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ON LIPSCHITZ BEHAVIOUR OF SOME GENERALIZED DERIVATIVES

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ABSTRACT. The aim of the present paper is to compare various forms of stable properties of nonsmooth functions at some points. By stable property we mean the Lipschitz property of some generalized derivatives related only to the reference point. Namely we compare Lipschitz behaviour of lower Clarke derivative, lower Dini derivative and calmness of Clarke subdifferential. In this way, we continue our study of ℓ-stable functions.

1. Introduction and preliminaries

We will start with the definition of the calmness property of set-valued mappings. From now on, the symbol $\rightsquigarrow$ will be used for set-valued mappings.

Definition 1. A set-valued mapping $F\colon \mathbb{R}^N \rightsquigarrow \mathbb{R}^M$ is calm at $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$F(y) \subset F(x) + K\|y - x\| B_{\mathbb{R}^M}, \qquad \text{for all} \quad y \in U.$$

The symbol $B_{\mathbb{R}^M}$ denotes the unit ball $\{h \in \mathbb{R}^M : \ \|h\| \le 1\}$. In the paper, we will use also the symbol $S_{\mathbb{R}^M}$ for the unit sphere, i.e. the set $\{h \in \mathbb{R}^M : \|h\| = 1\}$. If $A \subset \mathbb{R}^M$, $B \subset \mathbb{R}^M$, then $A + B = \{a + b : \ a \in A, \ b \in B\}$.

Remark 1. We notice that a set-valued mapping $F\colon \mathbb{R}^N \rightsquigarrow \mathbb{R}^M$ is calm at $x \in \mathbb{R}^N$ if and only if there exist a neighbourhood U of x and $K > 0$ such that

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for every $y \in U$ and $v \in F(y)$ it holds that

$$\inf_{u \in F(x)} \|v - u\| \leq K\|y - x\|.$$

The concept of calmness property was studied e.g. in [R] and [RW]. In this paper we will deal with the calmness property of the Clarke generalized gradient. The following definitions can be found e.g. in [C] and [CLSW].

Definition 2. Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^N$, and let $h \in \mathbb{R}^N$. The Clarke upper and lower generalized directional derivatives of f at x in the direction h are defined, respectively, by

$$f^\circ(x;h) = \limsup_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t},$$
$$f_\circ(x;h) = \liminf_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t},$$

and the Clarke generalized gradient of f at x is defined by

$$\partial_c f(x) = \big\{x^* \in \mathbb{R}^N : \ \forall h \in \mathbb{R}^N \ \ \langle x^*, h\rangle \leq f^\circ(x;h)\big\}.$$

Lemma 1. ([PB]) *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$. Then*

$$f_\circ(x;h) = -f^\circ(x;-h).$$

We will denote the Fréchet derivative of a function $f\colon \mathbb{R}^N \to \mathbb{R}$ at $x \in \mathbb{R}^N$ by $f'(x)$. Recall that if Fréchet differentiable at x function $f\colon \mathbb{R}^N \to \mathbb{R}$ satisfies

$$f'(x)h = \lim_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t}, \qquad \text{for all} \quad h \in S_{\mathbb{R}^N},$$

and this convergence is uniform for $h \in S_{\mathbb{R}^N}$, then f is said to be strictly differentiable at x.

Proposition 1. ([C: Proposition 2.2.4]) *Let a function $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^N$. Then f is strictly differentiable at x if and only if $\partial_c f(x)$ is a singleton.*

Proposition 2. ([PB: Remark 2.1]) *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^N$. Then for every $h \in S_{\mathbb{R}^N}$ we have*

$$f^\circ(x;h) = \max\big\{\langle x^*, h\rangle : \ x^* \in \partial_c f(x)\big\},$$
$$f_\circ(x;h) = \min\big\{\langle x^*, h\rangle : \ x^* \in \partial_c f(x)\big\}.$$

The first formula only expreses that $f^\circ(x;\cdot)$ is supporting Clarke subdifferential and the second one can be obtained by Lemma 1.

The class of $C^{1,1}$ fuctions, i.e. the functions with locally Lipschitz derivative, was intensively studied during last 30 years because, among the others, these functions appear in several problems of applied mathematics including variational inequalities, the penalty function method and the proximal point method, see e.g. [CC, CHN, GGR, GJN, GL, GZ, HSN, JL, K, Q, RW, TR, Y].

It seems to be useful to weaken the $C^{1,1}$ assumption and study so called ℓ-stable at the point functions which were introduced in [BP1].

For a function $f\colon \mathbb{R}^N \to \mathbb{R}$, we define *the Dini lower and upper first-order directional derivative of f at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$*, respectively, by

$$f^{\ell}(x;h) = \liminf_{t \downarrow 0} \frac{f(x+th) - f(x)}{t},$$

and

$$f^{u}(x;h) = \limsup_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}.$$

Definition 3. We say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^{\ell}(y;h) - f^{\ell}(x;h)| \le K\|y - x\|, \qquad \text{for all} \quad y \in U, \quad h \in S_{\mathbb{R}^N}.$$

Analogously, we say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is u-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood V of x and $L > 0$ such that

$$|f^{u}(y;h) - f^{u}(x;h)| \le L\|y - x\|, \qquad \text{for all} \quad y \in V, \quad h \in S_{\mathbb{R}^N}.$$

The properties of ℓ-stable functions were then studied e.g. in [BP2, BP3, BP4, BP5, G, PB, LX].

Proposition 3. ([BP4]) *Let a function $f\colon \mathbb{R}^N \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^N$. Then f is Lipschitz on a neighbourhood of x.*

Proposition 4. ([BP4]) *Let a function $f\colon \mathbb{R}^N \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^N$. Then f is strictly differentiable at x.*

The following proposition is now a consequence of Proposition 3 and [BP2: Corollary 1].

Proposition 5. *A function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$ if and only if f is u-stable at x.*

Now, it is natural to introduce the following stable properties.

Definition 4. We say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓc-stable at $x \in \mathbb{R}^N$ if there exists a neighbourhood U of x and $K > 0$ such that

$$|f_{\circ}(y;h) - f_{\circ}(x;h)| \le K\|y - x\|, \qquad \text{for all} \quad y \in U, \quad h \in S_{\mathbb{R}^N}.$$

Analogously, we say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is uc-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood V of x and $L > 0$ such that

$$|f^{\circ}(y;h) - f^{\circ}(x;h)| \le L\|y - x\|, \qquad \text{for all} \quad y \in V, \quad h \in S_{\mathbb{R}^N}.$$

We will compare the notions of ℓc-stability and uc-stability. From Lemma 1 it follows immediately.

Proposition 6. *A function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓc-stable at $x \in \mathbb{R}^N$ if and only if f is uc-stable at x.*

Remark 2. We notice that the conditions of stability in Definition 4 and Proposition 6 imply that e.g. ℓc-stable at some point function is Lipschitz near this point.

Summarizing the previous notions of stable properties and having in mind Propositions 5 and 6, it is interesting to deal with the following.

Problem 1. What is a relation among ℓ-stability, ℓc-stability and the calmness property of the Clarke generalized gradient of locally Lipschitz function at some point?

Solving the previous problem, we will use a particular case of more generalize results given in [BF] and [BMW].

Proposition 7. *Let α and β be real-valued continuous functions defined on an open interval (a, b). Then there exists a real-valued locally Lipschitz function f defined on (α, β) such that*

$$\partial_c f(y) = [\alpha(y), \beta(y)], \qquad \text{for all} \quad y \in (a, b).$$

2. ℓ-stability

At first, we will show an example of such function which is ℓc-stable at some point but not ℓ-stable at this point.

Example 1. We set $\alpha(y) = 0$ and $\beta(y) = 1$ for every $y \in (-1, 1)$. By Proposition 7, there exists a locally Lipschitz function $g\colon (-1, 1) \to \mathbb{R}$ such that

$$\partial_c g(y) = [0, 1], \qquad \text{for every} \quad y \in (-1, 1).$$

Using Proposition 2, we obtain that

$$g_\circ(y; 1) = 0, \qquad \text{for every} \quad y \in (-1, 1),$$

and

$$g_\circ(y; -1) = -1, \qquad \text{for every} \quad y \in (-1, 1).$$

Thus, the function g is ℓc-stable at 0.

On the other hand, since $\partial_c g(0) = [0, 1]$ is not a singleton, Propositions 1 and 4 imply that the function g is not ℓ-stable at 0. ♣

For the proof of main result of this section (Theorem 1), we will use the following corollary.

Lemma 2. ([CLSW: page 98]) *Let a function $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $y \in \mathbb{R}^N$. Then*

$$f^{\circ}(y;h) = \limsup_{z\to x} f^{\ell}(y;h).$$

Using liminf and limsup calculus, the previous lemma immediately imply.

Corollary 1. *Let a function $f\colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $y \in \mathbb{R}^N$. Then*

$$f_{\circ}(y;h) = \liminf_{z\to x} f^{u}(y;h).$$

Theorem 1. *If a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$, then it is ℓc-stable at x.*

Proof. Using Propositions 4 and 5, we can suppose that there exists a neighbourhood U of x and $K > 0$ such that

$$|f^{u}(y;h) - f'(x)h| \le K\|y-x\|, \qquad \text{for all} \quad y \in U, \;\; h \in S_{\mathbb{R}^N}. \tag{1}$$

Now, we consider an arbitratry $y \in U$ and $h \in S_{\mathbb{R}^N}$. Due to Corollary 1, we can find a sequence $\{z_n\}_{n=1}^{+\infty}$, $\lim\limits_{n\to+\infty} z_n = y$, satisfying

$$f_{\circ}(y;h) = \lim_{n\to+\infty} f^{u}(z_n;h).$$

Then, by formula (1), it holds that

$$\begin{aligned} |f_{\circ}(y;h) - f'(x)h| &= \lim_{n\to+\infty} |f^{u}(z_n;h) - f'(x)h| \\ &\le \lim_{n\to+\infty} K\|z_n - x\| = K\|y-x\|. \end{aligned}$$

Hence, the function f is ℓc-stable at x. □

Remark 3. Note that it is easy to see that actually we can reverse the previous theorem if we add an extra assumption that f is strictly differentiable. For instance, if $f_{\circ}(x,h) = 0$ for any h,then f must be strictly differentiable at x due to Lemma 1.

3. ℓc-stability and calmness property

Starting the comparison between the class of ℓc-stable at some point functions and the class of functions having the calmness property of their Clarke's generalized gradient at some point, we show an example of such function for which the Clarke generalized gradient is calm at some point but not ℓc-stable at this point.

Example 2. We consider the function $\alpha\colon(-1,1)\to\mathbb{R}$, $\alpha(y)=\sqrt{|y|}$ for every $y\in(-1,1)$, and the function $\beta\colon(-1,1)\to\mathbb{R}$, $\beta(y)=1$ for every $y\in(-1,1)$. By Proposition 7 there exists a locally Lipschitz function $g\colon(-1,1)\to\mathbb{R}$ such that

$$\partial_c g(y)=[\sqrt{|y|},1],\qquad \text{for all}\quad y\in(-1,1). \tag{2}$$

Using Proposition 2, we obtain that

$$g_\circ(y;1)=\sqrt{y},\qquad \text{for every}\quad y\in[0,1),$$

specially $g_\circ(0;1)=0$. Since

$$\lim_{y\downarrow 0}\frac{\sqrt{y}}{y}=+\infty,$$

the function g is not ℓc-stable at 0.

On the other hand, setting $K=1$ in Definition 1 and using formula (2), we have that the Clarke generalized gradient of function g is calm at 0. ♣

Theorem 2. *Let $f\colon\mathbb{R}^N\to\mathbb{R}$ be lc-stable at $x\in\mathbb{R}^N$. Then the set-valued mapping $\partial_c f\colon\mathbb{R}^N\rightsquigarrow\mathbb{R}^N$ is calm at x.*

Proof. Since f is lc-stable at x, by Proposition 6 f is also uc-stable at x and there exist a neighborhood U of x and $K>0$ such that

$$|f^\circ(y;h)-f^\circ(x;h)|\le K\|y-x\|,\qquad \text{for all}\quad y\in U,\ \ h\in S_{\mathbb{R}^N}. \tag{3}$$

Due to Remark 1, it suffices to show that for every $y\in U$ and for every $y^*\in\partial_c f(y)$, it holds that

$$\inf_{x^*\in\partial_c f(x)}\max_{h\in S_{\mathbb{R}^N}}\langle y^*-x^*,h\rangle\le K\|y-x\|. \tag{4}$$

We fix $y\in U$ and $y^*\in\partial_c f(y)$. Let $\lambda\in\mathbb{R}$ satisfy the inequality

$$\lambda<\inf_{x^*\in\partial_c f(x)}\max_{h\in S_{\mathbb{R}^N}}\langle y^*-x^*,h\rangle. \tag{5}$$

We show that $\lambda\le K\|y-x\|$. So, we consider an arbitrary $x_0^*\in\partial_c f(x)$. Using formula (5), we can find $h_1\in S_{\mathbb{R}^N}$ such that

$$\lambda<\langle y^*-x_0^*,h_1\rangle.$$

We take $x_1^*\in\partial_c f(x)$ such that $\langle x_1^*,h_1\rangle=f^\circ(x;h_1)$. Using formula (5) again, there exist $h_2\in S_{\mathbb{R}^N}$ such that

$$\lambda<\langle y^*-x_1^*,h_2\rangle.$$

Repeating the previous consideration, we can obtain, without any loss of generality, sequences $\{x_k^*\}_{k=1}^{+\infty}\subset\partial_c f(x)$ and $\{h_k\}_{k=1}^{+\infty}\subset S_{\mathbb{R}^N}$ satisfying

$$\langle x_k^*,h_k\rangle=f^\circ(x;h_k),\qquad \text{for all}\quad k\in\mathbb{N}, \tag{6}$$

and

$$\lambda<\langle y^*-x_k^*,h_{k+1}\rangle,\qquad \text{for all}\quad k\in\mathbb{N}. \tag{7}$$

Because of the compactness of $S_{\mathbb{R}^N}$ and $\partial_c f(x)$, we can find subsequences $\{h_l\}_{l=1}^{+\infty} \subset \{h_k\}_{k=1}^{+\infty}$, $\{x_l\}_{l=1}^{+\infty} \subset \{x_k\}_{k=1}^{+\infty}$ and $h \in S_{\mathbb{R}^N}$, $x^* \in \partial_c f(x)$ such that

$$\lim_{l \to +\infty} h_l = h \quad \text{and} \quad \lim_{l \to +\infty} x_l^* = x^*.$$

Using formulas (6), (7), and the lipschitzness of mapping $h \mapsto f^\circ(x;h)$, and limiting for $l \to +\infty$, we obtain

$$\langle x^*, h \rangle = f^\circ(x;h),$$

$$\begin{aligned} \lambda &\le \langle y^* - x^*, h \rangle = \langle y^*, h \rangle - f^\circ(x;h) \\ &\le f^\circ(y;h) - f^\circ(x;h) \le K\|y - x\|, \end{aligned} \tag{8}$$

where the last inequality follows from uc-stability at x, see formula (3).

Since λ has been chosen arbitrary, formula (8) implies that inequality (4) is true and the mapping $\partial_c f$ is calm at x. □

We finish our paper with a direct consequence of Theorems 1 and 2.

Corollary 2. *If a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$, then the set-valued mapping $\partial_c f\colon \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is calm at x.*

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