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STRONGLY NONATOMIC DENSITIES DEFINED BY CERTAIN MATRICES

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ABSTRACT. Drewnowski and Paúl proved about ten years ago that for any strongly nonatomic submeasure η on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of all natural numbers the ideal of all null sets of η has the Nikodym property (NP). They stated the problem whether the converse is true in general. By presenting an example, Alon, Drewnowski and Łuczak proved recently that the answer is negative. Nevertheless, it is of mathematical interest to identify classes of submeasures η such that η is strongly nonatomic if and only if the set of all null sets of η has the Nikodym property. In this context, the authors proved some years ago that this equivalence holds, for instance, if one restricts the attention to the case of densities defined by regular Riesz matrices or by nonnegative regular Hausdorff methods. Also sufficient and necessary conditions in terms of the matrix coefficients are given, that the defined density is strongly nonatomic. In this paper we extend these investigations to the class of generalized Riesz matrices, introduced by Drewnowski, Florencio and Paúl in 1994.

1. Preliminaries and introduction

We start with few preliminaries. (Otherwise, the terminology is standard, we refer to Wilansky [10,11] and Boos [3]).

ω denotes the space of all sequences $x = (x_k)$ in $\mathbb{R}$, and any vector subspace of ω is called a *sequence space*. Let χ be the set of all sequences of 0's and 1's and, if E is any sequence space, let $\chi(E)$ denote the linear hull of the sequences

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of 0's and 1's contained in E. A sequence space E is said to have the *Hahn property* (HP), if $\chi(E) \subset F$ implies $E \subset F$ for every FK-space F (cf. [2]).

Let $A = (a_{nk})$ be an infinite matrix with real entries. The *domain* c_A of A is defined as $c_A = \left\{x \in \omega \mid Ax := \left(\sum_k a_{nk}x_k\right)_n \in c\right\}$ where the definition of Ax implies the convergence of the series. Moreover, $c_{0A} = \{x \in \omega \mid Ax \in c_0\}$ is called the *null domain* of A. By definition, A is *regular for null sequences* if $c_0 \subset c_{0A}$, and it is called *regular* if $c \subset c_A$ and $\lim \circ A|_c = \lim$. Note that $A = (a_{nk})$ is regular for null sequences if all columns of A converge to 0 and the row norm $\|A\| := \sup_n \sum_k |a_{nk}|$ is finite. A regular matrix A is called *strongly regular*, if $ac \subset c_A$ where ac denotes the set of all almost convergent sequences.

Let $\mathcal{R}$ be a ring of sets. By a *measure* (respectively *σ-measure*) on $\mathcal{R}$ we mean a finitely (respectively countably) additive scalar valued set function defined on $\mathcal{R}$. A *submeasure on* $\mathcal{R}$ is a nondecreasing, subadditive, nonnegative (in general, $\overline{\mathbb{R}}_+$-valued) set function defined on $\mathcal{R}$ and vanishing on the empty set. We denote by $\mathrm{ba}(\mathcal{R})$ the Banach space of all bounded measures on $\mathcal{R}$ with the supremum-norm. By definition, $\mathcal{R}$ has the *Nikodym Property (NP)* if every pointwise bounded subset M of $\mathrm{ba}(\mathcal{R})$ is uniformly bounded (or norm bounded in $\mathrm{ba}(\mathcal{R})$).

Let $\mathcal{A}$ be a class of subsets of $\mathbb{N}$. A sequence of scalars (x_n) is said to be summable over $\mathcal{A}$ if the subseries $\sum_{n\in E} x_n$ converges for every $E \in \mathcal{A}$. By definition, $\mathcal{A}$ has the *Absolute Summability Property (ASP)* if $(x_n) \in \ell^1$ for any sequence (x_n) of scalars being summable over $\mathcal{A}$.

Given a nonnegative matrix $A = (a_{nk})$, we define for each $E \subset \mathbb{N}$,

$$\tau_n(E) := \sum_{k=1}^{\infty} a_{nk}\chi_E(k) \quad (n \in \mathbb{N}) \qquad \text{and} \qquad d_A(E) := \limsup_{n\to\infty} \tau_n(E).$$

Clearly, each τ_n is a nonnegative (not necessarily finite) σ-measure on $\mathcal{P}(\mathbb{N})$ and d_A is a submeasure on $\mathcal{P}(\mathbb{N})$. We will often refer to d_A as the submeasure or density defined by the matrix A. The elements of the ideal $\mathcal{Z}_A := \{E \subset \mathbb{N} \mid d_A(E) = 0\}$ are called *d_A-null set*. Obviously, if A is regular for null sequences, then $\{k\} \in \mathcal{Z}_A$ $(k \in \mathbb{N})$.

From the view of sequence space theory we consider the sequence space $|A|_0 := \left\{x \in \ell^\infty \mid \sum_k a_{nk}|x_k| \longrightarrow 0 \ (n \to \infty)\right\}$ of all sequences being *strongly A-summable to* 0 and call it *strong null domain* of A. Obviously we have $\chi \cap |A|_0 = \{\chi_E \mid E \in \mathcal{Z}_A\}$ and $\chi(|A|_0) = \mathrm{lin}\{\chi_E \mid E \in \mathcal{Z}_A\}$.

We will make use of the following results:

Proposition 1.1. (cf. [8: Proposition 5.2]) *Let A be a nonnegative matrix being regular for null sequences. If $\mathcal{Z}_A$ has the ASP, then A has spreading rows, that is,* $\lim_n \sup_k a_{nk} = 0$.

Proposition 1.2. (cf. [4: Theorem 1.5]) *Let A be any nonnegative matrix being regular for null sequences. Then the following conditions are equivalent:*

(a) *$\mathcal{Z}_A$ has the NP.*

(b) *$\mathcal{Z}_A$ has the ASP.*

(c) $\chi(|A|_0)^\beta = \ell^1$.

(d) *$|A|_0$ has the HP.*

The problem with Proposition 1.2 is that its hypothesis, even (b) and (c), are difficult to check. Thus, further necessary and sufficient conditions for NP are mathematically interesting.

Proposition 1.3. (cf. [8: Proposition 6.3]) *If η is a strongly nonatomic submeasure on $\mathcal{P}(\mathbb{N})$, then the ideal $\mathcal{Z}(\eta) := \{N \in \mathcal{P}(\mathbb{N}) \mid \eta(N) = 0\}$ has the NP. Thereby, η is called* strongly nonatomic (on $\mathcal{P}(\mathbb{N})$)*, if for all $\varepsilon > 0$ there exists a finite partition $E_1, \dots, E_N$ of $\mathbb{N}$ with*[1] $\eta(E_\nu) \leq \varepsilon$ $(\nu \in \mathbb{N}_N)$.

In connection with the last result, Drewnowski and Paúl stated in [8] the problem whether the converse is true in general. Recently, Alon, Drewnowski and Łuczak have shown in [1: Theorem 2.3] — by presenting a deep example — that the answer is negative. Nevertheless, having this result, it is of mathematical interest to identify classes of submeasures η such that η is strongly nonatomic if and only if $\mathcal{Z}(\eta)$ has the Nikodym property. Before the negative result of Alon et al., Drewnowski, Florencio and Paúl — aiming to a negative answer — considered in [7] generalized Riesz matrices[2] $R_{p,m}$ and gave sufficient conditions in terms of the coefficients of $R_{p,m}$ such that the generated density $d_{R_{p,m}}$ is strongly nonatomic. In this context the authors proved in [4] that this equivalence holds, for instance, if one restricts the attention to the case of densities defined by regular Riesz matrices R_p or by nonnegative regular Hausdorff methods H_p. Furthermore, the authors investigated in [5] the more general situation that the densities are defined by sequences of matrices. Also sufficient and necessary conditions in terms of the matrix coefficients are given in [4], that the defined density is strongly nonatomic. In the next section we will continue Drewnowski, Florencio and Paúl's investigations in the case of densities defined by $R_{p,m}$ by using methods and results presented in [4].

[1] We use the notation $\mathbb{N}^0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N}_n := \{1, \dots, n\}$ $(n \in \mathbb{N})$.

[2] For a definition of $R_{p,m}$ and R_p see Section 2.

In [4] we introduced the following notation: Let $(\nu_t)_t$ be a sequence in $\mathbb{N}$ and, for each $t \in \mathbb{N}$, $\mathcal{N}_t = \{N_{\nu t} \mid \nu \in \mathbb{N}_{\nu_t}\}$ a partition of $\mathbb{N}$, that is, $\mathbb{N} = \bigcup\limits_{\nu=1}^{\nu_t} N_{\nu t}$ and $N_{\nu t} \cap N_{\mu t} = \emptyset$ if $\nu \neq \mu$. Then $(\mathcal{N}_t)$ is called an admissible partition sequence of $\mathbb{N}$.

Example 1.4. If we put $\nu_t := t$ $(t \in \mathbb{N})$ and $N_{\nu t} := \{\nu + rt \mid r \in \mathbb{N}^0\}$ $(t \in \mathbb{N}, \nu \in \mathbb{N}_t)$, then $\mathcal{N}_t = \{N_{\nu t} \mid \nu \in \mathbb{N}_t\}$ is an admissible partition sequence of $\mathbb{N}$.

Proposition 1.5. (cf. [4: Proposition 2.3]) *Let A be a nonnegative matrix being regular for null sequences. Then the following statements are equivalent:*

(a) *d_A is strongly nonatomic.*

(b) *There exists an admissible partition sequence $(\mathcal{N}_t)$ of $\mathbb{N}$ such that*[3]

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \sup_{1\le\nu\le\nu_t} A_{n\nu t} = 0 \qquad \textit{where} \quad A_{n\nu t} := \sum_{k\in N_{\nu t}} a_{nk}. \tag{1.1}$$

Therefore, if (a) *or* (b) *is fulfilled, then $\mathcal{Z}_A$ has the NP by* Proposition 1.3 (*and, equivalently, $|A|_0$ has the Hahn property and $\chi(|A|_0)^\beta = \ell^1$ by* Proposition 1.2).

The following result due to Kuttner and Parameswaran is an essential tool for constructing suitable admissible partition sequences $(\mathcal{N}_t)$ of $\mathbb{N}$.

Lemma 1.6. (cf. [9: Lemma 2] or [3: Lemma 3.2.15][4]) *Let $\{z_1, z_2, \dots, z_n\}$ for any $n \in \mathbb{N}$ be a set of non-negative real numbers. Let $Z_n := \sum\limits_{\nu=1}^{n} z_\nu$ and suppose that $B_n > 0$ with $z_\nu \le B_n$ $(\nu \in \mathbb{N}_n)$ and let $t \in \mathbb{N}$ be arbitrarily given. Then we can divide the set $\mathbb{N}_n$ into t (pairwise disjoint) subsets $N_1, N_2, \dots, N_t$ (some of them may be empty), such that*

$$\sum_{\nu\in N_s} z_\nu \le \frac{1}{t} Z_n + B_n \qquad (s = 1, 2, \dots, t). \tag{1.2}$$

2. Generalized Riesz matrices with strongly nonatomic density

In [7: Example 4.3] Drewnowski, Florencio and Paúl considered the following example of a submeasure on $\mathcal{P}(\mathbb{N})$: Let $p = (p_i)$ be a sequence of positive reals with $p \notin \ell^1$, and let γ be the countably additive measure

$$\gamma\colon \mathcal{P}(\mathbb{N}) \longrightarrow \overline{\mathbb{R}}_+, \quad N \longmapsto \sum_{i\in N} p_i$$

[3] As usual, we put $\sum\limits_{k\in\emptyset} a_k := 0$.

[4] A similar version of this lemma is [8: Lemma 6.5].

and, for any index sequence $m = (m_n)$ and $S_n = \mathbb{N}_{m_n}$ $(n \in \mathbb{N})$, η be the submeasure

$$\eta\colon \mathcal{P}(\mathbb{N}) \longrightarrow [0,1], \quad N \longmapsto \limsup_n \frac{\gamma(N \cap S_n)}{\gamma(S_n)}.$$

They asked for precise conditions on p and m such that η is strongly nonatomic. Now, we give a partial answer containing the sufficient conditions given in [7: Example 4.3].

For that we remark that η is generated by the matrix $R_{p,m} = (a_{nk})$ defined by

$$a_{nk} := \begin{cases} \frac{p_k}{P_{m_n}} & \text{if } 1 \le k \le m_n, \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N}). \tag{2.1}$$

Thus we may ask for precise conditions on p and m such that $d_{R_{p,m}}$ is strongly nonatomic (or satisfies the equivalent condition stated in Proposition 1.2).

We call $R_{p,m}$ *generalized Riesz matrix*, and get in the case $m_n = n$ $(n \in \mathbb{N})$ the (ordinary) *Riesz matrix* R_p $(= R_{p,(n)})$ (cf. [3: Section 3.2]). In general, $R_{p,m}$ may be understood as the row submatrix of R_p that may be obtained from R_p by deleting all the rows of R_p with index unequal to m_n $(n \in \mathbb{N})$. In the special case $R_p = R_{p,m}$, the stated problem is completely answered by [4: Theorem 3.1.1].

Theorem 2.1. (cf. also Remark 2.2(c) and the footnote in Problem 2.3) *Let $p = (p_i)$ be a sequence of positive reals with $p \notin \ell^1$.*

We consider the following conditions:

(a) *$d_{R_{p,m}}$ is strongly nonatomic (on $\mathcal{Z}_{R_{p,m}}$).*

(b) *$R_{p,m}$ has spreading rows, that is, $\lim_n \frac{1}{P_{m_n}} \sup_{1 \le k \le m_n} p_k = 0$ where $P_\nu := \sum_{k=1}^{\nu} p_k$.*

Then (a) $\Longrightarrow$ (b) *holds in general and* (b) $\Longrightarrow$ (a) *is true if at least one of the following conditions is fulfilled:*

(i) $\lim_n \frac{1}{P_n} \sup_{1 \le k \le n} p_k = 0$.

(ii) $\liminf_n \frac{P_{m_n}}{P_{m_{n+1}}} > 0$.

(iii) $\limsup_n \frac{P_{m_n}}{P_{m_{n+1}}} < 1$.

(iv) *There exists an index sequence (μ_j) and an $\alpha \in]0,1[$ such that*

$$\lim_j \frac{P_{m_{\mu_j - 1}}}{P_{m_{\mu_j}}} = 0 \qquad \textit{and} \qquad \frac{P_{m_{n-1}}}{P_{m_n}} > \alpha \quad (n \in \mathbb{N} \setminus \{\mu_j \mid j \in \mathbb{N}\}).$$

(v) *There exists an index sequence* (ν_j) *with* $\nu_j + 1 < \nu_{j+1}$ $(j \in \mathbb{N})$ *and an* $\alpha \in]0,1[$ *such that*

$$\lim_j \frac{P_{m_{\nu_j}}}{P_{m_{\nu_j+1}}} = 1 \qquad \textit{and} \qquad \frac{P_{m_n}}{P_{m_{n+1}}} < \alpha \quad (n \in \mathbb{N} \setminus \{\nu_j \mid j \in \mathbb{N}\}).$$

Remark 2.2.

(a) In Theorem 2.1 we did not mention the (general) inclusion condition $\mathcal{T} \subset c_{0R_{p,m}}$ which implies (a) by [4: Proposition 2.6] and holds for any nonnegative strongly regular matrix A (instead $R_{p,m}$). Thereby $\mathcal{T}$ denotes the set of all thin sequences (cf. [3: 1.2.4]).

(b) Drewnowski, Florencio and Paúl proved in [7: Example 4.3] that $\eta = d_{R_{p,m}}$ is strongly nonatomic if (p_k) is bounded or if (p_k) is nondecreasing and (b) holds. In both cases we are in the case (b)(i) of Theorem 2.1, so that these cases are covered by the results in 2.1.

(c) (cf. also the footnote in Problem 2.3). The possible cases (i)–(iv) in 2.1 would be complete if we replace (iv) with the more general condition

$$\limsup_n \frac{P_{m_n}}{P_{m_{n+1}}} = 1 \quad \text{and} \quad \liminf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0 \tag{2.2}$$

(that contains (v)), but the authors are not yet able to manage this general case.

(d) In the case of regular Riesz matrices R_p we have that R_p has spreading rows if and only if $(\frac{p_n}{P_n}) \in c_0$. The corresponding condition in the case of regular matrices $R_{p,m}$ is condition (b). However, in general, this condition is not sufficient for spreading rows of the matrix $R_{p,m}$ as the following example shows:

Let $m_n = 2n-1$ $(n \in \mathbb{N})$, and let $p = (p_n)$ be defined by

$$p_k = \begin{cases} 1 & \text{if } k = 2n-1, \\ P_{k-1} & \text{if } k = 2n \end{cases} \qquad (k, n \in \mathbb{N}).$$

Obviously, $p \notin \ell^1$, $\left(\frac{p_{m_n}}{P_{m_n}}\right) \in c_0$, and $\left(\frac{1}{P_{m_n}} \sup_{1 \le k \le m_n} p_k\right) \notin c_0$ since we have $\limsup_n \frac{p_{2n}}{P_{2n+1}} = \frac{1}{2}$. In particular (cf. 1.1, 1.2, and 1.3), $|R_{p,m}|_0$ does not have HP and $d_{R_{p,m}}$ is not strongly nonatomic.

P r o o f o f T h e o r e m 2.1.

(a) $\Longrightarrow$ (b): Let $d_{R_{p,m}}$ be strongly nonatomic. Then, by [4: Proposition 2.8, (c*)$\Longrightarrow$(i*)$\Longrightarrow$c)], the matrix $R_{p,m}$ has spreading rows, that is, (b) holds.

(b) $\Longrightarrow$ (a): Now, we assume that $R_{p,m}$ satisfies (b).

Case (i):

If $\lim_n \sup_{1 \le k \le n} \frac{p_k}{P_n} = 0$, then, by the proof of [4: Theorem 3.1.1, (b)$\Longrightarrow$(c*)], the condition (1.1) is fulfilled for R_p and some admissible partition sequence $(\mathcal{N}_t)$ of $\mathbb{N}$.

Then the condition (1.1) is obviously fulfilled for the submatrix $R_{p,m}$ of R_p. Consequently, $d_{R_{p,m}}$ is strongly nonatomic.
Case (ii):
In the remaining proof, we set $A_n := P_{m_n}$ ($n \in \mathbb{N}$). Now, let

$$\liminf_n \frac{A_n}{A_{n+1}} > 0.$$

Then

$$\limsup_n \frac{A_{n+1} - A_n}{A_{n+1}} = \limsup_n \left(1 - \frac{A_n}{A_{n+1}}\right) < 1$$

and therefore we may choose $u \in]0,1[$ and $n_0 \in \mathbb{N}$ such that

$$\frac{A_{n+1} - A_n}{A_{n+1}} \leq u \qquad \text{for each} \quad n \geq n_0. \tag{2.3}$$

Let $t \in \mathbb{N}$ be fixed. By (b), let $\alpha_t \in \mathbb{N}$ be chosen such that $\alpha_t \geq n_0$ and

$$\frac{p_k}{A_n} < \frac{1}{t} \qquad \text{for each} \quad k \in \mathbb{N}_{m_n} \ \text{ and } \ n \geq \alpha_t. \tag{2.4}$$

Let (n_s) be the index sequence with $n_1 = 1$ and (note $p \notin \ell^1$)

$$n_{s+1} = \min\left\{\nu \in \mathbb{N} \mid A_\nu \geq 2A_{n_s}\right\}. \tag{2.5}$$

Then, from (2.3) and (2.5) in combination with

$$\frac{A_{n_{s+1}}}{A_{n_s}} = \frac{A_{n_{s+1}-1}}{A_{n_s}} + \frac{A_{n_{s+1}} - A_{n_{s+1}-1}}{A_{n_{s+1}}}\frac{A_{n_{s+1}}}{A_{n_s}},$$

we get

$$\frac{A_{n_{s+1}}}{A_{n_s}} \leq \frac{2}{1-u} \qquad (s \geq \alpha_t). \tag{2.6}$$

By Lemma 1.6, for any positive integer $s \geq \alpha_t$ the set $I_s := \left]m_{n_s}, m_{n_{s+1}}\right] \cap \mathbb{N}$ can be divided into t disjoint subsets (some of them may be empty), say $N_{st}(1)$, $N_{st}(2), \dots, N_{st}(t)$, such that for each $\nu \in \mathbb{N}_t$ we have (cf. (2.4))

$$\sum_{k \in N_{st}(\nu)} p_k \leq \frac{1}{t} \sum_{k \in I_s} p_k + \frac{1}{t} A_{n_{s+1}} \leq \frac{2}{t} A_{n_{s+1}}. \tag{2.7}$$

Now, setting

$$N_{\nu t} := \begin{cases} \mathbb{N}_{m_{n_{\alpha_t}}} \cup \bigcup\limits_{s=\alpha_t}^{\infty} N_{st}(1) & \text{if } \nu = 1, \\ \bigcup\limits_{s=\alpha_t}^{\infty} N_{st}(\nu) & \text{if } 1 < \nu \leq t, \end{cases} \quad \text{and} \quad \mathcal{N}_t := \left\{N_{\nu t} \mid \nu \in \mathbb{N}_t\right\}, \tag{2.8}$$

we get an admissible partition sequence $\mathcal{N} := (\mathcal{N}_t)_t$ of $\mathbb{N}$. (Note the remarks at the corresponding place in the proof of [4: Theorem 3.1.1, (b)$\Longrightarrow$(c*)].) The chosen partition $\mathcal{N}$ satisfies (1.1) as we verify now. For that, let $n \geq n_{\alpha_t}$, and let r be chosen such that $n_r < n \leq n_{r+1}$. Then, on account of (2.5), (2.6) and (2.7), for each $\nu \in \mathbb{N}_t$ we have

$$\frac{1}{A_n}\sum_{s=\alpha_t}^{r}\sum_{k\in N_{st}(\nu)} p_k \leq \frac{2}{t}\frac{1}{A_n}\sum_{s=\alpha_t}^{r} A_{n_{s+1}} \leq \frac{2}{t}\frac{A_{n_{r+1}}}{A_{n_r}}\sum_{s=0}^{\infty} 2^{-s} \leq \frac{8}{1-u}\frac{1}{t},$$

thus

$$\limsup_{t\to\infty}\limsup_{n\to\infty}\sup_{\nu\in\mathbb{N}_t}\frac{1}{A_n}\sum_{k\in N_{\nu t},\, k\leq m_n} p_k = 0.$$

Therefore, by Proposition 1.5, $d_{R_{p,m}}$ is strongly nonatomic.
Case (iii):
Now, we suppose

$$\limsup_n \frac{A_n}{A_{n+1}} < 1.$$

Then we may chose $v > 1$ and $n_0 \in \mathbb{N}$ such that

$$\frac{A_{n+1}}{A_n} \geq v \qquad \text{for every} \quad n \geq n_0. \tag{2.9}$$

Consequently, on account of (b) and (2.9), we have

$$\liminf_n (m_{n+1} - m_n) = \infty.$$

Now, let $t \in \mathbb{N}$ be fixed, and let $\alpha_t \geq n_0$ be chosen according to (2.4). If $s \geq \alpha_t$, then, by applying Lemma 1.6, we divide the set $I_s :=]m_s, m_{s+1}] \cap \mathbb{N}$ into disjoint subsets (some of them may be empty), say $N_{st}(1), N_{st}(2), \ldots, N_{st}(t)$, such that

$$\sum_{k\in N_{st}(\nu)} p_k \leq \frac{1}{t}\sum_{k\in I_s} p_k + \frac{1}{t}A_{s+1} \leq \frac{2}{t}A_{s+1} \tag{2.10}$$

is fulfilled for each $\nu \in \mathbb{N}_t$. Let $\mathcal{N} := (\mathcal{N}_t)_t$ be defined as in (2.8). For $n \geq \alpha_t$ and $\nu \in \mathbb{N}_t$, on account of (2.9) and (2.10), we have

$$\frac{1}{A_{n+1}}\sum_{s=\alpha_t}^{n}\sum_{k\in N_{st}(\nu)} p_k \leq \frac{2}{t}\frac{1}{A_{n+1}}\sum_{s=\alpha_t}^{n} A_{s+1} < \frac{2}{t}\sum_{s=0}^{\infty}\frac{1}{v^s} = \frac{v}{v-1}\frac{2}{t}.$$

Therefore

$$\limsup_{t\to\infty}\limsup_{n\to\infty}\sup_{\nu\in\mathbb{N}_t}\frac{1}{A_n}\sum_{k\in N_{\nu t}, k\leq m_n} p_k = 0.$$

Again, by Proposition 1.5, $d_{R_{p,m}}$ is strongly nonatomic.
Case (iv):
We choose an index sequence (μ_j) and an $\alpha \in]0,1[$ such that

$$\lim_j \frac{A_{\mu_j - 1}}{A_{\mu_j}} = 0 \quad \text{and} \qquad \frac{A_{n-1}}{A_n} > \alpha \quad (n \in \mathbb{N} \setminus \{\mu_j \mid j \in \mathbb{N}\}). \tag{2.11}$$

Let $t \in \mathbb{N} \setminus \{1\}$ be fixed. Then, by (2.11) and (b), we may choose a $j_0 \in \mathbb{N}$ with

$$\frac{A_{\mu_j - 1}}{A_{\mu_j}} < \frac{1}{t} \quad (j \geq j_0) \qquad \text{and} \qquad \frac{p_k}{A_n} < \frac{1}{t} \quad (k \in \mathbb{N}_{m_n},\ \ n \geq \mu_{j_0}). \tag{2.12}$$

For every $j \geq j_0$ we divide the set $\left]m_{\mu_j}, m_{\mu_{j+1}}\right] \cap \mathbb{N}$ into finitely many, say r_j, subsets $I_1^j, \ldots, I_{r_j}^j$ as follows: We put $n_1^j := \mu_j =: \widehat{n}_1^j$. Then, if $\mu_{j+1} = \mu_j + 1$, we set $r_j := 1$. If $\mu_{j+1} > \mu_j + 1$, then, having already defined $\widehat{n}_r^j$ and n_r^j for $1 \leq r \leq s$ we consider

$$\widehat{n}_{s+1}^j = \min\left\{\nu \in \mathbb{N} \mid A_\nu \geq 2A_{\widehat{n}_s^j}\right\} \tag{2.13}$$

and put

$$n_{s+1}^j := \begin{cases} \widehat{n}_{s+1}^j & \text{if } \widehat{n}_{s+1}^j < \mu_{j+1}, \\ \mu_{j+1} - 1 & \text{if } \widehat{n}_{s+1}^j \geq \mu_{j+1}, \end{cases}$$

as well (2.14)

$$r_j := s + 1 \quad \text{if } \widehat{n}_{s+1}^j \geq \mu_{j+1}.$$

Now, we define

$$I_s^j :=]m_{n_s^j}, m_{n_{s+1}^j}] \cap \mathbb{N} \qquad (s \in \mathbb{N}_{r_j})$$

where $n_{r_j+1}^j := \mu_{j+1}$. Then $I_{r_j}^j =]m_{\mu_{j+1}-1}, m_{\mu_{j+1}}] \cap \mathbb{N}$ and, by (2.13),

$$\frac{A_{n_{s+1}^j}}{A_{n_s^j}} = \frac{A_{n_{s+1}^j}}{A_{n_{s+1}^j - 1}} \frac{A_{n_{s+1}^j - 1}}{A_{n_s^j}} < \frac{2}{\alpha} \qquad (s = 1, \ldots, r_j - 1). \tag{2.15}$$

Applying Lemma 1.6, for $s = 1, \ldots, r_j$ we divide the set I_s^j into t disjoint subsets (some of them may be empty), say $N_{st}^j(1), N_{st}^j(2), \ldots, N_{st}^j(t)$, such that

$$\sum_{k \in N_{st}^j(\nu)} p_k \leq \frac{1}{t} \sum_{k \in I_s^j} p_k + \frac{1}{t} A_{n_{s+1}^j} \leq \frac{2}{t} A_{n_{s+1}^j}. \tag{2.16}$$

For $n_l^j < n \leq n_{l+1}^j$ and $\nu = 1, \ldots, t$ we define

$$T_{\nu t}(n) := \frac{1}{A_n} \sum_{i=j_0}^{j-1} \sum_{s=1}^{r_i} \sum_{k \in N_{st}^i(\nu)} p_k + \frac{1}{A_n} \sum_{s=1}^{l} \sum_{k \in N_{st}^j(\nu)} p_k.$$

If $n = \mu_{j+1} = n_{r_j+1}^j$, then, by (2.12) and (2.16),

$$0 \leq T_{\nu t}(n) \leq \frac{A_{n-1}}{A_n} + \frac{1}{A_n} \sum_{k \in N_{r_j t}^j(\nu)} p_k < \frac{1}{t} + \frac{2}{t} \frac{A_n}{A_n} = \frac{3}{t}. \tag{2.17}$$

Moreover, for $n_l^j < n \leq n_{l+1}^j$ and $1 \leq l \leq r_j - 2$ we get (cf. (2.17), (2.16), (2.13), (2.14), (2.15))

$$\begin{aligned} 0 \leq T_{\nu t}(n) &\leq T_{\nu t}(\mu_j) + \frac{1}{A_n} \sum_{s=1}^{l} \sum_{k \in N_{st}^j(\nu)} p_k \\ &< \frac{3}{t} + \frac{2}{t} \frac{1}{A_n} \sum_{s=1}^{l} A_{n_{s+1}^j} \\ &< \frac{3}{t} + \frac{4}{t} \frac{A_{n_{l+1}^j}}{A_{n_l^j}} < \frac{3}{t} + \frac{4}{t} \frac{2}{\alpha}. \end{aligned} \tag{2.18}$$

Finally, if $n_{r_j-1}^j < n \leq n_{r_j}^j$, then by (2.18) and (2.16) we obtain

$$\begin{aligned} 0 \leq T_{\nu t}(n) &\leq T_{\nu t}(n_{r_j-1}^j) + \frac{1}{A_n} \sum_{k \in N_{n_{r_j-1},t}^j(\nu)} p_k \\ &< \frac{3}{t} + \frac{4}{t} \frac{2}{\alpha} + \frac{2}{t} \frac{A_{n_{r_j}^j}}{A_{n_{r_j-1}^j}} \\ &< \frac{3}{t} + \frac{8}{t\alpha} + \frac{2}{t} \frac{2}{\alpha} = \frac{1}{t} \left(3 + \frac{12}{\alpha} \right). \end{aligned}$$

Now, if we put

$$N_{\nu t} := \begin{cases} \mathbb{N}_{m_{\mu_{j_0}}} \cup \bigcup_{j=j_0}^{\infty} \bigcup_{s=1}^{r_j} N_{st}^j(1) & \text{if } \nu = 1, \\ \bigcup_{j=j_0}^{\infty} \bigcup_{s=1}^{r_j} N_{st}^j(\nu) & \text{if } 1 < \nu \leq t, \end{cases}$$

and $\mathcal{N}_t := \{N_{\nu t} \mid \nu \in \mathbb{N}_t\}$, we get an admissible partition sequence $\mathcal{N} := (\mathcal{N}_t)_t$ of $\mathbb{N}$. Noting that $R_{p,m}$ is regular for null sequences and using the above estimations we get

$$\limsup_{t \to \infty} \limsup_{n \to \infty} \sup_{\nu \in \mathbb{N}_t} \frac{1}{A_n} \sum_{k \in N_{\nu t}, k \leq m_n} p_k = 0.$$

Therefore, by Proposition 1.5, $d_{R_{p,m}}$ is strongly nonatomic.

Case (v):

Let (s_i) be the subsequence of (n) being complementary to (ν_j). We consider the row submatrices of $R_{p,m}$, say $A^{(1)} = (a_{jk}^{(1)})$ and $A^{(2)} = (a_{ik}^{(2)})$, defined by

$$a_{jk}^{(1)} := \begin{cases} \frac{p_k}{P_{m_{\nu_j}}} & \text{if } k \leq m_{\nu_j}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$a_{ik}^{(2)} := \begin{cases} \frac{p_k}{P_{m_{s_i}}} & \text{if } k \le m_{s_i}, \\ 0 & \text{otherwise} \end{cases} \qquad (j, i, k \in \mathbb{N}).$$

It is sufficient to prove that both, the submeasure $d_{A^{(1)}}$ and the submeasure $d_{A^{(2)}}$ are strongly nonatomic. In the case of $A^{(1)}$, by the inequality $\frac{P_{m_{\nu_{j+1}-1}}}{P_{m_{\nu_{j+1}}}} < \alpha$ we get

$$\frac{P_{m_{\nu_j}}}{P_{m_{\nu_{j+1}}}} = \frac{P_{m_{\nu_j}}}{P_{m_{\nu_{j+1}-1}}} \frac{P_{m_{\nu_{j+1}-1}}}{P_{m_{\nu_{j+1}}}} < \alpha < 1 \qquad (j \in \mathbb{N}),$$

and, by (b)(iii), $d_{A^{(1)}}$ is strongly nonatomic. In the case of $A^{(2)}$, for every $i \in \mathbb{N}$, we have either $s_{i+1} = s_i + 1$ or $s_{i+1} = s_i + 2$. If $s_{i+1} = s_i + 1$, then $\frac{P_{m_{s_i}}}{P_{m_{s_{i+1}}}} = \frac{P_{m_{s_i}}}{P_{m_{s_i+1}}} < \alpha$, whereas, in the case $s_{i+1} = s_i + 2$, we obtain

$$\frac{P_{m_{s_i}}}{P_{m_{s_{i+1}}}} = \frac{P_{m_{s_i}}}{P_{m_{s_i+1}}} \frac{P_{m_{s_i+1}}}{P_{m_{s_{i+1}}}} < \alpha.$$

Therefore, again by (b)(iii), $d_{A^{(2)}}$ is strongly nonatomic. □

Problem 2.3. Complete the distinction of cases in Theorem 2.1 (cf. Remark 2.2(c), too) or answer the question whether $\mathcal{Z}_{R_{p,m}}$ has NP whenever (2.2) is satisfied (may be, $\mathcal{T} \subset c_{0R_{p,m}}$ holds).[5]

Remark 2.4. Let p and $m = (m_n)$ be given as in Theorem 2.1.

(a) If $\limsup_n \frac{1}{A_n} \sum_{m_n < \nu \le m_{n+1}} p_\nu = 0$, e.g., if $\limsup_n \frac{1}{A_n} \sup_{m_n < \nu \le m_{n+1}} p_\nu = 0$ and $\sup_n (m_n - m_{n+1}) < \infty$, then $\lim_n \frac{A_n}{A_{n+1}} = 1$, thus $\liminf_n \frac{A_n}{A_{n+1}} = 1 > 0$; consequently, $d_{R_{p,m}}$ is strongly nonatomic by case (b)(ii) in 2.1.

(b) If $\liminf_n \frac{1}{A_n} \sum_{m_n < \nu \le m_{n+1}} p_\nu > 0$, for instance, if — as in the Examples 2.5 and 2.6 — there exist an $\alpha > 0$ and a sequence (ν_n) in $\mathbb{N}$ with $m_n < \nu_n \le m_{n+1}$ and $\frac{p_{\nu_n}}{A_n} \ge \alpha$, then $\limsup_n \frac{A_n}{A_{n+1}} < 1$; in particular, $d_{R_{p,m}}$ is strongly nonatomic by case (b)(iii) in 2.1.

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[5]Some time after the reviewing process of this paper Maria Zeltser (Tallin University, Estonia) has communicated (with proof) to the authors that $\liminf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0$ implies that $d_{R_{p,m}}$ is strongly nonatomic (on $\mathcal{Z}_{R_{p,m}}$) so that in Theorem 2.1 the statements (a) and (b) are equivalent.

Example 2.5. (similar to [6: Example 3.12]) For any fixed index sequence (m_n) with $m_1 = 1$ and $m_n + 1 < m_{n+1}$ $(n \in \mathbb{N})$ we define $p = (p_n)$ inductively by

$$p_i = \begin{cases} 1 & \text{if } k \le m_2, \\ P_{m_n} & \text{if } i = m_n + 1, \\ p_{m_n+1} & \text{if } m_n + 1 < i \le m_{n+1} \end{cases} \qquad (i \in \mathbb{N}, \quad n \ge 2). \tag{2.19}$$

Using this definition of $p = (p_i)$ we finally define (m_n): Having fixed m_n for an $n \in \mathbb{N}$ we choose an m_{n+1} with $m_n + 1 < m_{n+1}$ such that $\frac{p_{m_n+1}}{P_{m_{n+1}}} < \frac{1}{n}$. Obviously, p is monotonically increasing, $p \notin \ell^1$, $(\frac{P_n}{p_n}) \notin \ell^\infty$ and $(\frac{p_n}{P_n}) \notin c_0$. Therefore, $c \subsetneq \ell^\infty \cap c_{R_p} \subsetneq c_{R_p} \subset c_{C_1}$ and $\ell^\infty \cap c_{R_p}$ as well $|R_p|_0$ do not have the Hahn property (cf. [4: Theorem 3.1.1]). However, if we consider $R_{p,m}$ defined by (2.1), then for $R_{p,m}$ condition (b) in 2.1 is fulfilled, and $R_{p,m}$ has non-decreasing rows (up to m_n in the n^{th} row). Therefore $R_{p,m}$ is strongly regular (cf. [3: Theorem 2.4.9]), thus $d_{R_{p,m}}$ is strongly nonatomic (cf. [4: Corollary 2.7]).

Now, we modify a little bit Example 2.5 such that p fulfills still the assumptions and condition (b) in 2.1, but $R_{p,m}$ is not strongly regular.

Example 2.6. For any fixed index sequence (m_n) with $m_1 = 1$ and $m_n + 2 < m_{n+1}$ $(n \in \mathbb{N})$ we define $p = (p_n)$ inductively by

$$p_i = \begin{cases} 1 & \text{if } k \le m_2, \\ P_{m_n} & \text{if } i = m_n + 1, \\ 1 & \text{if } m_n + 1 < i \le m_{n+1} \text{ and } i = m_n + 2\mu \ (\mu \in \mathbb{N}), \\ P_{m_n} & \text{if } m_n + 1 < i \le m_{n+1} \text{ and } i = m_n + 2\mu + 1 \ (\mu \in \mathbb{N}) \end{cases} \tag{2.20}$$

$(i, n \in \mathbb{N},\ n \ge 2)$. Using this definition of p we finally determine an index sequence (m_n): Having fixed m_n for an $n \in \mathbb{N}$ we choose an m_{n+1} with $m_n + 1 < m_{n+1}$ such that

$$\frac{P_{m_n}}{P_{m_{n+1}}} = \frac{p_{m_n+1}}{P_{m_{n+1}}} < \frac{1}{n} \qquad \text{and} \qquad \frac{1}{P_{m_{n+1}}} \left(p_{m_{n+1}} + \sum_{i=1}^{m_{n+1}-1} |p_i - p_{i+1}| \right) \ge \frac{1}{4}.$$

Obviously, $p \notin \ell^1$, $(\frac{P_n}{p_n}) \notin \ell^\infty$ and $(\frac{p_n}{P_n}) \notin c_0$. Therefore, $c \subsetneq \ell^\infty \cap c_{R_p}$ and $\ell^\infty \cap c_{R_p}$ as well $|R_p|_0$ do not have the Hahn property (cf. [4: Theorem 3.1.1]). However, if we consider $R_{p,m}$, then it is regular and fulfills condition (b) in 2.1. Moreover, in contrast to Example 2.5, A is not strongly regular. Since $\frac{P_{m_n}}{P_{m_{n+1}}} < \frac{1}{n}$ $(n \in \mathbb{N})$ we have $\limsup_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0$, that is, we are in case (b)(iii) of Theorem 2.1, so that $d_{R_{p,m}}$ is strongly nonatomic.

Example 2.7. For any fixed index sequence (m_n) with $m_1 = 1$ and $m_{2n} + 1 = m_{2n+1}$ and $m_{2n+1} + 1 < m_{2n+2}$ $(n \in \mathbb{N})$ we define $p = (p_n)$ inductively by

$$p_i = \begin{cases} 1 & \text{if } i \leq m_2, \\ 1 & \text{if } i = m_{2n+1}, \\ P_{m_{2n+1}} & \text{if } m_{2n+1} + 1 \leq i \leq m_{2n+2} \end{cases} \qquad (i, n \in \mathbb{N},\ \ n \geq 1). \tag{2.21}$$

Using this definition of $p = (p_i)$ we finally determine the index sequence (m_n): Having fixed m_{2n} for an $n \in \mathbb{N}$, we put $m_{2n+1} = m_{2n} + 1$ and choose an m_{2n+2} with $m_{2n+1} + 1 < m_{2n+2}$ such that $\frac{P_{m_{2n+1}}}{P_{m_{2n+2}}} < \frac{1}{n}$.

Obviously, $p \notin \ell^1$, $\left(\frac{P_n}{p_n}\right) \notin \ell^\infty$ and $\left(\frac{p_n}{P_n}\right) \notin c_0$. Therefore, $c \subsetneqq \ell^\infty \cap c_{R_p}$ and $\ell^\infty \cap c_{R_p}$ as well $|R_p|_0$ do not have the Hahn property (cf. [4: Theorem 3.1.1]). However, if we define A by (2.1), then A is regular and satisfies condition (b) in 2.1. Note, A is not strongly regular. Because of $\frac{A_{2n+1}}{A_{2n+2}} = \frac{P_{m_{2n+1}}}{P_{m_{2n+2}}} < \frac{1}{n}$ $(n \in \mathbb{N})$ we have $\liminf_n \frac{A_n}{A_{n+1}} = 0$, and because of $\lim_n \frac{A_{2n}}{A_{2n+1}} = 1$ we get $\limsup_n \frac{A_n}{A_{n+1}} = 1$. Therefore, we are obviously in case (b)(iv) of 2.1, so that $d_{R_{p,m}}$ is strongly nonatomic.

3. Remarks on the example of Alon, Drewnowski and Łuczak

Alon et al. consider in [1] the countably additive measure $\gamma\colon \mathcal{P}(\mathbb{N}) \longrightarrow \overline{\mathbb{R}}_+$, $N \longmapsto |N|$ and the submeasure $\eta\colon \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$, $N \longmapsto \limsup_n \frac{|N \cap F_n|}{|F_n|}$ where (F_n) is a suitable fixed family of distinct finite subsets of $\mathbb{N}$. They stated the following main result:

Theorem 3.1. (cf. [1: Theorem 2.3]) *There exists a family (F_n) of distinct finite subsets of $\mathbb{N}$ such that the ideal $\mathcal{Z}(\eta)$ has the NP, but η is not strongly nonatomic.*

Correspondingly to the considerations in Section 2 the submeasure η can be generated by the matrix $A = (a_{nk})$ defined by

$$a_{nk} := \begin{cases} \frac{1}{|F_n|} & \text{if } k \in F_n, \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N}). \tag{3.1}$$

Thus, translating 3.1 into the language of matrix transformations, we get

Corollary 3.2. *There exists a nonnegative regular matrix A such that $\mathcal{Z}_A$ has NP and d_A is not strongly nonatomic.*

The ideal $\mathcal{Z}(\boldsymbol{F})$ constructed in the proof of [1: Theorem 2.3] can be described as $\mathcal{Z}_A$ where A is the matrix defined by (3.1). Obviously, A can not be a generalized Riesz mean $R_{p,m}$ because otherwise we would have $p_k = 1$ ($k \in \mathbb{N}$), thus condition (i) in Theorem 2.1 would be satisfied, implying that d_A, hence $\mathcal{Z}(\boldsymbol{F})$, would be strongly nonatomic.

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