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INVARIANCE OF THE BAJRAKTAREVIĆ MEANS WITH RESPECT TO THE BECKENBACH-GINI MEANS

Janusz Matkowski

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ABSTRACT. Assuming that the involved functions are three times differentiable, we determine all Bajraktarević means which are invariant respect to the mean-type mapping of the Beckenbach-Gini mean type. Some applications in iteration theory and functional equation are presented.

1. Introduction

A function $M\colon I^2 \to \mathbb{R}$ is called a *mean* in an interval $I \subseteq \mathbb{R}$, if

$$\min(x,y) \le M(x,y) \le \max(x,y), \qquad x, y \in I.$$

The mean M is called *strict*, if these inequalities are sharp for all $x, y \in I$, $x \neq y$; and M is called *symmetric*, if $M(x,y) = M(y,x)$ for all $x, y \in I$. If M is a mean in I then $M(J^2) = J$ for every subinterval $J \subseteq I$, and, obviously, M is *reflexive*, i.e.

$$M(x,x) = x, \qquad x \in I.$$

Let $M, N\colon I^2 \to I$ be means. A mean $K\colon I^2 \to I$ is called *invariant with respect to the mean-type mapping* $(M,N)\colon I^2 \to I^2$ (briefly, K is (M,N)*-invariant*, if

$$K(M(x,y), N(x,y)) = K(x,y), \qquad x, y \in I.$$

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The mean K is also referred to as the Gauss composition of the means M and N. The invariant mean is useful when we are looking for the limits of the sequence of iterates of the mean-type mapping $(M, N): I^2 \to I^2$ (cf. Borwein–Borwein [3], Bullen–Mitrinović–Vasić [4], also [13], [14], [15]). For example, note that

$$G(A(x,y), H(x,y)) = G(x,y), \qquad x, y > 0,$$

where A, G, H denote, respectively, the arithmetic, geometric and harmonic mean, that is G is invariant with respect to the mean-type mapping (A, H). This allows to obtain the following nontrivial fact:

$$\lim_{n\to\infty} (A,H)^n = (G,G)\,,$$

where $(A, H)^n$ denotes the nth iterate of the mapping (A, H). Note also that the (A, H)-invariance of G is equivalent to the classical Pythagorean harmony proportion.

Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{K}$ be three classes of means in the interval I. Consider the following problem. Determine all means means $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $K \in \mathcal{K}$ such that K is (M, N)-invariant. In the case when $\mathcal{M} = \mathcal{N} = \mathcal{K} = \mathcal{A}$ where $\mathcal{A}$ is the class of weighted quasi-arithmetic means this problem leads to the functional equation

$$A_p^{[f]}(x,y) + A_{1-p}^{[g]}(x,y) = x + y, \qquad x, y \in I,$$

where $f, g: I \to \mathbb{R}$ are the continuous and strictly monotonic functions, $p \in (0, 1)$ is a fixed number and $A_p^{[f]}: I^2 \to I$ defined by

$$A_p^{[f]}(x,y) := f^{-1}\left(pf(x) + (1-p)f(y)\right), \qquad x, y \in I,$$

is a *weighted quasi-arithmetic mean* of a *generator* f and the *weight* p. In the case when $p = \frac{1}{2}$, the analytic solutions f, g of this equation were examined by O. Sutô [15], twice continuously differentiable solutions by J. Matkowski [14], and the continuously differentiable solutions by Z. Daróczy and Gy. Maksa [5]. A complete solution in the case $p = \frac{1}{2}$ was done by Z. Daróczy and Zs. Páles [7]. The invariance problem in the case when $p \in (0, 1)$ is arbitrary was first treated in J. Jarczyk and J. Matkowski [11] in the class of twice continuously differentiable functions and, recently, has been completely solved by J. Jarczyk [9].

Let the functions $f, g: I \to \mathbb{R}$ be continuous, $g(x) \neq 0$ for $x \in I$, and such that $\frac{f}{g}$ is one-to-one. Then the function $B^{[f,g]}: I^2 \to I$ defined by

$$B^{[f,g]}(x,y) = \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right), \qquad x, y \in I,$$

is a mean in I and it is called Bajraktarević mean of generators f and g (cf. [2]). $B^{[f,g]}$ is a strict mean, and it is a generalization of quasi-arithmetic mean. Note that in the case when $\frac{f}{g} = \mathrm{id}\,|_I$ we have $B^{[f,g]} = B^{[g]}$ where

$$B^{[g]}(x,y) := \frac{xg(x) + yg(y)}{g(x) + g(y)}, \qquad x, y \in I.$$

The mean $B^{[g]}$ is called *Beckenbach-Gini mean* of a *generator* g (cf. [4]).

In the present paper we deal with the invariance problem $K \circ (M, N) = K$ in the case when $K = B^{[f,g]}$, $M = B^{[f]}$ and $N = B^{[g]}$, i.e. we consider the composite functional equation

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]}.$$

Solving some differential equations, we prove that three times differentiable functions f and g satisfy this equation if, and only if, there is $p \in \mathbb{R}$, $p \neq 0$, such that

$$f(x) = a\mathrm{e}^{px}, \quad g(x) = b\mathrm{e}^{-px}, \qquad x \in I.$$

Applying this result we show that, for a fixed $p \in \mathbb{R}$, $p \neq 0$, a function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$, continuous on the diagonal $\{(x,x) : \ x \in \mathbb{R}\}$, satisfies the functional equation

$$\Phi\left(\frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px} + \mathrm{e}^{py}}, \frac{x\mathrm{e}^{-px} + y\mathrm{e}^{-py}}{\mathrm{e}^{-px} + \mathrm{e}^{-py}}\right) = \Phi\left(x, y\right), \qquad x, y \in \mathbb{R},$$

if, and only if, there is a continuous single variable function $\varphi\colon I \to \mathbb{R}$ such that

$$\Phi(x,y) = \varphi\left(x + y\right), \qquad x, y \in \mathbb{R}.$$

2. Main result

In the proof of the main result we need the following

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, g\colon I \to (0,\infty)$ are twice differentiable and $\frac{f}{g}$ is one-to-one. If $B^{[f,g]}$ is invariant with respect to the mean-type mapping $(B^{[f]}, B^{[g]})$ i.e., if*

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]}, \tag{1}$$

then there are $a, r \in \mathbb{R}$, $r \neq 0$, $a > 0$, such that

$$f(x) = a\mathrm{e}^{rx} g(x), \qquad x \in I. \tag{2}$$

P r o o f. Assume that the functions $f, g\colon I \to (0,\infty)$ satisfy (1). From the definition of the mean $B^{[f,g]}$ and (1) we have

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{f(B^{[f]}(x,y)) + f(B^{[g]}(x,y))}{g(B^{[f]}(x,y)) + g(B^{[g]}(x,y))}\right) = \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right), \qquad x,y \in I,$$

whence, for all $x, y \in I$,

$$\begin{aligned} &\left[f(B^{[f]}(x,y)) + f(B^{[g]}(x,y))\right][g(x)+g(y)] \\ = &\left[g(B^{[f]}(x,y)) + g(B^{[g]}(x,y))\right][f(x)+f(y)]. \end{aligned} \tag{3}$$

Differentiation of both sides with respect to x gives

$$\begin{aligned} &\left[f'(B^{[f]})\frac{\partial B^{[f]}}{\partial x} + f'(B^{[g]})\frac{\partial B^{[g]}}{\partial x}\right][g(x)+g(y)] + \left[f(B^{[f]}) + f(B^{[g]})\right]g'(x) \\ = &\left[g'(B^{[f]})\frac{\partial B^{[f]}}{\partial x} + g'(B^{[g]})\frac{\partial B^{[g]}}{\partial x}\right][f(x)+f(y)] + \left[g(B^{[f]}) + g(B^{[g]})\right]f'(x) \end{aligned}$$

and differentiation with respect to y gives

$$\begin{aligned} &\left[f'(B^{[f]})\frac{\partial B^{[f]}}{\partial y} + f'(B^{[g]})\frac{\partial B^{[g]}}{\partial y}\right][g(x)+g(y)] + \left[f(B^{[f]}) + f(B^{[g]})\right]g'(y) \\ = &\left[g'(B^{[f]})\frac{\partial B^{[f]}}{\partial y} + g'(B^{[g]})\frac{\partial B^{[g]}}{\partial y}\right][f(x)+f(y)] + \left[g(B^{[f]}) + g(B^{[g]})\right]f'(y), \end{aligned}$$

where $B^{[f]}$ and $B^{[g]}$ stand for $B^{[f]}(x,y)$ and $B^{[g]}(x,y)$. Subtracting the respective sides of these equalities and then dividing by $x-y$ we obtain

$$\begin{aligned} &\left[f'(B^{[f]})\frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x-y} + f'(B^{[g]})\frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x-y}\right] \\ &\cdot [g(x)+g(y)] + \left[f(B^{[f]}) + f(B^{[g]})\right]\frac{g'(x)-g'(y)}{x-y} \\ = &\left[g'(B^{[f]})\frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x-y} + g'(B^{[g]})\frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x-y}\right] \\ &\cdot [f(x)+f(y)] + \left[g(B^{[f]}) + g(B^{[g]})\right]\frac{f'(x)-f'(y)}{x-y}. \end{aligned} \tag{4}$$

Since

$$\frac{\partial B^{[f]}}{\partial x} = \frac{f(x)^2 + f(x)f(y) + xf'(x)f(y) - yf'(x)f(y)}{[f(x)+f(y)]^2}$$

and

$$\frac{\partial B^{[f]}}{\partial y} = \frac{f(y)^2 + f(x)f(y) + yf'(y)f(x) - xf'(y)f(x)}{[f(x)+f(y)]^2}$$

it is easy to verify that, for all $x \in I$,

$$\lim_{y\to x} B^{[f]}(x,y) = \lim_{y\to x} B^{[g]}(x,y) = x,$$

$$\begin{aligned} &\lim_{y\to x} \frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x-y} \\ &= \lim_{y\to x} \frac{\left[f(x)^2+xf'(x)f(y)-yf'(x)f(y)\right]-\left[f(y)^2+yf'(y)f(x)-xf'(y)f(x)\right]}{[f(x)+f(y)]^2(x-y)} \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

and

$$\lim_{y\to x} \frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x-y} = \frac{g'(x)}{g(x)}.$$

Thus, letting $y \to x$ in (4), we obtain, for all $x \in I$,

$$\begin{aligned} &2\left[f'(x)\frac{f'(x)}{f(x)} + f'(x)\frac{g'(x)}{g(x)}\right]g(x) + 2f(x)g''(x) \\ &= 2\left[g'(x)\frac{f'(x)}{f(x)} + g'(x)\frac{g'(x)}{g(x)}\right]f(x) + 2g(x)f''(x), \end{aligned}$$

whence, after obvious simplification, we get

$$\frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 = \frac{g''(x)}{g(x)} - \left(\frac{g'(x)}{g(x)}\right)^2, \qquad x \in I.$$

Note that this differential equation, with two unknown functions, can be written in the form

$$\left(\frac{f'}{f}\right)' = \left(\frac{g'}{g}\right)'.$$

It follows that, for some $r \in \mathbb{R}$,

$$\frac{f'}{f} = \frac{g'}{g} + r,$$

whence, for some $a > 0$,

$$\log \circ f(x) = \log \circ g(x) + rx + \log a, \qquad x \in I,$$

and, consequently,

$$f(x) = a\mathrm{e}^{rx} g(x), \qquad x \in I.$$

The injectivity of $\frac{f}{g}$ implies that $r \neq 0$. This completes the proof. □

In the proof we have obtained the following:

Remark 1. Under the assumption of Lemma 1,

$$\left(\frac{f'}{f}\right)' = \left(\frac{g'}{g}\right)'.$$

The main result reads as follows.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, g\colon I \to (0, \infty)$ are three times differentiable and $\frac{f}{g}$ is one-to-one. Then the following conditions are equivalent:*

(i) *$B^{[f,g]}$ is invariant with respect to the mean-type mapping $(B^{[f]}, B^{[g]})$, i.e.,*

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]};$$

(ii) *there are $a, b, p \in \mathbb{R}$, $p \neq 0$, $a, b > 0$, such that*

$$f(x) = a\mathrm{e}^{px}, \quad g(x) = b\mathrm{e}^{-px}, \qquad x \in I;$$

(iii) *there is $p \in \mathbb{R}$, $p \neq 0$, such that*

$$B^{[f,g]}(x,y) = \frac{x+y}{2}, \quad B^{[f]}(x,y) = \frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px} + \mathrm{e}^{py}}, \quad B^{[g]}(x,y) = \frac{x\mathrm{e}^{-px} + y\mathrm{e}^{-py}}{\mathrm{e}^{-px} + \mathrm{e}^{-py}}$$

for all $x, y \in \mathbb{R}$.

Proof. Assume that $B^{[f,g]}$ is invariant with respect to the mean-type mapping $(B^{[f]}, B^{[g]})$. Then equality (3) is satisfied. Differentiating three times both sides (3) with respect to x and then letting $y \to x$, we get the equation

$$\left(\frac{g'}{g}\right)^3 - \left(\frac{f'}{f}\right)^3 - \frac{g'}{g}\left(\frac{f'}{f}\right)' - \frac{f'}{f}\left(\frac{g'}{g}\right)' + 2\frac{f'}{f}\frac{f''}{f} - 2\frac{g'}{g}\frac{g''}{g} + \frac{g'''}{g} - \frac{f'''}{f} = 0. \tag{5}$$

From formula (2) of Lemma 1 we have

$$f(x) = a\mathrm{e}^{rx} g(x), \qquad x \in I,$$

for some $a, r \in \mathbb{R}$, $r \neq 0$, $a > 0$, whence, after obvious calculations,

$$\frac{f'}{f} = r + \frac{g'}{g}, \qquad \frac{f''}{f} = r^2 + 2r\frac{g'}{g} + \frac{g''}{g}, \qquad \frac{f'''}{f} = r^3 + 3r^2\frac{g'}{g} + 3r\frac{g''}{g} + \frac{g'''}{g}.$$

Setting these functions into equation (5) we obtain the differential equation

$$\left(\frac{g'}{g} + r\right)\left(\frac{g'}{g}\right)' = 0.$$

Note that if $\frac{g'}{g} + r = 0$ on a nonempty open subinterval $J \subset I$, then, clearly, $\left(\frac{g'}{g}\right)' = 0$ on J. It follows that g satisfies this differential equation iff

$$\left(\frac{g'}{g}\right)' = 0.$$

Solving this equation we obtain, for some $b, q \in \mathbb{R}$, $q \neq 0$, $b > 0$,

$$g(x) = b\mathrm{e}^{qx}, \qquad x \in I. \tag{6}$$

Since, by Remark 1,

$$\left(\frac{f'}{f}\right)' = 0,$$

we infer that, for some $a, p \in \mathbb{R}$, $p \neq 0$, $a > 0$,

$$f(x) = a\mathrm{e}^{px}, \qquad x \in I. \tag{7}$$

Substituting the functions f and g of the forms (6) and (7) into equation (3), by the analyticity of the involved function, we obtain the equality valid for all $x, y \in \mathbb{R}$. Differentiating both sides of this equality with respect to x and then setting $x = y = 0$ we get $p + q = 0$, which completes the proof of the implication (i) $\Longrightarrow$ (ii). Since the remaining implications are easy to verify, the proof is completed. □

Remark 2. Note that the case

$$B^{[f,g]} = B^{[f]} = B^{[g]}$$

cannot happen. Indeed, assume that there are $f, g \colon I \to (0, \infty)$ such that the $B^{[f]} = B^{[g]}$. Then

$$\frac{xf(x) + yf(y)}{f(x) + f(y)} = \frac{xg(x) + yg(y)}{g(x) + g(y)}, \qquad x, y \in I,$$

whence

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(y)}, \qquad x, y \in I.$$

Thus $\frac{f}{g}$ is a constant function and, consequently, the Bajraktarević mean $B^{[f,g]}$ does not exist.

Remark 3. For every $p \in \mathbb{R}$, the function $\mathcal{B}^{[p]} \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{B}^{[p]}(x, y) := \frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px} + \mathrm{e}^{py}}, \qquad x, y \in \mathbb{R},$$

is a mean, and $\mathcal{B}^{[0]}$ is the arithmetic mean. Obviously, for every $p \in \mathbb{R}$, the mean $\mathcal{B}^{[p]}$ is of the Beckenbach-Gini type.

3. An application

Let us quote the following ([13], [15]):

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval. If $M, N\colon I^2 \to I$ are continuous means such that at least one of them is strict, then*

(i) *the sequence of iterates $\left((M,N)^n\right)_{n\in\mathbb{N}}$ of the mean-type mapping (M,N) : $I^2 \to I^2$ converges to a continuous mean-type mapping $(K,K)\colon I^2 \to I^2$ where $K\colon I^2 \to I$ is a continuous mean;*

(ii) *K is (M,N)-invariant, i.e. $K \circ (M,N) = K$;*

(iii) *a continuous (M,N)-invariant mean is unique;*

(iv) *if the means M, N are strict, then so is K.*

Now we can prove the following:

Corollary 1. *Let $p \in \mathbb{R}$ be fixed.*

(i) *The arithmetic mean $A\colon \mathbb{R}^2 \to \mathbb{R}$, $A(x,y) = \frac{x+y}{2}$, is invariant with respect to the mean-type mapping*

$$\mathbb{R}^2 \ni (x,y) \longrightarrow \left(\frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px}+\mathrm{e}^{py}}, \frac{x\mathrm{e}^{-px} + y\mathrm{e}^{-py}}{\mathrm{e}^{-px}+\mathrm{e}^{-py}}\right);$$

(ii) *the sequence of iterates of this mapping converges pointwise to the mean-type mapping (A,A), that is*

$$\lim_{n\to\infty}\left(\frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px}+\mathrm{e}^{py}}, \frac{x\mathrm{e}^{-px} + y\mathrm{e}^{-py}}{\mathrm{e}^{-px}+\mathrm{e}^{-py}}\right)^n = \left(\frac{x+y}{2}, \frac{x+y}{2}\right), \qquad (x,y) \in \mathbb{R}^2; \tag{7}$$

(iii) *a function $\Phi\colon I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta := \left\{(x,x) :\ x \in \mathbb{R}\right\}$, satisfies the functional equation*

$$\Phi\left(\frac{x\mathrm{e}^{px} + y\mathrm{e}^{py}}{\mathrm{e}^{px}+\mathrm{e}^{py}}, \frac{x\mathrm{e}^{-px} + y\mathrm{e}^{-py}}{\mathrm{e}^{-px}+\mathrm{e}^{-py}}\right) = \Phi\left(x,y\right), \qquad x,y \in \mathbb{R}, \tag{8}$$

if, and only if, there is a continuous function in a single variable $\varphi\colon I \to \mathbb{R}$ such that

$$\Phi(x,y) = \varphi\left(x+y\right), \qquad x,y \in \mathbb{R}.$$

P r o o f. For $p = 0$ all parts are obvious.

Assume that $p \neq 0$. Part (i) follows from Theorem 1 and part (ii) follows from Theorem 1 and Theorem 2. To prove part (iii), note that from equation

(7), by induction, we get

$$\Phi(x,y)=\Phi\left(\left(\frac{xe^{px}+ye^{py}}{e^{px}+e^{py}},\frac{xe^{-px}+ye^{-py}}{e^{-px}+e^{-py}}\right)^n\right),\qquad n\in\mathbb{N}\ \ x,y\in\mathbb{R}.$$

Letting $n\to\infty$ and making use of (7) and the assumed continuity of Φ on the diagonal, we obtain

$$\Phi(x,y)=\Phi\left(\frac{x+y}{2},\frac{x+y}{2}\right),\qquad x,y\in\mathbb{R},$$

whence, setting $\varphi(u):=\Phi(\frac{u}{2},\frac{u}{2})$ for $u\in\mathbb{R}$, we get the desired form of Φ. Since the converse implication is easy to verify, the proof is completed. □

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Faculty of Mathematics
Informatics and Econometrics
University of Zielona Góra
Podgórna 50
PL–65246 Zielona Góra
POLAND

E-mail: J.Matkowski@wmie.uz.zgora.pl