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Mathematica Slovaca

DOI: 10.2478/s12175-013-0109-2
Math. Slovaca **63** (2013), No. 3, 469–478

A COMPLETELY MONOTONIC FUNCTION INVOLVING THE TRI- AND TETRA-GAMMA FUNCTIONS

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(*Communicated by Ján Borsík*)

ABSTRACT. The di-gamma function $\psi(x)$ is defined on $(0,\infty)$ by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ denote the polygamma functions, where $\Gamma(x)$ is the classical Euler's gamma function. In this paper we prove that a function involving the difference between $[\psi'(x)]^2 + \psi''(x)$ and a proper fraction of x is completely monotonic on $(0,\infty)$.

1. Introduction

We recall from [6: Chapter XIII] and [18: Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $n \geq 0$.

The famous Bernstein-Widder Theorem (see [18: p. 160, Theorem 12a]) states that a function $f(x)$ on $[0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x) = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\alpha(t) \tag{2}$$

2010 Mathematics Subject Classification: Primary 26A48, 33B15; Secondary 26A51, 26D10, 65R10.
Keywords: completely monotonic function, tri-gamma function, tetra-gamma function, polygamma function, inequality.
The last two authors were partially supported by the Natural Science Basic Research Plan in Shaanxi Province of China under grant No. 2011JM1015.

converges for $x \in [0, \infty)$. This says that a completely monotonic function $f(x)$ on $[0, \infty)$ is a Laplace transform of the measure $\alpha(t)$.

We also recall that the classical Euler gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \qquad x > 0. \tag{3}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or di-gamma function, and the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are respectively called the polygamma functions. In particular, the functions $\psi'(x)$ and $\psi''(x)$ are called the tri-gamma and tetra-gamma functions.

In [2: p. 208, (4.39)], it was established that the inequality

$$[\psi'(x)]^2 + \psi''(x) > \frac{p(x)}{900x^4(x+1)^{10}} \tag{4}$$

holds for $x > 0$, where

$$\begin{aligned} p(x) = 75x^{10} &+ 900x^9 + 4840x^8 + 15370x^7 + 31865x^6 + 45050x^5 \\ &+ 44101x^4 + 29700x^3 + 13290x^2 + 3600x + 450. \end{aligned} \tag{5}$$

For more information on the background, motivation, and history of this topic, please refer to [3, 5, 13, 15], the survey and expository papers [7, 16] and plenty of references cited therein.

The aim of this paper is to prove the complete monotonicity of the difference between two functions on both sides of the inequality (4).

Our main result may be stated as the following theorem.

Theorem 1. *The function*

$$g(x) = [\psi'(x)]^2 + \psi''(x) - \frac{p(x)}{900x^4(x+1)^{10}} \tag{6}$$

is completely monotonic on $(0, \infty)$*, where the function* $p(x)$ *is defined by* (5).

Remark 1. By the definition of completely monotonic functions and the above recited Bernstein-Widder Theorem, it is easy to see that our Theorem 1 is stronger than the inequality (4), so our Theorem 1 generalizes the inequality (4).

2. Proof of Theorem 1

Now we are in a position to verify our Theorem 1 by a simple but effectual approach, which has been used in [3,5,13,15] and others, and by a large amount of calculating derivatives.

By the recursion formula

$$\psi^{(n-1)}(x+1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n} \tag{7}$$

for $x > 0$ and $n \in \mathbb{N}$, see [1: pp. 258, 260; 6.3.5, 6.4.6], a direct calculation produces

$$\begin{aligned}
g(x) - g(x+1) &= \big[\psi'(x) - \psi'(x+1)\big]\big[\psi'(x) + \psi'(x+1)\big] \\
&\quad + \big[\psi''(x) - \psi''(x+1)\big] - \left[\frac{p(x)}{900x^4(x+1)^{10}} - \frac{p(x+1)}{900(x+1)^4(x+2)^{10}}\right] \\
&= \frac{1}{x^2}\left[2\psi'(x) - \frac{1}{x^2}\right] - \frac{2}{x^3} - \left[\frac{p(x)}{900x^4(x+1)^{10}} - \frac{p(x+1)}{900(x+1)^4(x+2)^{10}}\right] \\
&= \frac{2}{x^2}\left[\psi'(x) - \frac{1}{2x} - \frac{3}{4x^2} - \frac{28}{5(x+1)} + \frac{51}{10(x+2)} + \frac{251}{120(x+1)^2}\right. \\
&\quad + \frac{331}{120(x+2)^2} - \frac{7}{6(x+1)^3} + \frac{17}{12(x+2)^3} + \frac{13}{90(x+1)^4} + \frac{49}{72(x+2)^4} \\
&\quad - \frac{13}{180(x+1)^5} + \frac{47}{180(x+2)^5} - \frac{1}{120(x+1)^6} - \frac{1}{60(x+2)^6} \\
&\quad + \frac{1}{180(x+1)^7} - \frac{2}{45(x+2)^7} + \frac{1}{200(x+1)^8} - \frac{13}{600(x+2)^8} \\
&\quad \left. + \frac{1}{900(x+1)^9} - \frac{1}{450(x+2)^9} - \frac{1}{1800(x+1)^{10}} + \frac{1}{450(x+2)^{10}}\right] \\
&\triangleq \frac{2}{x^2}H(x).
\end{aligned}$$

Using the formula

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}\mathrm{e}^{-xt}\,\mathrm{d}t \tag{8}$$

for $r > 0$ and $x > 0$, see [1: p. 255, 6.1.1], and the integral representations

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\,\mathrm{d}t \tag{9}$$

for $n \in \mathbb{N}$ and $x > 0$, see [1: p. 260, 6.4.1], gives

$$\begin{aligned}
H(x) = \int_0^\infty \bigg(&\frac{t}{1-\mathrm{e}^{-t}} - \frac{1}{2} - \frac{3}{4}t - \frac{28}{5}\mathrm{e}^{-t} + \frac{51}{10}\mathrm{e}^{-2t} + \frac{251}{120}t\mathrm{e}^{-t} \\
&+ \frac{331}{120}t\mathrm{e}^{-2t} - \frac{7}{12}t^2\mathrm{e}^{-t} + \frac{17}{24}t^2\mathrm{e}^{-2t} + \frac{13}{540}t^3\mathrm{e}^{-t} + \frac{49}{432}t^3\mathrm{e}^{-2t} \\
&- \frac{13}{4320}t^4\mathrm{e}^{-t} + \frac{47}{4320}t^4\mathrm{e}^{-2t} - \frac{1}{14400}t^5\mathrm{e}^{-t} - \frac{1}{7200}t^5\mathrm{e}^{-2t} \\
&+ \frac{1}{129600}t^6\mathrm{e}^{-t} - \frac{2}{32400}t^6\mathrm{e}^{-2t} + \frac{1}{1008000}t^7\mathrm{e}^{-t} - \frac{13}{3024000}t^7\mathrm{e}^{-2t}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{36288000}t^8\mathrm{e}^{-t}-\frac{1}{18144000}t^8\mathrm{e}^{-2t}-\frac{1}{653184000}t^9\mathrm{e}^{-t}\\
&+\frac{1}{163296000}t^9\mathrm{e}^{-2t}\biggr)\mathrm{e}^{-xt}\,\mathrm{d}t\\
=&\frac{1}{653184000}\int_0^\infty\frac{1}{\mathrm{e}^t-1}\bigl[163296000(t-2)\mathrm{e}^{3t}-\mathrm{e}^{2t}\bigl(t^9-18t^8\\
&-648t^7-5040t^6+45360t^5+1965600t^4-15724800t^3\\
&+381024000t^2-1856131200t+3331238400\bigr)+\mathrm{e}^t\bigl(5t^9-54t^8\\
&-3456t^7-45360t^6-45360t^5+9072000t^4+58363200t^3\\
&+843696000t^2+435456000t+6989068800\bigr)-4\bigl(t^9-9t^8\\
&-702t^7-10080t^6-22680t^5+1776600t^4+18522000t^3\\
&+115668000t^2+450424800t+832809600\bigr)\bigr]\mathrm{e}^{-(x+2)t}\,\mathrm{d}t\\
\triangleq&\frac{1}{653184000}\int_0^\infty\frac{1}{\mathrm{e}^t-1}\theta(t)\mathrm{e}^{-(x+2)t}\,\mathrm{d}t.
\end{aligned}
$$

A straightforward computation yields

$$
\begin{aligned}
\theta'(t)&=163296000(3t-5)\mathrm{e}^{3t}-\bigl(4806345600-2950214400t\\
&\quad+714873600t^2-23587200t^3+4158000t^4+60480t^5-14616t^6\\
&\quad-1440t^7-27t^8+2t^9\bigr)\mathrm{e}^{2t}+\bigl(7424524800+2122848000t\\
&\quad+1018785600t^2+94651200t^3+8845200t^4-317520t^5-69552t^6\\
&\quad-3888t^7-9t^8+5t^9\bigr)\mathrm{e}^{t}-36\bigl(50047200+25704000t+6174000t^2\\
&\quad+789600t^3-12600t^4-6720t^5-546t^6-8t^7+t^8\bigr),\\
\theta''(t)&=489888000(3t-4)\mathrm{e}^{3t}-4\bigl(1665619200-1117670400t\\
&\quad+339746400t^2-7635600t^3+2154600t^4+8316t^5-9828t^6\\
&\quad-774t^7-9t^8+t^9\bigr)\mathrm{e}^{2t}+\bigl(9547372800+4160419200t\\
&\quad+1302739200t^2+130032000t^3+7257600t^4-734832t^5\\
&\quad-96768t^6-3960t^7+36t^8+5t^9\bigr)\mathrm{e}^{t}-144\bigl(6426000+3087000t\\
&\quad+592200t^2-12600t^3-8400t^4-819t^5-14t^6+2t^7\bigr),\\
\theta^{(3)}(t)&=4408992000(t-1)\mathrm{e}^{3t}-4\bigl(2213568000-1555848000t\\
&\quad+656586000t^2-6652800t^3+4350780t^4-42336t^5-25074t^6\\
&\quad-1620t^7-9t^8+2t^9\bigr)\mathrm{e}^{2t}+\bigl(13707792000+6765897600t\\
&\quad+1692835200t^2+159062400t^3+3583440t^4-1315440t^5
\end{aligned}
$$

$$
\begin{aligned}
&- 124488t^6 - 3672t^7 + 81t^8 + 5t^9\big)e^t - 1008\big(441000 + 169200t\\
&- 5400t^2 - 4800t^3 - 585t^4 - 12t^5 + 2t^6\big),\\
\theta^{(4)}(t) ={}& 4408992000(3t-2)e^{3t} - 16\big(717822000 - 449631000t\\
&+ 323303400t^2 + 1024380t^3 + 2122470t^4 - 58779t^5 - 15372t^6\\
&- 828t^7 + t^9\big)e^{2t} + \big(20473689600 + 10151568000t + 2170022400t^2\\
&+ 173396160t^3 - 2993760t^4 - 2062368t^5 - 150192t^6 - 3024t^7\\
&+ 126t^8 + 5t^9\big)e^t - 12096\big(14100 - 900t - 1200t^2 - 195t^3 - 5t^4 + t^5\big),\\
\theta^{(5)}(t) ={}& 13226976000(3t-1)e^{3t} - 16\big(986013000 - 252655200t\\
&+ 649679940t^2 + 10538640t^3 + 3951045t^4 - 209790t^5\\
&- 36540t^6 - 1656t^7 + 9t^8 + 2t^9\big)e^{2t} + \big(30625257600\\
&+ 14491612800t + 2690210880t^2 + 161421120t^3 - 13305600t^4\\
&- 2963520t^5 - 171360t^6 - 2016t^7 + 171t^8 + 5t^9\big)e^t\\
&+ 60480\big(180 + 480t + 117t^2 + 4t^3 - t^4\big),\\
\theta^{(6)}(t) ={}& 119042784000te^{3t} - 64\big(429842700 + 198512370t\\
&+ 332743950t^2 + 9220365t^3 + 1713285t^4 - 159705t^5\\
&- 21168t^6 - 810t^7 + 9t^8 + t^9\big)e^{2t} + \big(45116870400\\
&+ 19872034560t + 3174474240t^2 + 108198720t^3 - 28123200t^4\\
&- 3991680t^5 - 185472t^6 - 648t^7 + 216t^8 + 5t^9\big)e^t\\
&+ 120960\big(240 + 117t + 6t^2 - 2t^3\big),\\
\theta^{(7)}(t) ={}& 119042784000(1+3t)e^{3t} - 64\big(1058197770 + 1062512640t\\
&+ 693148995t^2 + 25293870t^3 + 2628045t^4 - 446418t^5 - 48006t^6\\
&- 1548t^7 + 27t^8 + 2t^9\big)e^{2t} + \big(64988904960 + 26220983040t\\
&+ 3499070400t^2 - 4294080t^3 - 48081600t^4 - 5104512t^5\\
&- 190008t^6 + 1080t^7 + 261t^8 + 5t^9\big)e^t - 362880\big(2t^2 - 39 - 4t\big),\\
\theta^{(8)}(t) ={}& 357128352000(2+3t)e^{3t} - 128\big(1589454090 + 1755661635t\\
&+ 731089800t^2 + 30549960t^3 + 1512000t^4 - 590436t^5 - 53424t^6\\
&- 1440t^7 + 36t^8 + 2t^9\big)e^{2t} + \big(91209888000 + 33219123840t\\
&+ 3486188160t^2 - 196620480t^3 - 73604160t^4 - 6244560t^5\\
&- 182448t^6 + 3168t^7 + 306t^8 + 5t^9\big)e^t - 1451520(t-1),\\
\theta^{(9)}(t) ={}& 3214155168000(1+t)e^{3t} - 128\big(4934569815 + 4973502870t\\
&+ 1553829480t^2 + 67147920t^3 + 71820t^4 - 1501416t^5 - 116928t^6
\end{aligned}
$$

$$
\begin{aligned}
&- 2592t^7 + 90t^8 + 4t^9\big)\mathrm{e}^{2t} + \big(124429011840 + 40191500160t\\
&+ 2896326720t^2 - 491037120t^3 - 104826960t^4 - 7339248t^5\\
&- 160272t^6 + 5616t^7 + 351t^8 + 5t^9\big)\mathrm{e}^t - 1451520,\\
\theta^{(10)}(t) &= \mathrm{e}^t\big[164620512000 + 45984153600t + 1423215360t^2 - 910344960t^3\\
&- 141523200t^4 - 8300880t^5 - 120960t^6 + 8424t^7 + 396t^8 + 5t^9\\
&- 512\big(3710660625 + 3263666175t + 827275680t^2 + 33645780t^3\\
&- 1840860t^4 - 926100t^5 - 63000t^6 - 1116t^7 + 54t^8 + 2t^9\big)\mathrm{e}^t\\
&+ 3214155168000(4 + 3t)\mathrm{e}^{2t}\big]\\
&\triangleq \mathrm{e}^t\theta_1(t),\\
\theta_1'(t) &= 3214155168000(11 + 6t)\mathrm{e}^{2t} - 512\big(6974326800 + 4918217535t\\
&+ 928213020t^2 + 26282340t^3 - 6471360t^4 - 1304100t^5 - 70812t^6\\
&- 684t^7 + 72t^8 + 2t^9\big)\mathrm{e}^t + 9\big(5109350400 + 316270080t\\
&- 303448320t^2 - 62899200t^3 - 4611600t^4 - 80640t^5\\
&+ 6552t^6 + 352t^7 + 5t^8\big),\\
\theta_1''(t) &= 8\big[1607077584000(7 + 3t)\mathrm{e}^{2t} - 64\big(11892544335 + 6774643575t\\
&+ 1007060040t^2 + 396900t^3 - 12991860t^4 - 1728972t^5 - 75600t^6\\
&- 108t^7 + 90t^8 + 2t^9\big)\mathrm{e}^t + 9\big(39533760 - 75862080t - 23587200t^2\\
&- 2305800t^3 - 50400t^4 + 4914t^5 + 308t^6 + 5t^7\big)\big],\\
\theta_1^{(3)}(t) &= 8\big[1607077584000(17 + 6t)\mathrm{e}^{2t} - 64\big(18667187910 + 8788763655t\\
&+ 1008250740t^2 - 51570540t^3 - 21636720t^4 - 2182572t^5\\
&- 76356t^6 + 612t^7 + 108t^8 + 2t^9\big)\mathrm{e}^t - 63\big(10837440 + 6739200t\\
&+ 988200t^2 + 28800t^3 - 3510t^4 - 264t^5 - 5t^6\big)\big],\\
\theta_1^{(4)}(t) &= 16\big[3214155168000(10 + 3t)\mathrm{e}^{2t} - 32\big(27455951565\\
&+ 10805265135t + 853539120t^2 - 138117420t^3 - 32549580t^4\\
&- 2640708t^5 - 72072t^6 + 1476t^7 + 126t^8 + 2t^9\big)\mathrm{e}^t\\
&- 945\big(224640 + 65880t + 2880t^2 - 468t^3 - 44t^4 - t^5\big)\big],\\
\theta_1^{(5)}(t) &= 16\big[3214155168000(23 + 6t)\mathrm{e}^{2t} - 32\big(38261216700\\
&+ 12512343375t + 439186860t^2 - 268315740t^3 - 45753120t^4\\
&- 3073140t^5 - 61740t^6 + 2484t^7 + 144t^8 + 2t^9\big)\mathrm{e}^t\\
&- 945\big(65880 + 5760t - 1404t^2 - 176t^3 - 5t^4\big)\big],\\
\theta_1^{(6)}(t) &= 64\big[3214155168000(13 + 3t)\mathrm{e}^{2t} - 8\big(50773560075
\end{aligned}
$$

$$
\begin{aligned}
&+ 13390717095t - 365760360t^2 - 451328220t^3 - 61118820t^4 \\
&- 3443580t^5 - 44352t^6 + 3636t^7 + 162t^8 + 2t^9\big)e^t \\
&- 945\big(1440 - 702t - 132t^2 - 5t^3\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
\theta_1^{(7)}(t) &= 64\big[3214155168000(29+6t)e^{2t} - 8\big(64164277170 \\
&\quad + 12659196375t - 1719745020t^2 - 695803500t^3 - 78336720t^4 \\
&\quad - 3709692t^5 - 18900t^6 + 4932t^7 + 180t^8 + 2t^9\big)e^t \\
&\quad + 2835\big(234 + 88t + 5t^2\big)\big],\\
\theta_1^{(8)}(t) &= 128\big[6428310336000(16+3t)e^{2t} - 4\big(76823473545 \\
&\quad + 9219706335t - 3807155520t^2 - 1009150380t^3 - 96885180t^4 \\
&\quad - 3823092t^5 + 15624t^6 + 6372t^7 + 198t^8 + 2t^9\big)e^t + 2835(44+5t)\big],\\
\theta_1^{(9)}(t) &= 128\big[6428310336000(35+6t)e^{2t} - 4\big(86043179880 \\
&\quad + 1605395295t - 6834606660t^2 - 1396691100t^3 - 116000640t^4 \\
&\quad - 3729348t^5 + 60228t^6 + 7956t^7 + 216t^8 + 2t^9\big)e^t + 14175\big],\\
\theta_1^{(10)}(t) &= 512e^t\big[6428310336000(19+3t)e^t - 87648575175 + 12063818025t \\
&\quad + 11024679960t^2 + 1860693660t^3 + 134647380t^4 + 3367980t^5 \\
&\quad - 115920t^6 - 9684t^7 - 234t^8 - 2t^9\big] \\
&\triangleq 512e^t\theta_2(t),\\
\theta_2'(t) &= 9\big[1340424225 + 2449928880t + 620231220t^2 + 59843280t^3 \\
&\quad + 1871100t^4 - 77280t^5 - 7532t^6 - 208t^7 - 2t^8 \\
&\quad + 714256704000(22+3t)e^t\big],\\
\theta_2''(t) &= 72\big[306241110 + 155057805t + 22441230t^2 + 935550t^3 \\
&\quad - 48300t^4 - 5649t^5 - 182t^6 - 2t^7 + 89282088000(25+3t)e^t\big],\\
\theta_2^{(3)}(t) &= 504\big[22151115 + 6411780t + 400950t^2 - 27600t^3 - 4035t^4 \\
&\quad - 156t^5 - 2t^6 + 12754584000(28+3t)e^t\big],\\
\theta_2^{(4)}(t) &= 6048\big[534315 + 66825t - 6900t^2 - 1345t^3 - 65t^4 - t^5 \\
&\quad + 1062882000(31+3t)e^t\big],\\
\theta_2^{(5)}(t) &= 30240\big[13365 - 2760t - 807t^2 - 52t^3 - t^4 \\
&\quad + 212576400(34+3t)e^t\big],\\
\theta_2^{(6)}(t) &= 60480\big[106288200(37+3t)e^t - 1380 - 807t - 78t^2 - 2t^3\big],\\
\theta_2^{(7)}(t) &= 181440\big[35429400(40+3t)e^t - 269 - 52t - 2t^2\big],
\end{aligned}
$$

$$\theta_2^{(8)}(t) = 725760\big[8857350(43+3t)\mathrm{e}^t - 13 - t\big],$$
$$\theta_2^{(9)}(t) = 725760\big[8857350(46+3t)\mathrm{e}^t - 1\big].$$

It is easy to calculate that

$$\begin{aligned}
\theta'(0) &= 0, & \theta''(0) &= 0,\\
\theta^{(3)}(0) &= 0, & \theta^{(4)}(0) &= 0,\\
\theta^{(5)}(0) &= 1632960000, & \theta^{(6)}(0) &= 17635968000,\\
\theta^{(7)}(0) &= 116321184000, & \theta^{(8)}(0) &= 602017920000,\\
\theta^{(9)}(0) &= 2706957792000, & \theta^{(10)}(0) &= 11121382944000,\\
\theta_1'(0) &= 31830835680000, & \theta_1''(0) &= 83910208435200,\\
\theta_1^{(3)}(0) &= 208999489144320, & \theta_1^{(4)}(0) &= 500203983121920,\\
\theta_1^{(5)}(0) &= 1163218362768000, & \theta_1^{(6)}(0) &= 2648180949926400,\\
\theta_1^{(7)}(0) &= 5932619924353920, & \theta_1^{(8)}(0) &= 13125845965639680,\\
\theta_1^{(9)}(0) &= 28754776198995840, & \theta_1^{(10)}(0) &= 62489726878118400,\\
\theta_2'(0) &= 141434891210025, & \theta_2''(0) &= 160729807759920,\\
\theta_2^{(3)}(0) &= 180003853569960, & \theta_2^{(4)}(0) &= 199280851953120,\\
\theta_2^{(5)}(0) &= 218562955581600, & \theta_2^{(6)}(0) &= 237847398969600,\\
\theta_2^{(7)}(0) &= 257132364632640, & \theta_2^{(8)}(0) &= 276417335013120,\\
\theta_2^{(9)}(0) &= 295702274730240.
\end{aligned}$$

Since $\theta_2^{(9)}(t)$ is increasing, so $\theta_2^{(9)}(t) > 0$ on $(0,\infty)$, which means that $\theta_2^{(8)}(t)$ is increasing and positive on $(0,\infty)$. By the same argument, it is derived that the functions $\theta_2^{(i)}(t)$ for $1 \le i \le 9$, $\theta_1^{(i)}(t)$ and $\theta^{(i)}(t)$ for $1 \le i \le 10$ are increasing and positive on $(0,\infty)$. Therefore, the function $\theta(t)$ is increasing and positive on $(0,\infty)$, which implies that the function $H(x)$ is completely monotonic on $(0,\infty)$. Because the function $\frac{2}{x^2}$ is completely monotonic on $(0,\infty)$ and the product of finitely many completely monotonic functions are also completely monotonic, we obtain that the function $g(x) - g(x+1)$ is completely monotonic on $(0,\infty)$, which is equivalent to

$$(-1)^k[g(x) - g(x+1)]^{(k)} = (-1)^k g^{(k)}(x) - (-1)^k g^{(k)}(x+1) \ge 0$$

for $k \le 0$ on $(0,\infty)$. By induction, we have

$$\begin{aligned}
(-1)^k g^{(k)}(x) &\ge (-1)^k g^{(k)}(x+1) \ge (-1)^k g^{(k)}(x+2) \ge \cdots\\
&\ge (-1)^k g^{(k)}(x+m) \ge \cdots \ge \lim_{m\to\infty}[(-1)^k g^{(k)}(x+m)] = 0
\end{aligned}$$

for $k \ge 0$ on $(0,\infty)$. The proof of Theorem 1 is complete.

Remark 2. An easy simplification yields that the function

$$H(x) = \psi'(x) - \frac{Q(x)}{1800x^2(1+x)^{10}(2+x)^{10}} \tag{10}$$

is completely monotonic on $(0, \infty)$, where

$$\begin{aligned} Q(x) = {} & 1382400 + 21657600x + 162792960x^2 + 778137600x^3 \\ & + 2645782983x^4 + 6789381590x^5 + 13626443025x^6 \\ & + 21889330810x^7 + 28579049475x^8 + 30634381522x^9 \\ & + 27125436630x^{10} + 19896883200x^{11} + 12088287630x^{12} \\ & + 6063596590x^{13} + 2494770300x^{14} + 832958400x^{15} \\ & + 222060150x^{16} + 46134540x^{17} + 7195500x^{18} \\ & + 792300x^{19} + 54900x^{20} + 1800x^{21} \end{aligned}$$

for $x \in (0, \infty)$.

Remark 3. This paper is a revised version of the preprint [12] and has been further developed in [3–5, 8–11, 13–15, 17, 19].

Acknowledgement. The authors appreciate the editor and anonymous referees for their valuable comments on this manuscript.

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Received 19. 11. 2010
Accepted 30. 5. 2011

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