\}$ does not satisfy (2.4). This means that the basis $\{e_i\}$ is a Ricci eigenbasis but not a Chern basis.

Now, we recall the definition of a Singer-Thorpe basis. An orthonormal basis $\{e_i\}$ of T_pM $(p \in M)$ is called a *Singer-Thorpe basis* if the basis $\{e_i\}$ satisfies (2.4) and

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323}. \tag{2.9}$$

We here give a generalization of the Singer-Thorpe basis based on the above observation.

Definition 2.2. Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and $\{e_i\}$ be an orthonormal basis of T_pM at $p \in M$. If the basis $\{e_i\}$ satisfies (2.4) and

$$R_{1212}^2 = R_{3434}^2, \quad R_{1313}^2 = R_{2424}^2, \quad R_{1414}^2 = R_{2323}^2, \tag{2.10}$$

then the orthonormal basis is called a *generalized Singer-Thorpe basis* of T_pM.

3. Proof of Theorem B

First, we shall prove the following proposition which gives a necessary condition for a 4-dimensional Riemannian manifold to be weakly Einstein.

Proposition 3.1. *Let $M = (M, g)$ be a weakly Einstein manifold and $\{e_i\}$ $(1 \leqq i \leqq 4)$ an orthonormal Ricci eigenbasis of T_pM corresponding to the eigenvalues λ_i $(1 \leqq i \leqq 4)$ at any point $p \in M$. Then, we see that the curvature condition*

$$R_{1212}^2 = R_{3434}^2, \quad R_{1313}^2 = R_{2424}^2, \quad R_{1414}^2 = R_{2323}^2 \tag{3.1}$$

holds and also the following cases (1) $\sim$ (4) *never occur:*

(1) $\lambda_1 = \lambda_2 = \lambda_3 (\neq 0)$, $\lambda_4 = 0$,

(2) $\lambda_1 = \lambda_2 = \lambda_4 (\neq 0)$, $\lambda_3 = 0$,

(3) $\lambda_1 = \lambda_3 = \lambda_4 (\neq 0)$, $\lambda_2 = 0$,

(4) $\lambda_2 = \lambda_3 = \lambda_4 (\neq 0)$, $\lambda_1 = 0$.

Especially, if M is Einstein, then

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323}$$

holds for any orthonormal basis $\{e_i\}$ of T_pM.

P r o o f. Let $M = (M, g)$ be a weakly Einstein manifold and p any point of M and $\{e_i\}$ $(1 \leqq i \leqq 4)$ an orthonormal Ricci eigenbasis of T_pM corresponding to the Ricci eigenvalues λ_i $(1 \leqq i \leqq 4)$ at p, namely, satisfying the following condition

$$Qe_i = \lambda_i e_i \qquad (1 \leqq i \leqq 4). \tag{3.2}$$

Then, from (3.2), using $\lambda_1 = \rho_{11} = -R_{1212} - R_{1313} - R_{1414}$, $0 = \rho_{12} = -R_{1323} - R_{1424}$, and so on, we get

$$\begin{aligned} |R|^2 = 4\big\{&R_{1212}^2 + R_{1313}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2 \\ &+ 4R_{1213}^2 + 4R_{1214}^2 + 4R_{1223}^2 + 4R_{1224}^2 + 4R_{1314}^2 \\ &+ 4R_{1323}^2 + 2R_{1234}^2 + 2R_{1342}^2 + 2R_{1423}^2\big\}. \end{aligned} \tag{3.3}$$

On the other hand, setting $i = j = 1$ in the left hand side of (1.3), we get

$$\begin{aligned}\check{R}_{11} &= \sum_{a,b,c} R^2_{abc1} \\ &= 2\big\{R^2_{1212} + R^2_{1313} + R^2_{1414} + R^2_{1234} + R^2_{1342} + R^2_{1423} \\ &\qquad + 2(R^2_{1213} + R^2_{1214} + R^2_{1223} + R^2_{1224} + R^2_{1314} + R^2_{1323})\big\}.\end{aligned} \tag{3.4}$$

From (3.3), (3.4), and taking account of (1.3), we have the following equality

$$R^2_{1212} + R^2_{1313} + R^2_{1414} - R^2_{2323} - R^2_{2424} - R^2_{3434} = 0. \tag{3.5}$$

Similarly, we get

$$R^2_{1212} + R^2_{2323} + R^2_{2424} - R^2_{1313} - R^2_{1414} - R^2_{3434} = 0, \tag{3.6}$$

$$R^2_{1212} + R^2_{1414} + R^2_{2424} - R^2_{1313} - R^2_{2323} - R^2_{3434} = 0, \tag{3.7}$$

$$R^2_{1212} + R^2_{1313} + R^2_{2323} - R^2_{1414} - R^2_{2424} - R^2_{3434} = 0. \tag{3.8}$$

From (3.5) and (3.6), we have

$$R^2_{1212} - R^2_{3434} = 0. \tag{3.9}$$

Similarly, from (3.5) and (3.7), we have

$$R^2_{1313} - R^2_{2424} = 0. \tag{3.10}$$

From (3.5) and (3.8), we have

$$R^2_{1414} - R^2_{2323} = 0. \tag{3.11}$$

Thus, from (3.9)$\sim$(3.11), we have (3.1).

Next, from (3.1), we see that the following eight cases can be taken into consideration;

Case (i): $R_{1212} = R_{3434}$, $R_{1313} = R_{2424}$, $R_{1414} = R_{2323}$.
Then, $\lambda_1 - \lambda_2 = 0$, $\lambda_1 - \lambda_3 = 0$, $\lambda_1 - \lambda_4 = 0$, and hence, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

Case (ii): $R_{1212} = -R_{3434}$, $R_{1313} = R_{2424}$, $R_{1414} = R_{2323}$.
Then, we get also

$$\begin{aligned}&\lambda_1 - \lambda_2 = 0, && \lambda_1 - \lambda_3 = -2R_{1212}, && \lambda_1 - \lambda_4 = -2R_{1212},\\ &\lambda_2 - \lambda_3 = -2R_{1212}, && \lambda_2 - \lambda_4 = -2R_{1212}, && \lambda_3 - \lambda_4 = 0,\end{aligned}$$

and hence, $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$.

Case (iii): $R_{1212} = R_{3434}$, $R_{1313} = -R_{2424}$, $R_{1414} = R_{2323}$.
Then, we get

$$\begin{aligned}&\lambda_1 - \lambda_2 = -2R_{1313}, && \lambda_1 - \lambda_3 = 0, && \lambda_1 - \lambda_4 = -2R_{1313},\\ &\lambda_2 - \lambda_3 = 2R_{1313}, && \lambda_2 - \lambda_4 = 0, && \lambda_3 - \lambda_4 = -2R_{1313},\end{aligned}$$

and hence, $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$.

Case (iv): $R_{1212} = R_{3434}$, $R_{1313} = R_{2424}$, $R_{1414} = -R_{2323}$.
Then, we get

$$\begin{array}{lll} \lambda_1 - \lambda_2 = -2R_{1414}, & \lambda_1 - \lambda_3 = -2R_{1414}, & \lambda_1 - \lambda_4 = 0, \\ \lambda_2 - \lambda_3 = 0, & \lambda_2 - \lambda_4 = 2R_{1414}, & \lambda_3 - \lambda_4 = 2R_{1414}, \end{array}$$

and hence, $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_3$.
Case (v): $R_{1212} = R_{3434}$, $R_{1313} = -R_{2424}$, $R_{1414} = -R_{2323}$.
Then, we get

$$\begin{array}{lll} \lambda_1 + \lambda_2 = -2R_{1212}, & \lambda_1 - \lambda_3 = -2R_{1414}, & \lambda_1 - \lambda_4 = -2R_{1313}, \\ \lambda_2 - \lambda_3 = 2R_{1313}, & \lambda_2 - \lambda_4 = 2R_{1414}, & \lambda_3 + \lambda_4 = -2R_{1212}, \end{array}$$

and hence, $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$.
Case (vi): $R_{1212} = -R_{3434}$, $R_{1313} = R_{2424}$, $R_{1414} = -R_{2323}$.
Then, we get

$$\begin{array}{lll} \lambda_1 - \lambda_2 = -2R_{1414}, & \lambda_1 + \lambda_3 = -2R_{1313}, & \lambda_1 - \lambda_4 = -2R_{1212}, \\ \lambda_2 - \lambda_3 = -2R_{1212}, & \lambda_2 + \lambda_4 = -2R_{1313}, & \lambda_3 - \lambda_4 = 2R_{1414}, \end{array}$$

and hence, $\lambda_1 + \lambda_3 = \lambda_2 + \lambda_4$.
Case (vii): $R_{1212} = -R_{3434}$, $R_{1313} = -R_{2424}$, $R_{1414} = R_{2323}$.
Then, we get

$$\begin{array}{lll} \lambda_1 - \lambda_2 = -2R_{1313}, & \lambda_1 - \lambda_3 = -2R_{1212}, & \lambda_1 + \lambda_4 = -2R_{1414}, \\ \lambda_2 + \lambda_3 = -2R_{1414}, & \lambda_2 - \lambda_4 = -2R_{1212}, & \lambda_3 - \lambda_4 = -2R_{1313}, \end{array}$$

and hence, $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$.
Case (viii): $R_{1212} = -R_{3434}$, $R_{1313} = -R_{2424}$, $R_{1414} = -R_{2323}$.
Then, we get

$$\begin{array}{lll} \lambda_1 + \lambda_2 = -2R_{1212}, & \lambda_1 + \lambda_3 = -2R_{1313}, & \lambda_1 + \lambda_4 = -2R_{1414}, \\ \lambda_2 + \lambda_3 = 2R_{1414}, & \lambda_2 + \lambda_4 = 2R_{1313}, & \lambda_3 + \lambda_4 = 2R_{1212}, \end{array}$$

and hence, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ (i.e., $\tau = 0$).

Thus, from the above arguments in Cases (i)∼(viii), we see that the cases (1)∼(4) in Proposition 3.1 do not occur. □

Remark 3.2. In the proof of Proposition 3.1, we may note that Cases (ii) to (iv) (also for Cases (v) to (vii), respectively) are all essentially equivalent.

The following examples illustrate Proposition 3.1. Then, from the examples we can easily check that M is not a weakly Einstein manifold.

Example 3.3. Let M be a Riemannian product manifold of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c_1 and c_2 satisfying $c_1^2 \neq c_2^2$. Then this implies that M is not a weakly Einstein manifold.

Example 3.4. Let $M = (M, g)$ be a Riemannian product manifold of a 3-dimensional space of constant sectional curvature c ($\neq 0$) and a real line $\mathbb{R}$. From Proposition 3.1, we see that M is not a weakly Einstein manifold.

Remark 3.5. Based on Proposition 3.1 and the related Examples 3.3 and 3.4, it may be seen that the statement "for any 4-dimensional Riemannian manifold one always gets (1.3)" ([1: pp. 165]), is incorrect.

The following examples show that a weakly Einstein manifold is not necessarily Einstein.

Example 3.6. ([5]) Let M be a Riemannian product manifold of 2-dimensional Riemannian manifolds $M_1(c)$ and $M_2(-c)$ of constant Gaussian curvatures c and $-c$ ($c \neq 0$), respectively. Then we can easily check that M is not Einstein. We can also easily check that M satisfies (1.3), thus M is weakly Einstein. Further, M belongs to Cases (ii), (vi), (vii) and (viii).

Example 3.7. Let $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$ be a 4-dimensional real Lie algebra equipped with the following Lie bracket operation:

$$\begin{aligned} &[e_1, e_2] = ae_2, \qquad &&[e_1, e_3] = -ae_3 - be_4, \qquad &&[e_1, e_4] = be_3 - ae_4, \\ &[e_2, e_3] = 0, &&[e_2, e_4] = 0, &&[e_3, e_4] = 0, \end{aligned} \tag{3.12}$$

where a ($\neq 0$), b are constant. We define an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ by $\langle e_i, e_j\rangle = \delta_{ij}$. Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$ of G and g the G-invariant Riemannian metric on G determined by $\langle\cdot,\cdot\rangle$. From (3.12),

$$\Gamma_{134} = -b, \quad \Gamma_{212} = -a, \quad \Gamma_{313} = a, \quad \Gamma_{414} = a, \tag{3.13}$$

and otherwise being zero up to sign. From (2.6) and (3.13), by direct calculations, we have

$$\begin{aligned} &R_{1212} = a^2, \qquad &&R_{1313} = a^2, \qquad &&R_{1414} = a^2, \\ &R_{2323} = -a^2, &&R_{2424} = -a^2, &&R_{3434} = a^2, \end{aligned} \tag{3.14}$$

and otherwise being zero up to sign. From this, we can easily check that G is not Einstein since the Ricci curvature components satisfy $\rho_{11} = -3a^2$ but $\rho_{22} = a^2$. We also can easily check that G satisfies (1.3), thus G is weakly Einstein. Then, we see that (G, g) belongs to Case (v).

Remark 3.8. Jensen [8] proved that a 4-dimensional homogeneous Einstein manifold is locally symmetric. We may easily check that Example 3.7 is homogeneous but not locally symmetric. Thus, Example 3.7 shows that Jensen's result does not necessarily hold for weakly Einstein manifolds in general.

In the remainder of this section, we shall give a proof of Theorem B.

Proof of Theorem B.

Necessity: From Proposition 3.1, it suffices to prove that there exists an orthonormal Ricci eigenbasis $\{e_i\}$ of T_pM at each point $p \in M$ which satisfies (2.4). Let $M = (M, g)$ be a weakly Einstein manifold. We take any orthonormal Ricci eigenbasis $\{e_i\}$ and fix it. We shall change this basis to another orthonormal eigenbasis satisfying (2.4). From (1.3) and (2.1), we have the following equality

$$2\sum_{a} \rho_{ai}\rho_{aj} + 2\sum_{a,b} \rho_{ab}R_{iabj} - \tau\rho_{ij} - |\rho|^2\delta_{ij} + \frac{\tau^2}{4}\delta_{ij} = 0. \tag{3.15}$$

Setting $i = j = 1$ in (3.15), we get

$$2\lambda_1^2 + 2\sum_i \lambda_i R_{1ii1} - \Big(\sum_i \lambda_i\Big)\lambda_1 - \sum_i \lambda_i^2 + \frac{1}{4}\Big(\sum_i \lambda_i\Big)^2 = 0. \tag{3.16}$$

Similarly, we get

$$\begin{aligned} 2\lambda_2^2 + 2\sum_i \lambda_i R_{2ii2} - \Big(\sum_i \lambda_i\Big)\lambda_2 - \sum_i \lambda_i^2 + \frac{1}{4}\Big(\sum_i \lambda_i\Big)^2 &= 0,\\ 2\lambda_3^2 + 2\sum_i \lambda_i R_{3ii3} - \Big(\sum_i \lambda_i\Big)\lambda_3 - \sum_i \lambda_i^2 + \frac{1}{4}\Big(\sum_i \lambda_i\Big)^2 &= 0,\\ 2\lambda_4^2 + 2\sum_i \lambda_i R_{4ii4} - \Big(\sum_i \lambda_i\Big)\lambda_4 - \sum_i \lambda_i^2 + \frac{1}{4}\Big(\sum_i \lambda_i\Big)^2 &= 0. \end{aligned} \tag{3.17}$$

Further, setting $i = 1$, $j = 2$ in (3.15), we get the following

$$(\lambda_3 - \lambda_4)R_{1323} = 0. \tag{3.18}$$

Similarly, we get

$$\begin{aligned} &(\lambda_2 - \lambda_4)R_{1223} = 0, \qquad (\lambda_2 - \lambda_3)R_{1224} = 0, \qquad (\lambda_1 - \lambda_4)R_{1213} = 0,\\ &(\lambda_1 - \lambda_3)R_{1214} = 0, \qquad (\lambda_1 - \lambda_2)R_{1314} = 0. \end{aligned} \tag{3.19}$$

Then, the following cases are to be considered:

Case I:	$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.		
Case II-1:	$\lambda_1 = \lambda_2 (\equiv \lambda)$,	$\lambda_3 \neq \lambda_4$,	$(\lambda_3, \lambda_4 \neq \lambda)$.
Case II-2:	$\lambda_1 = \lambda_3 (\equiv \lambda)$,	$\lambda_2 \neq \lambda_4$,	$(\lambda_2, \lambda_4 \neq \lambda)$.
Case II-3:	$\lambda_1 = \lambda_4 (\equiv \lambda)$,	$\lambda_2 \neq \lambda_3$,	$(\lambda_2, \lambda_3 \neq \lambda)$.
Case II-4:	$\lambda_2 = \lambda_3 (\equiv \lambda)$,	$\lambda_1 \neq \lambda_4$,	$(\lambda_1, \lambda_4 \neq \lambda)$.
Case II-5:	$\lambda_2 = \lambda_4 (\equiv \lambda)$,	$\lambda_1 \neq \lambda_3$,	$(\lambda_1, \lambda_3 \neq \lambda)$.
Case II-6:	$\lambda_3 = \lambda_4 (\equiv \lambda)$,	$\lambda_1 \neq \lambda_2$,	$(\lambda_1, \lambda_2 \neq \lambda)$.
Case III-1:	$\lambda_1 = \lambda_2 (\equiv \lambda)$,	$\lambda_3 = \lambda_4 (\equiv \mu)$,	$(\lambda \neq \mu)$.
Case III-2:	$\lambda_1 = \lambda_3 (\equiv \lambda)$,	$\lambda_2 = \lambda_4 (\equiv \mu)$,	$(\lambda \neq \mu)$.
Case III-3:	$\lambda_1 = \lambda_4 (\equiv \lambda)$,	$\lambda_2 = \lambda_3 (\equiv \mu)$,	$(\lambda \neq \mu)$.
Case IV-1:	$\lambda_1 = \lambda_2 = \lambda_3 (\equiv \lambda)$,	$\lambda_4 \neq \lambda$.	
Case IV-2:	$\lambda_1 = \lambda_2 = \lambda_4 (\equiv \lambda)$,	$\lambda_3 \neq \lambda$.	
Case IV-3:	$\lambda_1 = \lambda_3 = \lambda_4 (\equiv \lambda)$,	$\lambda_2 \neq \lambda$.	
Case IV-4:	$\lambda_2 = \lambda_3 = \lambda_4 (\equiv \lambda)$,	$\lambda_1 \neq \lambda$.	
Case V:	$\lambda_i \neq \lambda_j, \quad (i \neq j)$.		

Case I. The existence of a generalized Singer-Thorpe basis follows immediately from the construction of a Singer-Thorpe basis.

Cases II. We shall consider first the *Case II-1*. Then, it suffices to consider Cases (v) and (viii). First, we deal with Case (v). From (3.18) and (3.19), taking account of the equalities in Case (v), we have

$$\begin{aligned} &R_{1323} = 0, \qquad R_{1223} = 0, \qquad R_{1224} = 0, \\ &R_{1213} = 0, \qquad R_{1214} = 0, \qquad R_{1313} = R_{2323}. \end{aligned} \tag{3.20}$$

Here, we note that all of the relations in (3.20) and Case (v) are preserved under the changes of the orthonormal basis satisfying the conditions of Case II-1. We denote the 2-dimensional subspace of T_pM spanned $\{e_1, e_2\}$ by V. For any non-zero vector $x \in V$, we denote by $x^\perp$ the vector in V such that $|x^\perp| = |x|$, $g(x, x^\perp) = 0$, and the ordered pair $\{x, x^\perp\}$ and $\{e_1, e_2\}$ determine the same orientation on V. We define a unit vector $e \in V$ by

$$R(e, e_3, e^\perp, e_4) = \max_{x \in V, \, |x|=1} R(x, e_3, x^\perp, e_4). \tag{3.21}$$

We set $e'_1 = e$, $e'_2 = e^\perp$, $e'_3 = e_3$, $e'_4 = e_4$ and define a function $\phi(t)$ by

$$\phi(t) = R\big(\cos t e'_1 + \sin t e'_2, e'_3, -\sin t e'_1 + \cos t e'_2, e'_4\big). \tag{3.22}$$

Then, from (3.21) and (3.22), we have $\phi'(0) = 0$, and hence,

$$0 = -R'_{1314} + R'_{2324} = -2R'_{1314} \qquad (\text{and hence,} \quad R'_{2324} = 0), \tag{3.23}$$

where $R'_{ijkl} = R(e'_i, e'_j, e'_k, e'_l)$, $1 \leqq i, j, k, l \leqq 4$. Then together with (3.23), the respective equalities in (3.20) and Case (v) corresponding to the orthonormal basis $\{e'_i\}$, we see that the orthonormal basis $\{e'_i\}$ is a generalized Singer-Thorpe basis. Similarly, we may also choose a generalized Singer-Thorpe basis for Case (viii). The existence of the generalized Singer-Thorpe basis in the Cases II-2, II-3, II-4, II-5, and II-6 can be now proved analogously.

Cases III. We shall consider first the *Case III-1.* Then it suffices to consider Cases (ii), (vi), (vii), (viii). First, we shall consider Case (ii). Then, from (3.18) and (3.19), we have

$$R_{1223} = 0, \qquad R_{1224} = 0, \qquad R_{1213} = 0, \qquad R_{1214} = 0. \tag{3.24}$$

Here, we may note that each of the relations in (3.24) and Case (ii) is preserved under the changes of the orthonormal basis satisfying the conditions of Case III-1. Let V be a 2-dimensional subspace of T_pM spanned by $\{e_1, e_2\}$ and $V^\perp$ be the orthogonal complement of V in T_pM. Then $V^\perp$ is spanned by $\{e_3, e_4\}$. We define $e_1' \in V$ and $e_3' \in V^\perp$ by

$$R(e_1', e_3', e_1', e_3') = \max_{\substack{x\in V,\, y\in V^\perp \\ |x|=|y|=1}} R(x, y, x, y). \tag{3.25}$$

Further, we choose unit vectors $e_2' \in V$ and $e_4' \in V^\perp$ in such a way that $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ ($\{e_3, e_4\}$ and $\{e_3', e_4'\}$) define the same orientation on V (on $V^\perp$, respectively). We define the function $\phi(t)$ by

$$\phi(t) = R(e_1', \cos t e_3' + \sin t e_4', e_1', \cos t e_3' + \sin t e_4').$$

Then, we have $\phi'(0) = 0$, and hence

$$R'_{1314} = 0. \tag{3.26}$$

Similarly, considering the function $\psi(t)$ defined by

$$\psi(t) = R(\cos t e_1' + \sin t e_2', e_3', \cos t e_1' + \sin t e_2', e_3'),$$

we have $\psi'(0) = 0$, and hence,

$$R'_{1323} = 0. \tag{3.27}$$

Then, from (3.24), (3.26) and (3.27), we see that the orthonormal basis $\{e_i'\}$ is a generalized Singer-Thorpe basis. Similarly to Case (ii), we may choose a generalized Singer-Thorpe basis for Cases (vi), (vii), (viii). The existence of the generalized Singer-Thorpe basis in the Cases III-2 and III-3 can be now proved analogously.

Cases IV. We shall consider first the *Case IV-1.* Then, it suffices to consider Case (viii) with $\lambda \neq 0$. Then from (3.18) and (3.19), we have

$$R_{1223} = 0, \qquad R_{1213} = 0, \qquad R_{1323} = 0. \tag{3.28}$$

Further, from Case (viii), we have

$$\begin{aligned} R_{1212} = R_{1313} = R_{2323} = -\lambda, \\ R_{1414} = R_{2424} = R_{3434} = \lambda. \end{aligned} \tag{3.29}$$

Here, we note that each of the relations in (3.28) and (3.29) is preserved under the changes of the orthonormal basis satisfying the conditions of Case IV-1. Let V be a 3-dimensional subspace of T_pM spanned by $\{e_1, e_2, e_3\}$ satisfying that V is orthogonal complement of $\{e_4\}$. We define

$$R(e_1', e_2', e_2', e_4') = \max_{\substack{x,\, y \in V,\, x \perp y \\ |x|=|y|=1}} R(x, y, y, e_4), \tag{3.30}$$

where $e_3' \in V$ such that $e_3' \perp e_1'$, $e_3' \perp e_2'$, $|e_3'| = 1$, $e_4' = e_4$. First, we define the function $\phi(t)$ by

$$\phi(t) = R(e_1', \cos t e_2' + \sin t e_3', \cos t e_2' + \sin t e_3', e_4'). \tag{3.31}$$

Then, by the hypothesis (3.30), we have $\phi'(0) = 0$, and hence,

$$R'_{1234} + R'_{1324} = 0. \tag{3.32}$$

Next, we consider the function $\psi(t)$ defined by

$$\psi(t) = R(\cos t e_1' + \sin t e_3', e_2', e_2', e_4'). \tag{3.33}$$

Then we have $0 = \psi'(0) = R'_{3224}$, and hence,

$$R'_{1314} = 0. \tag{3.34}$$

Next, we consider the function $\zeta(t)$ defined by

$$\zeta(t) = R(\cos t e_1' + \sin t e_2', -\sin t e_1' + \cos t e_2', -\sin t e_1' + \cos t e_2', e_4'). \tag{3.35}$$

Then, by the hypothesis we have also $0 = \zeta'(0) = -R'_{1214}$, and hence

$$R'_{1214} = 0. \tag{3.36}$$

Now, we set

$$\begin{aligned} e_2'' &= \frac{1}{\sqrt{2}} e_2' + \frac{1}{\sqrt{2}} e_3', \\ e_3'' &= -\frac{1}{\sqrt{2}} e_2' + \frac{1}{\sqrt{2}} e_3', \\ e_1'' &= e_1', \quad e_4'' = e_4'. \end{aligned} \tag{3.37}$$

Then, we have

$$\begin{aligned} R(e_1'', e_2'', e_2'', e_4'') &= \frac{1}{2} R(e_1', e_2' + e_3', e_2' + e_3', e_4') \\ &= \frac{1}{2}\{R'_{1224} + R'_{1324} + R'_{1234} + R'_{1334}\} = 0 \end{aligned}$$

by virtue of (3.32), and hence,

$$R''_{1224} = 0. \tag{3.38}$$

Here, we set $R''_{ijkl} = R(e''_i, e''_j, e''_k, e''_l)$, $1 \leqq i, j, k, l \leqq 4$. Similarly, from (3.37), we have

$$\begin{aligned} R''_{1214} &= \frac{1}{\sqrt{2}} R(e'_1, e'_2 + e'_3, e'_1, e'_4) = \frac{1}{\sqrt{2}}(R'_{1214} + R'_{1314}) = 0, \\ R''_{1314} &= \frac{1}{\sqrt{2}} R(e'_1, -e'_2 + e'_3, e'_1, e'_4) = \frac{1}{\sqrt{2}}(-R'_{1214} + R'_{1314}) = 0 \end{aligned} \tag{3.39}$$

by virtue of (3.34) and (3.36). Thus, from (3.38) and (3.39), we see that the orthonormal basis $\{e''_i\}$ is a generalized Singer-Thorpe basis. The existence of the generalized Singer-Thorpe basis in Cases IV-2, IV-3, and IV-4 can be now proved analogously.

Case V. Here, according to (3.18) and (3.19), we may immediately choose a generalized Singer-Thorpe basis.

Sufficiency: We assume that $M = (M, g)$ admits a generalized Singer-Thorpe basis $\{e_i\}$. From the condition (2.4), we see that (3.18) and (3.19) hold on M. Further, by substituting $\lambda_i = \sum_k R_{ikki}$ $(1 \leqq i \leqq 4)$ to the left hand sides of (3.16) and (3.17), and taking account of (3.1), we see also that each equation in (3.16) and (3.17) holds. Therefore we see that M satisfies the curvature condition (3.15). Thus M is a weakly Einstein manifold by virtue of (2.1). This completes the proof of Theorem B. □

4. Proof of Theorem C

In this section, we shall prove first a topological inequality (Proposition 4.1, (4.17)) from which Theorem C will follow immediately.

Let $M = (M, g)$ be a compact oriented weakly Einstein manifold. Then, from Theorem B, we may choose an generalized Singer-Thorpe basis $\{e_i\}$ of T_pM at any point $p \in M$ compatible with the orientation of M. We set

$$\begin{aligned} &\alpha'_1 = R_{1212}, \quad &\alpha'_2 = R_{1313}, \quad &\alpha'_3 = R_{1414}, \\ &\alpha''_1 = R_{3434}, \quad &\alpha''_2 = R_{2424}, \quad &\alpha''_3 = R_{2323}, \\ &\beta_1 = R_{1234}, \quad &\beta_2 = R_{1342}, \quad &\beta_3 = R_{1423}. \end{aligned} \tag{4.1}$$

Then, from (4.1), by the first Bianchi identity,

$$\beta_1 + \beta_2 + \beta_3 = 0. \tag{4.2}$$

Further, we set $\mathbf{a}' = (\alpha'_1, \alpha'_2, \alpha'_3)$, $\mathbf{a}'' = (\alpha''_1, \alpha''_2, \alpha''_3)$ and $\mathbf{b} = (\beta_1, \beta_2, \beta_3)$ and denote the canonical inner product by $\langle \cdot, \cdot \rangle$ on the 3-dimensional Euclidean space $\mathbb{R}^3$. We set $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for any $\mathbf{x} \in \mathbb{R}^3$. Then we may note that $|\mathbf{a}'| = |\mathbf{a}''|$ by virtue of (2.10). Now, we denote the Euler number and the first Pontrjagin number of M by $\chi(M)$ and $p_1(M)$, respectively. Then, from (4.1), applying the similar arguments as in [7], we have the following equalities:

$$\chi(M) = \frac{1}{4\pi^2}\int_M \big\{\langle \mathbf{a}', \mathbf{a}''\rangle + |\mathbf{b}|^2\big\}\, \mathrm{d}v_g \tag{4.3}$$

and

$$p_1(M) = \frac{1}{2\pi^2}\int_M \langle \mathbf{a}' + \mathbf{a}'', \mathbf{b}\rangle\, \mathrm{d}v_g, \tag{4.4}$$

where $\mathrm{d}v_g$ is the volume element of M. Now, we set

$$\mathbf{a} = \frac{1}{2}(\mathbf{a}' + \mathbf{a}''). \tag{4.5}$$

Then, by (4.5), the equalities (4.3) and (4.4) are rewritten respectively by

$$\chi(M) = \frac{1}{4\pi^2}\int_M \big\{2|\mathbf{a}|^2 - |\mathbf{a}'|^2 + |\mathbf{b}|^2\big\}\, \mathrm{d}v_g, \tag{4.6}$$

$$p_1(M) = \frac{1}{2\pi^2}\int_M 2\langle \mathbf{a}, \mathbf{b}\rangle\, \mathrm{d}v_g. \tag{4.7}$$

Then, from (4.6) and (4.7), we have the following:

$$\begin{aligned} 2\chi(M) \pm p_1(M) &= \frac{1}{2\pi^2}\int_M \big\{|\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2\langle \mathbf{a}, \mathbf{b}\rangle + |\mathbf{a}|^2 - |\mathbf{a}'|^2\big\}\, \mathrm{d}v_g \\ &= \frac{1}{2\pi^2}\int_M \big\{|\mathbf{a} \pm \mathbf{b}|^2 + |\mathbf{a}|^2 - |\mathbf{a}'|^2\big\}\, \mathrm{d}v_g \\ &\geqq \frac{1}{2\pi^2}\int_M \big\{|\mathbf{a}|^2 - |\mathbf{a}'|^2\big\}\, \mathrm{d}v_g. \end{aligned} \tag{4.8}$$

We set $f = |\mathbf{a}|^2 - |\mathbf{a}'|^2$, Then, from the definition of the vectors $\mathbf{a}'$, $\mathbf{a}''$ and $\mathbf{a}$, taking account of the proof of Proposition 3.1, we have

$$f = 0 \qquad \text{for Case (i)}, \tag{4.9}$$

$$f = -\frac{1}{4}(\lambda_1 - \lambda_3)^2 \quad (\lambda_1 = \lambda_2,\ \ \lambda_3 = \lambda_4) \qquad \text{for Case (ii)}, \tag{4.10}$$

$$f = -\frac{1}{4}(\lambda_1 - \lambda_2)^2 \quad (\lambda_1 = \lambda_3,\ \ \lambda_2 = \lambda_4) \qquad \text{for Case (iii)}, \tag{4.11}$$

$$f = -\frac{1}{4}(\lambda_1 - \lambda_3)^2 \quad (\lambda_1 = \lambda_4,\ \ \lambda_2 = \lambda_3) \qquad \text{for Case (iv)}, \tag{4.12}$$

$$f = -\frac{1}{4}\Big\{(\lambda_1 - \lambda_3)^2 + (\lambda_1 - \lambda_4)^2\Big\} \quad (\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4) \qquad \text{for Case (v)}, \tag{4.13}$$

$$f = -\frac{1}{4}\Big\{(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_4)^2\Big\} \quad \text{for Case (vi)}, \quad (\lambda_1+\lambda_3=\lambda_2+\lambda_4) \tag{4.14}$$

$$f = -\frac{1}{4}\Big\{(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_3)^2\Big\} \quad \text{for Case (vii)}, \quad (\lambda_1+\lambda_4=\lambda_2+\lambda_3) \tag{4.15}$$

$$f = -\frac{1}{4}\Big\{(\lambda_1+\lambda_2)^2+(\lambda_1+\lambda_3)^2+(\lambda_1+\lambda_4)^2\Big\} \quad \text{for Case (viii)} \quad (\lambda_1+\lambda_2+\lambda_3+\lambda_4=0) \tag{4.16}$$

at $p \in M$. Then from (4.10)∼(4.16), we see that f gives rise a continuous function on M and further, $f = 0$ holds at p if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ holds at p (namely, M is Einstein at p). Therefore, summing up the above arguments, we have the following:

Proposition 4.1. *Let $M = (M, g)$ be a compact weakly Einstein manifold. Then, the following inequality holds on M:*

$$2\chi(M) \pm p_1(M) \geqq C, \tag{4.17}$$

where $C = \frac{1}{2\pi^2}\int_M \big\{|\mathbf{a}|^2 - |\mathbf{a}'|^2\big\}\,dv_g \leqq 0$.

Remark 4.2. Since $p_1(M) = 3\sigma(M)$ ($\sigma(M)$ is the Hirzebruch signature of M), from Proposition 4.1 together with the proof, we see that the inequality (4.17) reduces to the Hitchin inequality [7]

$$2\chi(M) \geqq 3|\sigma(M)|, \tag{4.18}$$

for the case where M is Einstein. Thus, the inequality (4.17) in Proposition 4.1 is regarded as the generalization of the Hitchin inequality (4.18).

Proof of Theorem C. In Proposition 4.1 we check easily that the constant C satisfies the inequality $C \geqq -\frac{1}{16\pi^2}\int_M |R|^2\,dv_g$. Hence our Theorem follows immediately. □

The following example illustrates Proposition 4.1, Theorem C and Remark 4.2.

Example 4.3. Let M_1 and M_2 be a unit 2-sphere and a compact oriented surface of genus m ($m \geqq 2$) with constant Gaussian curvature -1, respectively, and further, M be the Riemannian product of M_1 and M_2, $M = M_1 \times M_2$. Then, we may easily check that M is a compact, oriented weakly Einstein manifold which is a special case of Example 3.6. Then, by taking account of the Künneth formula, the Gauss-Bonnet formula and the formulas in [7], we have

$$\begin{aligned} &\chi(M) = 4(1-m), \quad p_1(M) = 0 \quad (\text{thus, } \sigma(M) = 0), \\ &\mathrm{Vol}(M) = 16(m-1)\pi^2, \quad C = 8(1-m). \end{aligned} \tag{4.19}$$

Therefore, from (4.19), we see that the equality signs of the inequalities (4.17) and (1.4) in Proposition 4.1 and Theorem C hold for M respectively, but M does not satisfy the Hitchin inequality (4.18).

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Received 3. 11. 2010
Accepted 23. 5. 2011

* *Department of Mathematics*
Sungkyunkwan University
Suwon 440-746
KOREA
E-mail: prettyfish@skku.edu

c *Correspondence author:*
E-mail: parkj@skku.edu

** *Department of Mathematics*
Faculty of Science
Niigata University
Niigata 950-2181
JAPAN
E-mail: sekigawa@math.sc.niigata-u.ac.jp