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EXTENDING ASYMMETRIC CONVERGENCE AND CAUCHY CONDITION USING IDEALS

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ABSTRACT. In this paper we use the notion of ideals to extend the convergence and Cauchy conditions in asymmetric metric spaces. The asymmetry (or rather, absence of symmetry) of these spaces makes the whole treatment different from the metric case and we use a genuinely asymmetric condition called (AMA) to prove many results and show that certain classic results fail in the asymmetric context if the assumption is dropped.

1. Introduction and background

Since 1951 when Steinhaus [28] and Fast [11] defined statistical convergence for sequences of real numbers, several generalizations and applications of this notion have been investigated (see [1], [2], [4]–[10], [12]–[14], [16], [20]–[22], [24]–[27], [29] where many more references can be found). In particular two interesting generalizations of statistical convergence were introduced by Kostyrko et al [13], using the notion of ideals of the set $\mathbb{N}$ of positive integers who named them as I and I^*-convergence. Corresponding I-Cauchy condition was first introduced and studied by Dems [10]. I^*-Cauchy sequences has been very recently introduced by Nabiev et al [22] where they showed that I^*-Cauchy sequences are I-Cauchy and they are equivalent if the ideal I satisfies the condition (AP)

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(this was also done to some extent by Balcerzak et. al. [1]). However two important questions remained unanswered. Firstly construction of an example of an I-Cauchy sequence which is not I^*-Cauchy and what is the necessary and sufficient condition for their equivalence. In our very recent works [7] we showed that under some general assumption, the condition (AP) is both necessary and sufficient for the equivalence of I and I^*-Cauchy conditions and constructed an example to prove that in general I-Cauchy sequences may not be I^*-Cauchy. Ideal convergence has also been studied in topological spaces [16], in 2-normed spaces [24], in (l) groups [2], in random 2-normed spaces [20], in probabilistic normed spaces [21] among others. Further ideal convergence has been extended to double sequences in [5].

Asymmetric metric spaces (sometimes called quasi-metric spaces) are defined as metric spaces, but without the requirement that the metric d has to satisfy $d(x, y) = d(y, x)$. These spaces were investigated in past including in [15, 18, 23, 30] and it was observed that an “embarrassing richness of material” [30] is revealed without symmetry. Asymmetric metric spaces have also found recent applications in different spheres of science as can be seen from [3], [17]–[19] etc. The uniqueness and primary difference with the metric case is that in an asymmetric metric space, corresponding to each notion (convergence, Cauchy condition etc.) one can actually consider two notions, namely forward and backward notions arising for the two natural topologies (forward and backward topologies) in the same space. In particular in [3] the notions of asymmetric convergence and Cauchy conditions were studied to some extent. Our investigation shows that many of these results can be extended using the more general ideas of I-convergence and I-Cauchy conditions. The investigation further reveals that as in [3] and recent studies on double sequences [8] (where the results of [6] had been investigated in asymmetric context), the symmetry requirement of a metric is not essential for most results and they can be obtained in the asymmetric metric structure under a genuinely asymmetric condition.

The paper is organized as follows. In Section 2, basic definitions and notions are described. In Section 3, forward and backward I and I^*-convergence are studied which is the basic requirement for our purpose. In Section 4, asymmetric I-Cauchy condition is discussed. Finally in Section 5, asymmetric I^*-Cauchy condition is introduced and the stated open problems arc dealt with.

2. Definitions and notations

We first recall the following basic concepts of an asymmetric metric space from [3, 23, 30].

Definition 1. A function $d\colon X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if

(i) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2. The forward topology τ_+ induced by d is the topology generated by the forward open balls $B^+(x, \varepsilon) = \{y \in X : \ d(x, y) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Like wise, the backward topology τ_- induced by d is the topology generated by the backward open balls $B^-(x, \varepsilon) = \{y \in X : \ d(y, x) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Definition 3. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be forward convergent to $x_0 \in X$, (respectively backward convergent to $x_0 \in X$) if and only if

$$\lim_{n\to\infty} d(x_0, x_n) = 0 \qquad \text{(respectively} \quad \lim_{n\to\infty} d(x_n, x_0) = 0)$$

and is denoted by $x_n \xrightarrow{F} x_0$ (respectively $x_n \xrightarrow{B} x_0$). In this case we write F-$\lim\limits_{n\to\infty} x_n = x_0$ (respectively B-$\lim\limits_{n\to\infty} x_n = x_0$).

Definition 4. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in an asymmetric metric space (X, d) is

(i) forward Cauchy if for each $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that for $m \geq n \geq k$, $d(x_n, x_m) < \varepsilon$ holds, and

(ii) backward Cauchy if for each $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that for $m \geq n \geq k$, $d(x_m, x_n) < \varepsilon$ holds.

Definition 5. An asymmetric metric space (X, d) is said to be forward complete if every forward Cauchy sequence is forward convergent in X. Similarly, we can define backward completeness.

Definition 6. An asymmetric metric space (X, d) is said to be forward (backward) sequentially compact if every sequence in X contains a forward (backward) convergent subsequence.

Definition 7. Let (X, d) be an asymmetric metric space and $A \subset X$. A point $x \in X$ is said to be a forward (backward) accumulation point of the set A, if for every $\varepsilon > 0$, $(B^+(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$ (respectively $(B^-(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$).

Now we state below a genuinely asymmetric property of an asymmetric metric space from [3] (see also [8] where this name is given) which will play a very important role throughout our paper.

Definition 8. An asymmetric metric space (X, d) is said to satisfy approximate metric axiom (or (AMA)) if there exists a function $c\colon X \times X \to \mathbb{R}$ such that for any $x, y \in X$,

$$d(y, x) \leq c(x, y)d(x, y)$$

where the function c is such that for any $x \in X$, there is a $\delta_x > 0$ with the property that

$$y \in B^+(x, \delta_x) \implies c(x, y) \leq C(x),$$

where $C(x) > 0$ depends only on x.

Remark 2.1. Note that if (X, d) is a metric space then evidently it satisfies (AMA) with the function c defined by $c(x, y) = 1$ for all $x, y \in X$. However the condition (AMA) is strictly weaker than the requirement of an asymmetric metric space to be a metric space, as shown by Examples 2.1 (more examples can be seen from [2]).

Example 2.1. We begin with the simplest example of an asymmetric metric space. Let $\alpha > 0$. Then $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ define by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x, \\ \alpha(x - y) & \text{if } y < x \end{cases}$$

is obviously an asymmetric metric. This metric satisfies $d(y, x) \leq C(x)d(x, y)$ for all $x, y \in \mathbb{R}$ where $C(x) = \max\{\alpha, \frac{1}{\alpha}\}$. This metric there satisfies the conditions for [2: Proposition 3.3]. Note that τ_+ and τ_- are the usual topologies on $\mathbb{R}$.

Finally we recall that a non-void class of sets I of a non-empty set X is called an ideal if

(i) $A, B \in I \implies A \cup B \in I$,

(ii) $A \in I$, $B \subset A \implies B \in I$.

If I is an ideal then $F(I) = \{A \subset X : A^c \in I\}$ is filter called the associated filter of I. I is said to be non-trivial if $X \notin I$. Further I is said to be admissible if $\{x\} \in I$ for all $x \in X$. Throughout the paper we assume I to be a non-trivial admissible ideal of $\mathbb{N}$, the set of all positive integers.

3. Asymmetric I and I^*-convergence

We first introduce the following definition.

Definition 9. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space X is said to converge forwardly to x with respect to the ideal I if for each $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N} : \ d(x, x_n) \geq \varepsilon\} \in I$. In this case we write $FI\text{-}\lim_{n\to\infty} x_n = x$ (or $x_n \xrightarrow{FI} x_0$).

Similarly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space X is said to converge backwardly to x with respect to the ideal I if for each $\varepsilon > 0$, $B(\varepsilon) = \{n \in \mathbb{N} : \ d(x_n, x) \geq \varepsilon\} \in I$. This is denoted by $BI\text{-}\lim_{n\to\infty} x_n = x$ (or $x_n \xrightarrow{BI} x_0$).

Note that if I is admissible then forward (backward) convergence of $\{x_n\}_{n\in\mathbb{N}}$ implies forward (backward) I-convergence.

Theorem 3.1. *If a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space (X, d) is forward I-convergent to $x_0 \in X$ and backward I-convergent to $y_0 \in X$ then $x_0 = y_0$.*

Proof. For each $\varepsilon > 0$, $A(\varepsilon)^c = \{n \in \mathbb{N} : \ d(x_0, x_n) < \frac{\varepsilon}{2}\}$ and $B(\varepsilon)^c = \{n \in \mathbb{N} : \ d(x_n, y_0) < \frac{\varepsilon}{2}\} \in F(I)$. Since $A(\varepsilon)^c \cap B(\varepsilon)^c \in F(I)$ and $\emptyset \notin F(I)$ so there exists $k \in \mathbb{N}$ such that $d(x_0, x_k) < \frac{\varepsilon}{2}$ and $d(x_k, y_0) < \frac{\varepsilon}{2}$ which implies

$$d(x_0, y_0) \leq d(x_0, x_k) + d(x_k, y_0) < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have $d(x_0, y_0) = 0$, which implies $x_0 = y_0$. □

The following example shows that in general forward I-convergence does not imply backward I-convergence and viceversa.

Example 3.1. (Sorgenfrey asymmetric metric) Let the function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be defined by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x, \\ 1 & \text{if } y < x. \end{cases}$$

Then d is an asymmetric metric on $\mathbb{R}$. Here, τ_+ is the lower limit topology on $\mathbb{R}$ and τ_- is the upper limit topology on $\mathbb{R}$, i.e. $B^+(x, \varepsilon) = [x, x + \varepsilon)$ and $B^-(x, \varepsilon) = (x - \varepsilon, x]$, provided $\varepsilon \leq 1$. Let I be any non-trivial admissible ideal of $\mathbb{N}$. Fix $x > 0$ in $\mathbb{R}$ and let $x_n = x(1 + \frac{1}{n})$ where $n \in \mathbb{N}$. Then clearly $\{x_n\}_{n\in\mathbb{N}}$ is forward convergent to x and so $\{x_n\}_{n\in\mathbb{N}}$ forward I-converges to x for any ideal I. But since $\{n \in \mathbb{N} : \ d(x_n, x) \geq \frac{1}{2}\} = \mathbb{N} \notin I$ hence $\{x_n\}_{n\in\mathbb{N}}$ is not backward I-convergent to x for any ideal I.

However we can prove the following result which improves [3: Proposition 3.3].

Theorem 3.2. *Let (X, d) be an asymmetric metric space satisfying the property* (AMA). *Then forward I-convergence of $\{x_n\}_{n\in\mathbb{N}}$ implies the backward I-convergence and the limits are same.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be FI-convergent and let $FI\text{-}\lim_{n\to\infty} x_n = x_0$. Now $d(x_n, x_0) \le c(x_0, x_n)d(x_0, x_n) \le C(x_0)d(x_0, x_n)$. Let $\varepsilon > 0$ be given. Then $\{n \in \mathbb{N} : d(x_n, x_0) \ge \varepsilon\} \subset \{n \in \mathbb{N} : c(x_0, x_n)d(x_0, x_n) \ge \varepsilon\} \subset \{n \in \mathbb{N} : C(x_0)d(x_0, x_n) \ge \varepsilon\} = \{n \in \mathbb{N} : d(x_0, x_n) \ge \frac{\varepsilon}{C(x_0)}\} \in I$ which implies that $BI\text{-}\lim_{n\to\infty} x_n = x_0$. $\square$

The next example shows that the condition (AMA) is only sufficient but not necessary.

Example 3.2. Let the function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined by

$$d(x, y) = \begin{cases} d(x, 0) + d(0, y) & \text{if } y \ne x, \\ 0 & \text{if } y = x, \end{cases}$$

where $d(x, 0) = \frac{1}{|x|}$ and $d(0, x) = \frac{1}{|x|^2}$ for $x \ne 0$.

Let $x \in \mathbb{R}^n$ and I be any non-trivial admissible ideal of $\mathbb{N}$. First let $x \ne 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be forward I-convergent to x. Then $\{n \in \mathbb{N} : x_n \ne x\} = \{n \in \mathbb{N} : d(x, 0) + d(0, x_n) \ge \frac{1}{|x|}\} = \{n \in \mathbb{N} : d(x, x_n) \ge \frac{1}{|x|}\} \in I$. Hence $\{n \in \mathbb{N} : x_n \ne x\} \in I$. In this case clearly $\{x_n\}_{n\in\mathbb{N}}$ is also backward I-convergent to x.

If $x = 0$, for each $\varepsilon > 0$, $\{n \in \mathbb{N} : d(x_n, 0) \ge \varepsilon\} = \{n \in \mathbb{N} : \frac{1}{|x_n|} \ge \varepsilon\} = \{n \in \mathbb{N} : \frac{1}{|x_n|^2} \ge \varepsilon^2\} = \{n \in \mathbb{N} : d(0, x_n) \ge \varepsilon^2\} \in I$. Thus in this case also forward I-convergence of $\{x_n\}_{n\in\mathbb{N}}$ implies backward I-convergence.

Now $B^+(0, \varepsilon) = \{0\} \cup \left\{y \in \mathbb{R}^n : |y| > \sqrt{\frac{1}{\varepsilon}}\right\}$. So if $y \in B^+(0, \varepsilon)$ then $|y|$ is not bounded above, and

$$\frac{d(y, 0)}{d(0, y)} = \frac{\frac{1}{|y|}}{\frac{1}{|y^2|}} = |y|.$$

Thus any function c satisfying $d(y, x) \le c(x, y)d(x, y)$ for every $x, y \in \mathbb{R}^n$ will be unbounded in any forward ball of 0. Hence $(\mathbb{R}^n, d)$ does not satisfy (AMA).

In [3] it was proved that if (X, d) is forward sequentially compact then $x_n \xrightarrow{B} x \implies x_n \xrightarrow{F} x$. It is not clear whether this result remains valid if convergence is replaced by I-convergence. However if we say that $S \subseteq X$ is forward I-sequentially compact if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in S has a forward convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with limit in S where $\{n_k\}_{k\in\mathbb{N}} \notin I$, then we have,

Theorem 3.3. *Let (X, d) be an asymmetric metric space which is forward I-sequentially compact. Then* $BI\text{-}\lim_{n\to\infty} x_n = x \implies FI\text{-}\lim_{n\to\infty} x_n = x$.

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{BI} x$ for some $x \in X$. If possible let $\{x_n\}_{n\in\mathbb{N}}$ be not FI-convergent to x. Then there exists $\varepsilon_0 > 0$ such that $A = \{n \in \mathbb{N} : \ d(x, x_n) \geq \varepsilon_0\} \notin I$. Consider the sequence $\{x_n\}_{n\in A}$. Now $\{x_n\}_{n\in A}$ is also backward I-convergent to x. Next by forward I-sequential compactness, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in A}$ and so of $\{x_n\}_{n\in\mathbb{N}}$ which is forward convergent where $\{n_k\}_{k\in\mathbb{N}} \notin I$. Let $x_{n_k} \xrightarrow{F} y \in X$ as $k \to \infty$. Then $x_{n_k} \xrightarrow{FI} y$. But since $x_{n_k} \xrightarrow{BI} x$, so by Theorem 3.1, $x = y$. So there exists $K \in \mathbb{N}$ such that for $k \geq K$ we have $d(x, x_{n_k}) < \varepsilon_0$ which is a contradiction. Hence $x_n \xrightarrow{FI} x$. □

Note 1. *If $I = I_0$ the ideal of all finite subset of $\mathbb{N}$, then* Theorem 3.3 *coincides with* [3: Lemma 4.2].

We now introduce the following definition.

Definition 10. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space X is said to be forward I^*-convergent to ζ if and only if there exists a set $M \in F(I)$ i.e. $\mathbb{N} \setminus M \in I$, $M = \{m_1 < m_2 < m_3 < \cdots < m_k < m_{k+1} < \dots\}$ such that $\lim_{k\to\infty} d(\zeta, x_{m_k}) = 0$, and we write $FI^*\text{-}\lim_{n\to\infty} x_n = \zeta$.

The notion of backward I^*-convergence can be similarly defined. If $\{x_n\}_{n\in\mathbb{N}}$ is forward I^*-convergent to x_0 and backward I^*-convergent to y_0 then it is easy to show that $x_0 = y_0$. Also in general forward I^*-convergence does not imply backward I^*-convergence as can be seen by taking the same sequence given in Example 3.1.

However we can easily prove the following result.

Theorem 3.4. *If (X, d) in an asymmetric metric space satisfying the property* (AMA)*, then $FI^*\text{-}\lim_{n\to\infty} x_n = \zeta$ implies $BI^*\text{-}\lim_{n\to\infty} x_n = \zeta$ for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.*

The converse of Theorem 3.4, is not generally true as can be seen by taking the same sequence as in Example 3.2.

Remark 3.1. It is easy to observe that (AMA) implies $\tau_+ \prec \tau_-$ i.e. the forward topology is finer than the backward topology (see [2: Proposition 3.3] for a possible outline of proof. Also the observation follows from the fact that the forward and backward topologies are both first countable T_1 topologies and so convergent sequences are sufficient to determine them entirely). Example 3.2 (which is same as [2: Example 3.7]) gives an example of an asymmetric metric space for which $\tau_+ \prec \tau_-$ but which does not satisfy (AMA). Hence the condition $'\tau_+ \prec \tau_-'$ is strictly weaker than (AMA). Since forward (backward) I and I^*-convergence are actually I and I^*-convergence with respect to forward (backward) topology, so it immediately follows that Theorem 3.2 and Theorem 3.4 can be proved under the weaker assumption that the forward topology is finer than the backward topology.

Theorem 3.5. *Let I be an admissible ideal. If $FI^*\text{-}\lim_{n\to\infty} x_n = \zeta$ then $FI\text{-}\lim_{n\to\infty} x_n = \zeta$. The same is true for backward case also.*

The proof is straightforward.

The general properties of $FI(BI)$-convergent sequences are similar to that of I-convergent sequences ([13], [16]). As far as the relationship between $FI(BI)$-convergence and $FI^*(BI^*)$-convergence is concerned we have the following.

Theorem 3.6. *Let (X, d) be an asymmetric metric space.*

(a) *If X has no forward (backward) accumulation point, then $FI(BI)$-convergence and $FI^*(BI^*)$-convergence coincide for each admissible ideal $I \subset 2^{\mathbb{N}}$.*
(b) *If X has an forward (backward) accumulation point ζ, then there exists an admissible ideal $I \subset 2^{\mathbb{N}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ of elements of X such that $FI(BI)\text{-}\lim_{n\to\infty} y_n = \zeta$ but $FI^*(BI^*)\text{-}\lim_{n\to\infty} y_n$ does not exist.*

The proof is parallel to [13: Theorem 3.1]. As in the case of metric spaces, the equivalence of $FI(BI)$-convergence and $FI^*(BI^*)$-convergence is governed by the condition (AP) as given below.

Condition (AP). An admissible ideal $I \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \dots\}$ belonging to I there exists a countable family of sets $\{B_1, B_2, B_3, \dots\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in I$.

Note that $B_j \in I$ for all $j \in \mathbb{N}$.

We can now prove the following result.

Theorem 3.7. *Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal.*

(i) *If the ideal I has the property* (AP) *and (X,d) is an arbitrary asymmetric metric space, then for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X, $FI(BI)$-$\lim_{n\to\infty} x_n = \zeta$ implies $FI^*(BI^*)$-$\lim_{n\to\infty} x_n = \zeta$.*

(ii) *If (X,d) has at least one forward (backward) accumulation point and for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X and for each $\zeta \in X$, $FI(BI)$-$\lim_{n\to\infty} x_n = \zeta$ implies $FI^*(BI^*)$-$\lim_{n\to\infty} x_n = \zeta$, then I has the property* (AP).

The proof is parallel to the proof of [13: Theorem 3.2] and so is omitted.

Note 2. *It should be mentioned that* Theorems 3.6 and 3.7 *are actually special cases of* [16: Theorem 8] *when applied to forward (backward) topology.*

4. Asymmetric I-Cauchy conditions

As mentioned before, here the I-Cauchy condition (by Dems [10], see also [1] and [22]) gives rise to two Cauchy conditions associated with forward and backward I-convergence in an asymmetric space, which naturally extends the notions of forward and backward Cauchy conditions ([23], see also [3]).

Definition 11. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space (X,d) is said to be forward I-Cauchy if for each $\varepsilon > 0$ there exists an $M = M(\varepsilon) \in F(I)$ such that for any $k \in M$, $\{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\} \in I$, denoted by FI-Cauchy.

Similarly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space (X,d) is said to be backward I-Cauchy if for each $\varepsilon > 0$ there exists an $M = M(\varepsilon) \in F(I)$ such that for any $k \in M$, $\{n \in \mathbb{N} : d(x_n, x_k) \geq \varepsilon\} \in I$, denoted by BI-Cauchy.

As in the case of I-convergence, one can observe that in general forward I-Cauchy condition does not imply backward I-Cauchy condition and they are equivalent if (X,d) satisfies the property (AMA) which is however not necessary.

Lemma 4.1. *Let (X,d) be an asymmetric metric space. Then for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,d), the following conditions are equivalent:*

(1) *$\{x_n\}_{n\in\mathbb{N}}$ is an FI-Cauchy sequence.*

(2) *For any $\varepsilon > 0$, $\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\} \in I$, where $E_k(\varepsilon) = \{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\}$ for $k \in \mathbb{N}$.*

Proof.

(1) $\implies$ (2): Let $\varepsilon > 0$ be given. Then there exists $M \in F(I)$ such that for any $k \in M, \{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\} \in I$ i.e. $E_k(\varepsilon) \in I$. So $E_k(\varepsilon) \notin I$ implies $k \notin M$. Consequently $\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\} \subset \mathbb{N} \setminus M \in I$.

(2) $\implies$ (1): Write $M = \{k \in \mathbb{N} : E_k(\varepsilon) \in I\}$. Then $M \in F(I)$. Clearly $k \in M$ implies $E_k(\varepsilon) \in I$ i.e. $\big\{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\big\} \in I$. □

The following example shows that as forward convergence does not imply forward Cauchy condition, also in general forward I-convergence does not imply forward I-Cauchy condition.

Example 4.1. As before let the function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ be defined by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x, \\ 1 & \text{if } y < x. \end{cases}$$

Let I be any non-trivial admissible ideal of $\mathbb{N}$. Then the sequence $\big\{\frac{1}{n}\big\}_{n\in\mathbb{N}}$ is forward I-convergent to 0 because for each $\varepsilon > 0$, $\big\{n \in \mathbb{N} : d(0, x_n) \geq \varepsilon\big\} = \big\{n \in \mathbb{N} : \frac{1}{n} \geq \varepsilon\big\} = \big\{1, 2, 3, \ldots, \big[\frac{1}{\varepsilon}\big]\big\} \in I$. But $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is not FI-Cauchy. Because for any $k \in \mathbb{N}$ and $\varepsilon = \frac{1}{2}$, $E_k(\varepsilon) = \big\{n \in \mathbb{N} : d(\frac{1}{k}, \frac{1}{n}) \geq \varepsilon\big\} \supset \big\{k + 1, k + 2, k + 3, \ldots\big\} \notin I$ since I is admissible and non-trivial. So $\big\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\big\} = \mathbb{N} \notin I$.

However we have:

Theorem 4.1. *Let (X, d) be an asymmetric metric space satisfying the property* (AMA). *In this situation, if a sequence $\{x_n\}_{n\in\mathbb{N}}$ is $FI(BI)$-convergent then it is $FI(BI)$-Cauchy.*

Proof. Let $FI\text{-}\lim_{n\to\infty} x_n = x_0$. Let $\varepsilon > 0$ be given. Since $C(x_0) > 0$, $A\big(\frac{\varepsilon}{2C(x_0)}\big) = \{n \in \mathbb{N} : d(x_0, x_n) \geq \frac{\varepsilon}{2C(x_0)}\} \in I$. Let $k \notin A(\frac{\varepsilon}{2C(x_0)})$. Then $d(x_0, x_k) < \frac{\varepsilon}{2C(x_0)}$. Consequently $d(x_k, x_n) \leq d(x_k, x_0) + d(x_0, x_n) \leq c(x_0, x_k)d(x_0, x_k) + d(x_0, x_n) \leq C(x_0)d(x_0, x_k) + d(x_0, x_n) < \frac{\varepsilon}{2} + d(x_0, x_n)$.

Hence $d(x_k, x_n) \geq \varepsilon \implies d(x_0, x_n) \geq \frac{\varepsilon}{2}$ and so we have $\big\{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\big\} \subset \big\{n \in \mathbb{N} : d(x_0, x_n) \geq \frac{\varepsilon}{2}\big\} \in I$. Therefore $E_k(\varepsilon) = \big\{n \in \mathbb{N} : d(x_k, x_n) \geq \varepsilon\big\} \in I$. Hence $\big\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\big\} \subset A\big(\frac{\varepsilon}{2C(x_0)}\big)$ and so belongs to I. This implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is forward I-Cauchy.

The proof for backward case is similar. □

The above condition is sufficient but not necessary as shown by:

Example 4.2. The asymmetric metric space of Example 3.2, does not satisfy the property (AMA). But we have already shown that if $\{x_n\}_{n\in\mathbb{N}}$ is forward I-convergent to x where $x \neq 0$ then $\{n \in \mathbb{N} : \ x_n \neq x\} \in I$. Then for any $\varepsilon > 0$, choosing $k \in \mathbb{N}$ for which $x_n = x$ it is easy to see that $\{n \in \mathbb{N} : \ n \geq k$ and $d(x_k, x_n) \geq \varepsilon\} \in I$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is also forward I-Cauchy. On the other hand if $x = 0$, first note that if $\{x_n\}_{n\in\mathbb{N}}$ is FI-convergent to 0 then it is also BI-convergent to 0. For $\varepsilon > 0$, choose $M = \big\{k \in \mathbb{N} : \ d(x_k, 0) < \frac{\varepsilon}{2}\big\} \in F(I)$. Then for $k \in M$, $\big\{n \in \mathbb{N} : \ d(x_k, x_n) \geq \varepsilon\big\} = \big\{n \in \mathbb{N} : \ d(x_k, 0) + d(0, x_n) \geq \varepsilon\big\} \subset \big\{n \in \mathbb{N} : \ d(0, x_n) \geq \frac{\varepsilon}{2}\big\} \in I$. This shows that $\{x_n\}_{n\in\mathbb{N}}$ is again forward I-Cauchy.

Remark 4.1. Theorem 4.1 can also be proved under the weaker assumption that $\tau_+ \prec \tau_-$. Note that as the sequence $\{x_n\}_{n\in\mathbb{N}}$ is FI-convergent to x_0 (say) so it is also $BI-$ convergent to x_0 by Remark 3.1. therefore

$$A = \big\{k \in \mathbb{N} : \ d(x_0, x_k) \geq \tfrac{\varepsilon}{2}\big\} \in I$$

and

$$B = \big\{n \in \mathbb{N} : \ d(x_n, x_0) \geq \tfrac{\varepsilon}{2}\big\} \in I$$

and for any $k \notin A$ we get $\big\{n \in \mathbb{N} : \ d(x_n, x_k) \geq \varepsilon\big\} \subset B \in I$. From this we can arrive at the conclusion.

Theorem 4.2. *If every $FI(BI)$-Cauchy sequence in X is $FI(BI)$-convergent in X then X is forward (backward) complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a forward Cauchy sequence in (X, d). Let $\varepsilon > 0$ be given. Then there is a $P \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n \geq P$. Then $d(x_P, x_n) < \varepsilon$ for all $n \geq P$. Now $E_p(\varepsilon) = \{n \in \mathbb{N} : \ d(x_P, x_n) \geq \varepsilon\} = \{1, 2, 3, \dots, P-1\} \in I$ and by same reasoning $E_k(\varepsilon) \in I$ for all $k > P$. Hence $\{k \in \mathbb{N} : \ E_k(\varepsilon) \notin I\} \subset \{1, 2, \dots, P-1\} \in I$, and so $\{x_n\}_{n\in\mathbb{N}}$ is a FI-Cauchy sequence. By our assumption $\{x_n\}_{n\in\mathbb{N}}$ is a FI-convergent sequence in X and so there exists a point $x_0 \in X$ such that $FI\text{-}\lim_{n\to\infty} x_n = x_0$. Then $A(\varepsilon) = \{n \in \mathbb{N} : \ d(x_0, x_n) \geq \varepsilon\} \in I$ for any $\varepsilon > 0$. Put $k_0 = 0$ and for $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, pick inductively $k_n \in \mathbb{N} \setminus (\{0, 1, 2, \dots, k_{n-1}\} \cup A(\varepsilon_n))$. Then $d(x_0, x_{k_n}) < \frac{1}{n}$ for all $n \in \mathbb{N}$ and so $\{x_{k_n}\}_{n\in\mathbb{N}}$ is a forward convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Therefore by [2: Lemma 4.3] (X, d) is forward complete.

The proof for backward case is similar. □

For the converse we have the following result (though we had initially proved the result using (AMA), the following proof was suggested by one of the referees which shows that the result can be proved without the requirement of (AMA)).

Theorem 4.3. *If (X,d) is a forward (backward) complete asymmetric metric space then every $FI(BI)$-Cauchy sequence is $FI(BI)$-convergent in X.*

Proof. For any $j \in \mathbb{N}$ we put $\varepsilon_j = \frac{1}{2^j}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is FI-Cauchy, for each $j \in \mathbb{N}$ there exists $M_j \in F(I)$ such that

$$(\forall k \in M_j)\Big(E_k(\varepsilon_j) = \big\{n \in \mathbb{N}:\ d(x_k, x_n) \geq \varepsilon_j\big\} \in I\Big).$$

We choose $k_1 \in M_1$, $k_2 \in M_2 \setminus [E_{k_1}(\varepsilon_1) \cup \{1,2,3,\dots,k_1\}]$, $k_3 \in M_3 \setminus [E_{k_2}(\varepsilon_2) \cup \{1,2,3,\dots,k_2\}]$ etc. Note that $M_{j+1} \setminus [E_{k_j}(\varepsilon_j) \cup \{1,2,3,\dots,k_j\}]$ is always non-empty, since $M_{j+1} \in F(I)$ and $E_{k_j}(\varepsilon_j) \cup \{1,2,3,\dots,k_j\} \in I$. In this way we obtain a sequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ with the property

$$d(x_{k_j}, x_{k_{j+1}}) < \varepsilon_j = \frac{1}{2^j}$$

which implies easily

$$d(x_{k_j}, x_{k_l}) \leq \frac{1}{2^{j-1}}$$

for each $l > j$. □

Hence the sequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ is forward Cauchy and, consequently, it is forward convergent to some point x_0.

Now, for any given $\varepsilon > 0$ we can choose j such that $d(x_0, x_{k_j}) < \frac{\varepsilon}{2}$ and $\varepsilon_j < \frac{\varepsilon}{2}$, and thus

$$d(x_0, x_n) \leq d(x_0, x_{k_j}) + d(x_{k_j}, x_n) < \varepsilon$$

whenever $n \notin E_{k_j}(\varepsilon_j)$. Since $E_{k_j}(\varepsilon_j) \in I$, this implies the forward I-convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to x_0.

Lemma 4.2. *Let (X,d) be an asymmetric metric space. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ is FI-Cauchy then for any $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $\big\{n \in \mathbb{N}:\ n \geq k,$ $d(x_k, x_n) \geq \varepsilon\big\} \in I$.*

Proof. For any $\varepsilon > 0$, $\big\{k \in \mathbb{N}:\ E_k(\varepsilon) \in I\big\} \neq \emptyset$ (by Lemma 4.1) which implies that $\{x_n\}_{n\in\mathbb{N}}$ satisfies the above condition. □

Combining Lemmas 4.1 and 4.2 in the following theorem we show that FI-Cauchy condition can be formulated in a different but equivalent form as in [10] under some additional assumption.

Theorem 4.4. *Let (X,d) be an asymmetric metric space which satisfies the property* (AMA) *where the corresponding function $C(x)$ is bounded. Then for a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in (X,d), the following conditions are equivalent:*

(1) *$\{x_n\}_{n\in\mathbb{N}}$ is an $FI(BI)$-Cauchy sequence.*

(2) *For any* $\varepsilon > 0$, *there is a* $k \in \mathbb{N}$ *such that* $\{n \in \mathbb{N} : n \geq k, \ d(x_k, x_n) \geq \varepsilon\} \in I$.

(3) *For any* $\varepsilon > 0$ *there exists a set* $D \in I$ *such that* $m, n \notin D \implies d(x_m, x_n) < \varepsilon$.

(4) *For any* $\varepsilon > 0$, $\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\} \in I$.

Proof.

(1) $\implies$ (2). Follows from Lemma 4.2.

(2) $\implies$ (3). Let $\varepsilon > 0$ be given. Since $C(x)$ bounded we can choose a positive real number M such that $C(x) < M$. Put $D = E_k\big(\frac{\varepsilon}{M+1}\big)$ where $k \in \mathbb{N}$ is chosen for $\frac{\varepsilon}{M+1}$ in the I-Cauchy condition for $\{x_n\}_{n\in\mathbb{N}}$. Thus $D \in I$ and any $m, n \notin D \implies d(x_k, x_m) < \frac{\varepsilon}{M+1}$ and $d(x_k, x_n) < \frac{\varepsilon}{M+1}$.

Now $d(x_m, x_n) \leq d(x_m, x_k) + d(x_k, x_n) \leq c(x_k, x_m)d(x_k, x_m) + d(x_k, x_n) \leq C(x_k)d(x_k, x_m) + d(x_k, x_n) \leq Md(x_k, x_m) + d(x_k, x_n) \leq \frac{M\varepsilon}{M+1} + \frac{\varepsilon}{M+1} = \varepsilon$.
(3) $\implies$ (4). Let $\varepsilon > 0$ and D is chosen as in (3). We shall show that $\{k \in \mathbb{N} : E_k(\varepsilon) \notin I\} \subset D$. Let $k \in \mathbb{N}$ be such that $E_k(\varepsilon) \notin I$. Suppose that $k \notin D$. Pick an $n \in E_k(\varepsilon) \setminus D$. Thus $d(x_k, x_n) \geq \varepsilon$ by definition of $E_k(\varepsilon)$. But $k, n \notin D$ implies $d(x_k, x_n) < \varepsilon$ by (3), a contradiction.

(4) $\implies$ (1). Follows from Lemma 4.1.

The proof for backward case is similar. □

Remark 4.2. It should be noted in Theorem 4.4 that the condition (3) implies the $FI(BI)$-Cauchy condition in any asymmetric metric space (this fact will be used in Theorem 5.3). However we need the additional assumption on the asymmetric metric space only to prove the implication (1) $\implies$ (3). That the assumption is essential can be easily observed by taking the sequence $\{\frac{n-1}{n}\}_{n\in\mathbb{N}}$ in the asymmetric metric space of Example 4.1, which is I-forward Cauchy but it does not satisfy the condition (3).

That the condition (AMA) is not necessary can be easily checked by taking the asymmetric metric space of Example 3.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be FI-Cauchy.

Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $\sqrt{\delta} < \varepsilon$. Then there is a $K \in \mathbb{N}$ such that $A = \{n \in \mathbb{N} : n \geq K \text{ and } d(x_K, x_n) \geq \frac{\delta}{16}\} \in I$. Choose $D = A \cup \{1, 2, 3, \dots, K-1\}$. Then $m, n \notin D \implies d(x_K, x_m) < \frac{\delta}{16}$, $d(x_K, x_n) < \frac{\delta}{16}$. But then $\frac{1}{|x_K|} < \frac{\delta}{16}$ and $\frac{1}{|x_m|} < \frac{\sqrt{\delta}}{4}$. Hence $d(x_m, x_n) \leq d(x_m, x_K) + d(x_K, x_n) < \frac{1}{|x_m|} + \frac{1}{|x_K|^2} + \frac{\delta}{16} < \sqrt{\delta} < \varepsilon$.

5. Asymmetric I^*-Cauchy conditions and condition (AP)

Just as $FI(BI)$-Cauchy condition can be formulated from the concept of $FI(BI)$-convergence, it seems natural to consider a Cauchy like condition associated with $FI^*(BI^*)$-convergence. In this section we do that and extend the results of [22] and [7] to asymmetric context. Our investigation reveals that symmetry is not required for any of these results and we can prove them under the condition (AMA).

We will now consider the following definition:

Definition 12. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space (X,d) is said to be forward I^*-Cauchy if there exists a $M \in F(I)$ such that the sequence $\{x_n\}_{n\in M}$ is forward Cauchy. It is denoted by FI^*-Cauchy.

Similarly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an asymmetric metric space (X,d) is said to be backward I^*-Cauchy if there exists a $M \in F(I)$ such that the sequence $\{x_n\}_{n\in M}$ is backward Cauchy. It is denoted by BI^*-Cauchy.

Theorem 5.1. *In an asymmetric metric space (X,d) an $FI^*(BI^*)$-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ is also $FI(BI)$-Cauchy.*

The proof is parallel to [22: Theorem 3].

The following example shows that the converse of the above theorem is not always true.

Example 5.1. Let (X,d) be the asymmetric metric space of Example 2.1. Let $\mathbb{N} = \bigcup_{j\in\mathbb{N}} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j \neq \emptyset$ for $i \neq j$. Let I be the class of all those subsets A of $\mathbb{N}$ that can intersect only finite number of Δ_i's. Then I is a non-trivial admissible ideal.

Now $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is forward Cauchy in (X,d). Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = \frac{1}{j}$ if $n \in \Delta_j$. Let $\varepsilon > 0$ be given. Then there is a $k \in \mathbb{N}$ such that $d(\frac{1}{n}, \frac{1}{m}) < \frac{\varepsilon}{2}$ whenever $n > m \geq k$. Now $B = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k \in I$ and clearly $m,n \notin B \implies d(x_m,x_n) < \varepsilon$. Thus for any $k \in B^c \in F(I)$, $\{n \in \mathbb{N} : d(x_k,x_n) \geq \varepsilon\} \subset B$ and so belongs to I. Hence $\{x_n\}_{n\in\mathbb{N}}$ is FI-Cauchy.

Next we shall show that $\{x_n\}_{n\in\mathbb{N}}$ is not FI^*-Cauchy. If possible assume that $\{x_n\}_{n\in\mathbb{N}}$ is FI^*-Cauchy. Then there is a $A \in F(I)$ such that $\{x_n\}_{n\in A}$ is forward Cauchy. Since $\mathbb{N}\setminus A \in I$ so there exists a $l \in \mathbb{N}$ such that $\mathbb{N}\setminus A \subset \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_l$. But then $\Delta_i \subset A$ for all $i > l$. In particular $\Delta_{l+1}, \Delta_{l+2} \subset A$. From the construction of Δ_j's it clearly follows that given any $k \in \mathbb{N}$ there are $m \in \Delta_{l+1}$ and $n \in \Delta_{l+2}$ such that $m,n \geq k$. Hence there is no $k \in \mathbb{N}$ such that whenever

$m, n \in A$ with $m, n \geq k$ then $d(x_m, x_n) > \varepsilon_0$ where $\varepsilon_0 = \frac{1}{3(l+1)(l+2)} > 0$. This contradicts the fact that $\{x_n\}_{n\in A}$ is forward Cauchy.

Observe that the above example is true for backward case also.

The following theorem may act as a converse.

Theorem 5.2. *Let (X, d) be an asymmetric metric space satisfying the condition of* Theorem 4.4 *and let I be an admissible ideal which satisfies the condition* (AP)*. Then for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, $FI(BI)$-Cauchy condition implies $FI^*(BI^*)$-Cauchy condition.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be FI-Cauchy. Then for each $p \in \mathbb{N}$, there exists a $K_p \in I$ such that $m, n \notin K_p$ implies $d(x_m, x_n) < \frac{1}{p}$. Let $A_1 = K_1$, $A_2 = K_2 \setminus K_1$, $A_3 = K_3 \setminus (K_1 \cup K_2)$, …, $A_i = K_i \setminus (K_1 \cup K_2 \cup K_3 \cup \dots \cup K_{i-1})$, and so on. Then $\{A_i : i = 1, 2, 3, \dots\}$ is a countable family of mutually disjoint sets in I. By the condition (AP) there exists a countable family of sets $\{B_i : i = 1, 2, 3, \dots\}$ in I such that $A_j \Delta B_j$ is finite for each $j \in \mathbb{N}$ and $B \in I$ where $\bigcup_j B_j = B$. Let $M = \mathbb{N} \setminus B$. Then $M \in F(I)$. We will show that $\{x_n\}_{n\in M}$ is forward Cauchy.

Let $\varepsilon > 0$ be given. Choose $l \in \mathbb{N}$ such that $\frac{1}{l} < \varepsilon$. Now

$$K_\ell \setminus B \subset \bigcup_{i=1}^{\ell} (A_i \setminus B) \subset \bigcup_{i=1}^{\ell} (A_i \setminus B_i).$$

Since by the condition (AP), $A_i \setminus B_i$ is finite for $i = 1$ to l, we can choose $n_0 \in \mathbb{N}$ such that $K_\ell \setminus B \subset \bigcup_{i=1}^{\ell} (A_i \setminus B_i) \subset \{1, 2, \dots, n_0\}$, i.e. $K_l \cap M \subset \{1, 2, \dots, n_0\}$. Since $M \in F(I)$ and I is admissible so we can choose $n_1 \in M$ such that $n_1 > n_0$. Now $i, j \in M$ and $i \geq j \geq n_1 \implies i, j \notin K_\ell \implies d(x_i, x_j) < \frac{1}{\ell} < \varepsilon$. This shows that $\{x_n : n \in M\}$ is forward Cauchy and so $\{x_n\}_{n\in N}$ is FI^*-Cauchy. The proof for backward case is similar. □

Theorem 5.3. *Let (X, d) be an asymmetric metric space containing at least one forward (backward) accumulation point, and satisfying the condition* (AMA)*. If for every sequence $\{x_n\}_{n\in\mathbb{N}}$, $FI(BI)$-Cauchy condition implies $FI^*(BI^*)$-Cauchy condition then I satisfies the condition* (AP)*.*

Proof. Let x_0 be a forward accumulation point of X. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct points in X such that $\{x_n\}_{n\in\mathbb{N}}$ forward convergent to x_0 and $x_n \neq x_0$ for all $n \in \mathbb{N}$. Suppose $\{A_i : i = 1, 2, 3, \dots\}$ is sequence of mutually disjoint non-empty sets from I. Define a sequence $\{y_n\}_{n\in\mathbb{N}}$ by $y_n = x_j$ if $n \in A_j$ and $y_n = x_0$ if $n \notin A_j$ for any $j \in \mathbb{N}$. Let $\delta > 0$ be given then there exists $\varepsilon > 0$ such that $\varepsilon < \min\{\delta, \delta C(x_0)\}$. Then there exists $l \in \mathbb{N}$ such that $d(x_0, x_n) < \frac{\varepsilon}{2C(x_0)}$ for all $n \geq l$. Then $A\left(\frac{\varepsilon}{2C(x_0)}\right) = \left\{n \in \mathbb{N} : d(x_0, y_n) \geq \frac{\varepsilon}{2C(x_0)}\right\} \subset A_1 \cup A_2 \cup \cdots \cup A_l$ and $A_1 \cup A_2 \cup \cdots \cup A_l \in I$. Now clearly $i, j \notin A\left(\frac{\varepsilon}{2C(x_0)}\right)$ implies that $d(x_0, y_i) < \frac{\varepsilon}{2C(x_0)}$ and $d(x_0, y_j) < \frac{\varepsilon}{2C(x_0)}$. So $d(y_i, y_j) \leq d(y_i, x_0) + d(x_0, y_j) \leq C(x_0)d(x_0, y_i) + d(x_0, y_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2C(x_0)} < \delta$. This shows that $\{y_n\}_{n\in\mathbb{N}}$ is FI-Cauchy sequence. By our assumption $\{y_n\}_{n\in\mathbb{N}}$ is then FI^*-Cauchy. Hence there exists $H \in I$ such that $B = \mathbb{N} \setminus H \in F(I)$ and $\{y_n\}_{n\in B}$ is forward Cauchy. Now let $B_j = A_j \cap H$ for $j \in \mathbb{N}$. Then each $B_j \in I$. Further $\bigcup_j B_j = H \cap \left(\bigcup_j A_j\right) \subset H$. Therefore $\bigcup_j B_j \in I$. Now for the sets $A_i \cap B$, $i \in \mathbb{N}$, the following three cases may arise:

Case I: Each $A_i \cap B$ is included in a finite subset of $\mathbb{N}$.

Case II: Only one of $A_i \cap B$'s, namely $A_k \cap B$ (say) is not included in a finite subset of $\mathbb{N}$.

Case III: More than one of $A_i \cap B$'s are not included in finite subsets of $\mathbb{N}$.

If (I) holds, then

$$A_i \triangle B_i = A_i \setminus B_i = A_i \setminus H = A_i \cap B$$

is included in a finite subset of $\mathbb{N}$ and this implies that I satisfies (AP) condition.

If (II) holds, then we redefine $B_k = A_k$ and $B_j = A_j \cap H$ for $j \neq k$. Then

$$\bigcup_{j\in\mathbb{N}} B_j \subset H \cup A_k$$

and so $\bigcup_j B_j \in I$. Also since $A_i \triangle B_i = A_i \cap B$ for $i \neq k$ and $A_k \triangle B_k = \emptyset$, so as in case (I) the criteria for (AP) condition are satisfied.

If (III) holds, then there exists $k, l \in \mathbb{N}$ with $k \neq l$ such that $A_k \cap B$ and $A_l \cap B$ are not finite subsets of $\mathbb{N}$. Let $\varepsilon_0 = \frac{d(x_k, x_l)}{2} > 0$. As $\{y_n\}_{n\in B}$ is forward Cauchy sequence, so for the above $\varepsilon_0 > 0$ there exists $k_0 \in \mathbb{N}$ such that $d(x_i, x_j) < \varepsilon_0$ for all $i \geq j \geq k_0$ and $i, j \in B$. Now since $A_k \cap B$ and $A_l \cap B$ are not finite subsets of $\mathbb{N}$, so we can choose $i \in A_k \cap B$ and $j \in A_l \cap B$ with $i, j > k_0$. But $y_i = x_k$ and $y_j = x_l$ and so $d(y_i, y_j) = d(x_k, x_l) > \varepsilon_0 > 0$ (in fact there is an infinite number of indices of B with that property). This contradicts the fact that $\{y_n\}_{n\in B}$ is forward Cauchy. Therefore case (III) cannot arise. And in view of case (I) and case (II) I satisfies (AP) condition. The proof for backward case is analogous. □

Open Problem. It is not clear whether Theorem 5.2 (where Theorem 4.4 is needed) and Theorem 5.3 can be proved without the condition (AMA). So it seems natural to ask whether these results can be proved under weaker condition or without any condition at all.

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