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REFLEXIVE RINGS AND THEIR EXTENSIONS

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ABSTRACT. A right ideal I is reflexive if $xRy \in I$ implies $yRx \in I$ for $x, y \in R$. We shall call a ring R a reflexive ring if $aRb = 0$ implies $bRa = 0$ for $a, b \in R$. We study the properties of reflexive rings and related concepts. We first consider basic extensions of reflexive rings. For a reduced iedal I of a ring R, if R/I is reflexive, we show that R is reflexive. We next discuss the reflexivity of some kinds of polynomial rings. For a quasi-Armendariz ring R, it is proved that R is reflexive if and only if $R[x]$ is reflexive if and only if $R[x; x^{-1}]$ is reflexive. For a right Ore ring R with Q its classical right quotient ring, we show that if R is a reflexive ring then Q is also reflexive. Moreover, we characterize weakly reflexive rings which is a weak form of reflexive rings and investigate its properties. Examples are given to show that weakly reflexive rings need not be semicommutative. It is shown that if R is a semicommutative ring, then $R[x]$ is weakly reflexive.

1. Introduction

It is well known that a subgroup H of a group G is normal if and only if $xy \in H$ implies $yx \in H$ for all $x, y \in H$. This property, as extended to arbitrary subsets of semi-group and rings, was called *réflectif* in [15]. Subsequently, this notion was extended to an ideal for a ring R in [12]. According to [12], a right ideal I is reflexive if $xRy \in I$ implies $yRx \in I$ for $x, y \in R$. Hence we shall call a ring R a reflexive ring if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$

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for $a, b \in R$). Moreover, a right ideal I is called completely reflexive if $xy \in I$ implies $yx \in I$. A ring R is completely reflexive if (0) has the corresponding property. It is clear that every completely reflexive ring is reflexive. We note that completely reflexive rings are just those rings that Cohn defined in [3], which are called reversible rings. Anderson-Camillo [1], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa-Niewieczerzal [8] took the term C_0 for it. It is well known that every reduced ring (i.e., rings without nonzero nilpotent elements in R) is a completely reflexive ring. According to [13], a ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator over R is an ideal of R. We shall call a right (or left) ideal I of a ring R a semicommutative right (or left) ideal if $ab \in I$ implies $aRb \in I$.

First we consider basic examples of reflexive rings and some related rings. We show that a ring R is a completely reflexive ring if and only if R is a semicommutative reflexive ring. Moreover, it is shown that if R is reduced, then $R[x]/(x^n)$ is a reflexive ring, where (x^n) is the ideal generated by x^n and n is any positive integer. Secondly, we discuss the reflexivity of some kinds of polynomial rings. We prove that:

(1) $R[x]$ is reflexive if and only if $R[x; x^{-1}]$ is reflexive.

(2) If a right Ore ring R with Q its classical right quotient ring is reflexive, then Q is also reflexive.

It is shown that for a quasi-Armendariz ring R, R is reflexive if and only if $R[x]$ is reflexive if and only if $R[x; x^{-1}]$ is reflexive. Finally, we introduce the concept of weakly reflexive rings which is a weak form of reflexive rings and consider its properties. Examples are given to show that weakly reflexive rings need not be semicommutative. It is shown that if R is a semicommutative ring, then $R[x]$ is weakly reflexive, and that if R is a weakly reflexive ring then the n-by-n upper triangular matrix ring $T_n(R)$ over R is weakly reflexive.

Throughout this paper, R denotes an associative ring with identity and α denotes a nonzero and non-identity endomorphism, unless specified otherwise. For a ring R, we denote by $\mathrm{nil}(R)$ the set of all nilpotent elements of R.

2. Basic examples of reflexive rings and related rings

In this section we observe properties and basic extensions of reflexive rings and related concepts to reflexive rings, including some kinds of examples needed in the process. Note that the class of reflexive rings is closed under direct products. We begin with the following.

Lemma 2.1. *A ring R is a completely reflexive ring if and only if R is a semicommutative reflexive ring.*

It is well known that every completely reflexive ring is semicommutative [7: Lemma 1.4]. The following example shows that there exists a semicommutative ring which is not reflexive.

Example 2.1. Let R be a reduced ring. Then

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is a semicommutative ring by [7: Proposition 1.2]. It follows from the following implications that S is not reflexive:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

but for any $a \neq 0$ we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

However, we have the following proposition which shows one way to build new reflexive rings from old ones.

Proposition 2.2. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}$$

is a reflexive ring.

P r o o f. Let $\begin{pmatrix} a_1 & 0 & b_1 \\ 0 & a_1 & c_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & a_2 & c_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S$. We can denote their addition and multiplication by

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2),$$
$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 a_2, a_1 b_2 + b_1 a_2, a_1 c_2 + c_1 a_2),$$

respectively. For all $(a, b, c) \in S$, suppose that $(a_1, b_1, c_1)(a, b, c)(a_2, b_2, c_2) = 0$. We shall prove that $(a_2, b_2, c_2)(a, b, c)(a_1, b_1, c_1) = 0$. By the assumption, we have the following system of equations:

$$a_1aa_2 = 0, \tag{1}$$

$$a_1ab_2 + (a_1b + b_1a)a_2 = 0, \tag{2}$$

$$a_1ac_2 + (a_1c + c_1a)a_2 = 0. \tag{3}$$

In the following computations we freely use Lemma 2.1 and the condition that every reduced ring is completely reflexive. From Eq. (1), we see that $a_1aa_2 = 0$, and so $a_2aa_1 = 0$. Multiply Eq. (2) on the right hand side by a_2, then $a_1ab_2a_2 = 0$ since every reduced ring is semicommutative and $a_1aa_2 = 0$, and so $0 = a_1ab_2a_2 + (a_1b + b_1a)a_2a_2 = (a_1b + b_1a)a_2a_2$. This implies that $(a_1b + b_1a)a_2(a_1b + b_1a)a_2 = ((a_1b + b_1a)a_2)^2 = 0$. Hence

$$(a_1b + b_1a)a_2 = 0. \tag{4}$$

Next multiply Eq. (4) on the left side by a_1, then $a_1a_1ba_2 + a_1b_1aa_2 = a_1a_1ba_2 = 0$ and so $a_1ba_2 = 0$. Hence we have an equation

$$a_1ab_2 + b_1aa_2 = 0. \tag{5}$$

Multiply Eq. (5) on the right side by a_2, then we have $0 = a_1ab_2a_2 + b_1aa_2a_2 = b_1aa_2a_2$ and so $b_1aa_2 = 0$. This shows that $a_1ab_2 = 0$, hence $a_2ba_1 = 0$, $a_2ab_1 = 0$ and $b_2aa_1 = 0$ by the reflexivity of R.

Similarly from Eq. (3) we obtain $a_1ac_2 = 0$, $a_1ca_2 = 0$ and $c_1aa_2 = 0$. This implies that $c_2aa_1 = 0$, $a_2ca_1 = 0$ and $a_2ac_1 = 0$. Now by the preceding results we have proved that for all $(a, b, c) \in S$ if $(a_1, b_1, c_1)(a, b, c)(a_2, b_2, c_2) = 0$, then $(a_2, b_2, c_2)(a, b, c)(a_1, b_1, c_1) = 0$. Therefore, S is a reflexive ring. □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrix $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$, $m \in M$ and the usual matrix operations are used.

Corollary 2.2.1. *If R is a reduced ring, then T(R, R) is a reflexive ring.*

The following result, similar to [7: Theorem 2.5], extends the class of reflexive rings.

Proposition 2.3. *Let R be a ring and n any positive integer. If R is reduced, then $R[x]/(x^n)$ is a reflexive ring, where (x^n) is the ideal generated by x^n.*

In [2], the reversible property of a ring is extended to a ring endomorphism as follows: an endomorphism α of a ring R is called right (resp., left) reversible if whenever $ab = 0$ for $a, b \in R$, we have $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called right (resp., left) α-reversible if there exists a right (resp., left) reversible endomorphism α of R. R is α-reversible if it is both right and left α-reversible. The next example shows that a right α-reversible ring need not be reflexive.

Example 2.4. Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z} \right\}.$$

Let $\alpha\colon R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is right α-reversible by [2: Example 2.2]. On the other hand, if $a \neq 0$ then it follows from the following implications that R is not reflexive:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

One may suspect that R is a reflexive ring if for any reflexive nonzero proper ideal I of R, R/I reflexive, where I is considered as a ring without identity. However the following example erases the possibility.

Example 2.5. Let S be a division ring and consider the ring $R = \left(\begin{smallmatrix} S & S \\ 0 & S \end{smallmatrix}\right)$. Note that R has only the following nonzero proper ideals:

$$I_1 = \begin{pmatrix} S & S \\ 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & S \\ 0 & S \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that I_1 and I_3 are reflexive, this implies that R/I_1, R/I_3 are reflexive since $R/I_1 \cong S$, $R/I_3 \cong S \bigoplus S$. Moreover, it can be easily checked that I_2 and R are not reflexive.

However, we have an affirmative answer as in the following.

Proposition 2.6. *Let R be a ring and I be a proper ideal of R. If R/I is reflexive and I is reduced (as a ring without identity), then R is reflexive.*

Proof. Let $r_1, r_2 \in R$, suppose that $r_1 r r_2 = 0$ for all $r \in R$. Then $\bar{r_1}\bar{r}\bar{r_2} = \bar{0}$ and so $\bar{r_2}\bar{r}\bar{r_1} = \bar{0}$ since R/I is reflexive. This shows that $r_2 r r_1 \in I$. Since

I is reduced and every reduced ring is completely reflexive, it follows from $(r_2rr_1)(rr_2rr_1) = 0$ that $(rr_2rr_1)(r_2rr_1) = 0$ and so

$$(rr_2rr_1)(r_2rr_1)r_2 = r(r_2rr_1)(r_2rr_1)r_2 = 0.$$

This implies that $(r_2rr_1)(r_2rr_1)r_2r = 0$, thus

$$(r_2rr_1)(r_2rr_1r_2r)r_1 = (r_2rr_1)^3 = 0.$$

Therefore, we obtain $r_2rr_1 = 0$ since $r_2rr_1 \in I$ and I is reduced. $\square$

Based on the preceding result, one may suspect that the ring R may be reduced, and that the condition "I is reduced" can be replaced by "I is reflexive" However the following example erases the possibility.

Example 2.7. Let S be the ring in Example 2.1. Consider

$$I = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d \in R \right\}.$$

Then $S/I \cong R$ is reduced (and so reflexive). It is straightforward to verify that I is reflexive. But I is not reduced, and S is not reflexive as shown by Example 2.1. We note that S is clearly not completely reflexive.

However, we obtain the following proposition.

Proposition 2.8. *Let R be a reduced ring and I be an ideal of R that is an annihilator in R. Then R/I is a reflexive ring.*

Proof. It follows from [7: Proposition 1.14(1)] and the fact that every completely reflexive ring is reflexive. $\square$

Following [9], a right ideal I of a ring R is called symmetric if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$, so we shall call R symmetric if 0 is a symmetric ideal (i.e., $rst = 0$ implies $rts = 0$ for $r, s, t \in R$). An equivalent condition on a ring with unity is that whenever a product of any number of elements is zero, any permutation of the factors still yields product zero). It is clear that every symmetric ring is reflexive. The next example shows that there exists a reflexive ring which is not symmetric.

Example 2.9. Let k be a field. Define the free algebra $F = k\langle x, y, z\rangle$, and let $I = (FxF)^2 + (FyF)^2 + (FzF)^2 + FxyzF + FyzxF + FzxyF \subset F$. Put $R = F/I$, then R is a local, 13-dimensional k-algebra with vector space basis: $v_0 = 1$, $v_1 = x$, $v_2 = y$, $v_3 = z$, $v_4 = xy$, $v_5 = yx$, $v_6 = xz$, $v_7 = zx$, $v_8 = yz$, $v_9 = zy$, $v_{10} = xzy$, $v_{11} = zyx$, $v_{12} = yxz$. It can be easily checked that R is not symmetric. However, R is a reflexive ring.

Proposition 2.10. *If R is a completely reflexive ring, then $I = \{a \mid a^n = 0$ for some $n\}$ is a semicommutative ideal.*

Proof. Let $a, b \in I$, then $a^n = 0, b^m = 0$ for some n, m. Let $k = \min(m, n) + s$ for some $s \in \mathbb{N}$, then $(ab)^k = 0$ since R is completely reflexive. This implies that there exists $k \in \mathbb{N}$ such that $(a - b)^k = 0$. For any $r \in R$, it is straightforward to verify that $(ar)^n \in I$ and $(ra)^n \in R$, hence I is an ideal of R. Moreover, if $ab \in I$ then $(ab)^n = 0$ for some n, hence $(arb)^n = 0$ since R is completely reflexive. This implies that I is semicommutative. $\square$

3. Extensions of reflexive rings

In this section, we consider some kinds of polynomial extensions of reflexive rings. Let R be a ring and $\triangle$ be a multiplicative monoid in R consisting of central regular elements, and let $\triangle^{-1}R = \{u^{-1}a \mid u \in \triangle,\ a \in R\}$, then $\triangle^{-1}R$ is a ring. First we give the following equivalence.

Proposition 3.1. *Let R be a ring, then $R[x]$ is reflexive if and only if $\triangle^{-1}R[x]$ is reflexive.*

Proof. It suffices to show that $\triangle^{-1}R[x]$ is reflexive if $R[x]$ is reflexive. Let $f(x) = \sum\limits_{i=0}^{m} u_i^{-1}a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} v_j^{-1} b_j x^j \in \triangle^{-1}R[x]$ with $f(x)h(x)g(x) = 0$, where $h(x) = \sum\limits_{k=0}^{t} \gamma_k^{-1} c_k x^k$ is any element in $\triangle^{-1}R[x]$. Then we have

$$\left.\begin{aligned} F(x) &= (u_m u_{m-1} \ldots u_0) f(x) \\ H(x) &= (\gamma_t \gamma_{t-1} \ldots \gamma_0)\, h(x) \\ G(x) &= (v_n v_{n-1} \ldots v_0) g(x) \end{aligned}\right\} \in R[x].$$

Since $R[x]$ is reflexive and $F(x)H(x)G(x) = 0$, this implies that $G(x)H(x)F(x) = 0$, and so $g(x)h(x)f(x) = 0$ since $\triangle$ is a multiplicative monoid in R consisting of central regular elements and $u_i, v_j \in \triangle$ for all i, j. This implies that $\triangle^{-1}R$ is reflexive. $\square$

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum\limits_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. Denote it by $R[x; x^{-1}]$. As a sequence we obtain the following corollary.

Corollary 3.1.1. *For a ring R, $R[x]$ is reflexive if and only if $R[x;x^{-1}]$ is reflexive.*

According to [4], a ring R is called to be quasi-Armendariz if whenever polynomials $f(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m, g(x)=b_0+b_1x+b_2x^2+\cdots+b_nx^n\in R[x]$ satisfy $f(x)R[x]g(x)=0$, then $a_iRb_j=0$ for each i,j. It was proved in [7: Proposition 2.4] that if R is an Armendariz ring, then R is completely reflexive if and only if $R[x]$ is completely reflexive if and only if $R[x;x^{-1}]$ is completely reflexive. Accordingly, we have the following equivalences on reflexive rings.

Proposition 3.2. *Let R be a quasi-Armendariz ring, then the following statements are equivalent:*

(1) *R is reflexive.*
(2) *$R[x]$ is reflexive.*
(3) *$R[x;x^{-1}]$ is reflexive.*

Proof. It suffices to show (1) implies (2). Let $f(x)=\sum\limits_{i=0}^{m}a_ix^i$, $g(x)=\sum\limits_{j=0}^{n}b_jx^j$ $\in R[x]$ such that $f(x)R[x]g(x)=0$. Since R is quai-Armendariz, we have $a_iRb_j=0$ for each i,j. But R is reflexive, so $b_jRa_i=0$ for all i,j. Consequently, we have $g(x)R[x]f(x)=0$ and hence $R[x]$ is reflexive. $\square$

Let R be an algebra over a commutative ring S. Recall that the Dorroh extension of R by S is the ring $R\times S$ with operations $(r_1,s_1)+(r_2,s_2)=(r_1+r_2,s_1+s_2)$ and $(r_1,s_1)(r_2,s_2)=(r_1r_2+s_1r_2+s_2r_1,s_1s_2)$, where $r_i\in R$ and $s_i\in S$. The next construction is due to Nagata [14]. Let R be a commutative ring, M be an R-module, and α be an endomorphism of R. Given $R\bigoplus M$ a (possibly noncommutative) ring structure with multiplication $(r_1,m_1)(r_2,m_2)=(r_1r_2,\alpha(r_1)m_2+r_2m_1)$, where $r_i\in R$ and $m_i\in M$. We shall call this extension the skewtrivial extension of R by M and α. Note that if $\alpha\equiv I_R$, then the skewtrivial extension of R by M and α is just the usual trivial extension of R by M. The following proposition extends [7: Proposition 1.14 (2, 3)] to reflexive rings.

Proposition 3.3.

(1) *Let R be a commutative domain and α an injective endomorphism of R. Then the skewtrivial extension of R by R and α is reflexive.*
(2) *Let R be an algebra over a commutative ring S with D the Dorroh extension of R by S. If R is reflexive and S is a domain, then D is reflexive.*

P r o o f.

(1) It follows directly from [7: Proposition 1.14(3)] and the fact that every completely reflexive ring is reflexive.

(2) Let $(r_1, s_1), (r_2, s_2) \in D$ with $(r_1, s_1)(r, s)(r_2, s_2) = 0$ for any (r, s). We shall prove $(r_2, s_2)(r, s)(r_1, s_1) = 0$. In fact, we have $(r_1rr_2 + s_1rr_2 + sr_1r_2 + s_1sr_2 + s_2r_1r + s_1s_2r + ss_2r_1, s_1ss_2) = 0$, and so $r_1rr_2 + s_1rr_2 + sr_1r_2 + s_1sr_2 + s_2r_1r+s_1s_2r+ss_2r_1 = 0$, $s_1ss_2 = 0$. Since S is a domain, $s_1 = 0, s = 0$ or $s_2 = 0$. If $s_1 = 0$, then $r_1rr_2+sr_1r_2+s_2r_1r+ss_2r_1 = 0$ and so we have $0 = r_1rr_2+sr_1r_2+s_2r_1r+ss_2r_1 = r_1(r+s)(r_2+s_2) = (r_2+s_2)(r+s)r_1 = r_2rr_1+r_2sr_1+s_2rr_1+s_2sr_1$ since R is a reflexive ring. This shows that $(r_2, s_2)(r, s)(r_1, s_1) = r_2rr_1 + r_2sr_1 + s_2rr_1+s_2sr_1 = (r_2rr_1+s_2rr_1+sr_2r_1+s_2sr_1+s_1r_2r+s_1s_2r+s_1sr_2, s_2ss_1) = 0$. If $s_2 = 0$, then $r_1rr_2+s_1rr_2+sr_1r_2+s_1sr_2 = (r_1+s_1)(r, s)r_2 = r_2(r, s)(r_1+s_1) = r_2rr_1+r_2rs_1+r_2sr_1+r_2ss_1 = r_2rr_1+s_2rr_1+sr_2r_1+s_2sr_1+s_1r_2r+s_1s_2r+s_1sr_2 = 0$. Therefore, we get $(r_2, s_2)(r, s)(r_1, s_1) = 0$ in any case and we are done. □

For Proposition 3.3(1), one may suspect that the result also holds for commutative reduced rings, however the following example eliminates the possibility.

Example 3.4. Let D be a domain of characteristic zero, and $R = D \bigoplus D$ with componentwise multiplication. It is clear that R is a commutative reduced ring but not a domain. If we define $\alpha\colon R \to R$ by $\alpha(s, t) = (t, s)$, then α is an automorphism of R. Let $r_1 = ((0, 1), (1, 0)), r_2 = ((1, 0), (0, 1)) \in R$. Then for $r = ((0, 1), (0, 1)) \in R$, we have $r_1rr_2 = ((0, 1), (1, 0))((0, 1), (0, 1))((1, 0), (0, 1)) = 0$, but $r_2rr_1 = ((1, 0), (0, 1))((0, 1), (0, 1))((0, 1), (1, 0)) = ((0, 0), (0, 2)) \neq 0$. Thus the skewtrivial extension of R by R and α is not reflexive.

Note that Corollary 2.2.1 is a special case of Proposition 3.3(1) if we let $\alpha = I_R$, where I_R is an identity endomorphism of R.

The next example shows that Proposition 3.3(1) need not hold when the endomorphism α is not injective.

Example 3.5. Let D be a commutative domain and $R = D(x)$ be the polynomial ring over D with an indeterminate x. Define $\alpha\colon D(x) \to D(x)$ by $\alpha(f(x)) = f(0)$, where $f(0)$ is the constant term of $f(x)$. Let N be the skewtrivial extension of R by R and α. Let $(1, x^2), (0, 1) \in N$, then we have $(1, x^2)(x, 1)(0, 1) = 0$ and $(0, 1)(x, 1)(1, x^2) = (0, x) \neq 0$ for $(x, 1) \in N$. This shows that N is not reflexive.

Recall that if T is a ring without identity, its Dorroh extension is $T' = Z \bigoplus T$ (as additive groups) with multiplication defined by $(n_1, t_1)(n_2, t_2) = (n_1n_2, t_1t_2+ n_1t_2 + n_2t_1)$. The following example illustrates the limits of Proposition 3.3(2).

Example 3.6. Let $S = \{a, b\}$ be the semigroup with multiplication $a^2 = ab = a$, $b^2 = ba = b$. Put $T = F_2S$, which is a four-element semigroup ring without identity. A quick calculation reveals that T is not completely reflexive. Let T' be the Dorroh extension of T, then we have $(0, a)(0, b)(1, a) = 0 \neq (1, a)(0, b)(0, a)$ and so T' is not reflexive.

Let $A(R, \alpha)$ or A be the subset $\{x^{-i}rx^i \mid r \in R,\ i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, where $\alpha\colon R \to R$ is an injective ring endomorphism of a ring R (see [6] for more details). Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-i}rx^i$ where $r \in R$ and i, j are non-negative integers. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Note that for each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set $A(R, \alpha)$ of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$$
$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)}$$

for $r, s \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$.

Proposition 3.7. *Let R be a reflexive ring. Then $A(R, \alpha)$ is reflexive.*

Proof. Let $a = x^{-i}rx^i$, $c = x^{-k}tx^k \in A(R, \alpha)$ and $b = x^{-j}sx^j$ be any element in $A(R, \alpha)$. Suppose that $abc = (x^{-i}rx^i)(x^{-j}sx^j)(x^{-k}tx^k) = 0$, then we have $x^{-(i+j+k)}(\alpha^{k+j}(r)\alpha^{k+i}(s)\alpha^{i+j}(t))x^{(i+j+k)} = 0$. This implies that

$$\alpha^{k+j}(r)\alpha^{k+i}(s)\alpha^{i+j}(t) = 0.$$

Since R is a reflexive ring, we have $\alpha^{i+j}(t)\alpha^{k+i}(s)\alpha^{k+j}(r) = 0$. It follows that $cba = (x^{-k}tx^k)(x^{-j}sx^j)(x^{-i}rx^i) = x^{-(i+j+k)}(\alpha^{i+j}(t)\alpha^{k+i}(s)\alpha^{k+j}(r))x^{(i+j+k)} = 0$. This shows that $A(R, \alpha)$ is a reflexive ring. □

Recall that a ring R is called right Ore if given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well known that R is a right Ore ring if and only if the classical right quotient ring $Q(R)$ of R exists. Let F be a field and $R = F\{x, y\}$ be the free algebra in two indeterminates over F. For x and y, there do not exist $a, b \in R$ such that $y^{-1}x = ab^{-1}(xy^{-1} = b^{-1}a)$. So the domain R cannot have its classical right (left) quotient ring and thus the hypothesis in the following proposition is not superfluous.

Proposition 3.8. *Let R be a right Ore ring with Q the classical right quotient ring of R. If R is reflexive, then Q is reflexive.*

Proof. Let $\alpha = au^{-1}$, $\gamma = cw^{-1} \in Q$ and suppose that R is a reflexive ring with $\alpha\beta\gamma = 0$ for all $\beta = bv^{-1} \in Q$. It follows that there exist $b_1, u_1 \in R$ with u_1 regular such that $u^{-1}b = b_1u_1^{-1}$, then $0 = \alpha\beta\gamma = au^{-1}bv^{-1}cw^{-1} = ab_1u_1^{-1}v^{-1}cw^{-1}$. Moreover, there exist $c_1, v_1 \in R$ with v_1 regular such that $v^{-1}c = c_1v_1^{-1}$, so we have $0 = \alpha\beta\gamma = ab_1u_1^{-1}c_1v_1^{-1}w^{-1}$. Also there exist $c_2, u_2 \in R$ with u_2 regular such that $u_1^{-1}c_1 = c_2u_2^{-1}$ and hence we obtain $0 = \alpha\beta\gamma = ab_1c_2u_2^{-1}v_1^{-1}w^{-1} = ab_1c_2(wv_1u_2)^{-1}$, which implies that $ab_1c_2 = 0$. In the following computations we freely use the condition that R is reflexive. Since $ab_1c_2 = 0$, we have $c_2b_1a = 0$ and so $c_2b_1au = 0$. This implies that $aub_1c_2 = abu_1c_2 = 0$ since $ub_1 = bu_1$, then $u_1c_2ba = 0$ and so $c_1u_2ba = 0$ since $u_1c_2 = c_1u_2$. This shows that $abc_1u_2 = 0$ and so $abc_1 = 0$. Hence $c_1ba = 0$ and thus $vc_1ba = 0$, so $abcv_1 = 0$. Hence we get $abc = 0$ and so $cba = 0$.

On the other hand, $\gamma\beta\alpha = cw^{-1}bv^{-1}au^{-1}$ and similarly there exist $a_3, a_4, b_3, w_3, v_3, v_4 \in R$ with w_3, v_3, v_4 regular such that $w^{-1}b = b_3w_3^{-1}$, $v^{-1}a = a_3v_3^{-1}$, $w_3^{-1}a_3 = a_4v_4^{-1}$. Then we have $\gamma\beta\alpha = cw^{-1}bv^{-1}au^{-1} = cb_3w_3^{-1}v^{-1}au^{-1} = cb_3w_3^{-1}a_3v_3^{-1}u^{-1} = cb_3a_4v_4^{-1}v_3^{-1}u^{-1} = cb_3a_4(uv_3v_4)^{-1}$. Since $cba = 0$, we have $w_3cba = 0$ and so $abw_3c = awb_3c = 0$. This implies that $cb_3aw = 0$ and thus $cb_3a = 0$. Then $cb_3av_3 = cb_3va_3 = 0$ and so $va_3b_3c = 0$. Hence $a_3b_3c = cb_3a_3 = 0$ and then $cb_3a_3v_4 = cb_3w_3a_4 = w_3a_4b_3c = 0$. It follows that $a_4b_3c = 0$ and we get $cb_3a_4 = 0$. Therefore $\gamma\beta\alpha = cw^{-1}bv^{-1}au^{-1} = cb_3a_4(uv_3v_4)^{-1} = 0$, proving that Q is reflexive. $\square$

4. Weakly reflexive rings

Now we investigate a weak form of reflexive rings in the sense of the following definition and we call them weakly reflexive rings. We do this by considering the nilpotent elements instead of the zero element in reflexive rings.

Definition 4.1. Let R be a ring, R is said to be a weakly reflexive ring if $arb = 0$ implies $bra \in \mathrm{nil}(R)$ for $a, b \in R$ and all $r \in R$.

Given a ring R, we use $N_*(R)$, $N^*(R)$ and $N(R)$ to denote the prime radical of R, the unique maximal nil ideal and the set of all nilpotent elements of R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. Marks [11] called a ring R NI when $N^*(R) = N(R)$ (equivalently, $N(R)$ forms an ideal in R). Reduced rings are clearly NI and it is obvious that a ring R is NI if and only if $R/N^*(R)$ is reduced. A ring R is called 2-primal if $N_*(R) = N(R)$. Clearly, every 2-primal ring is NI, but the converse need not true by [5]. It is clear that NI rings and reflexive rings are weakly reflexive. We shall give an example to show that there exists an example of weakly reflexive rings which is not completely reflexive.

Let R be the ring in Example 2.4. Then R is not a left α-reversible ring by [2: Example 2.2]. But R is weakly reflexive by Proposition 4.1. It is straightforward to verify that R is not completely reflexive. Note that the class of weakly reflexive rings is closed under subrings and finite direct products.

We denote by $T_n(R)$ the n-by-n upper triangular matrix ring over R. The following proposition gives more examples of weakly reflexive rings by matrix extensions. It also shows that weakly reflexive rings need not be reflexive by Example 2.1.

Proposition 4.1. *If R is a weakly reflexive ring, then for any n, $T_n(R)$ is weakly reflexive.*

Proof. Let $A = (a_{ij})$, $C = (c_{ij}) \in T_n(R)$ with $ABC = 0$ for all $B = (b_{ij}) \in T_n(R)$, where $1 \le i \le j \le n$. Then we have $a_{ii}b_{ii}c_{ii} = 0$ for any $1 \le i \le n$. Since R is weakly reflexive, there exists $m_i \in N$ such that $(c_{ii}b_{ii}a_{ii})^{m_i} = 0$ for any i, $i = 1, 2, \dots, n$. Let $m = \max\{m_1, m_2, \dots, m_n\}$, it follows from $((CBA)^m)^n = 0$ that $T_n(R)$ is weakly reflexive. $\square$

The next two propositions extend the class of weakly reflexive rings, and examples shall be given to show the relations between weakly reflexive rings and other related rings.

Proposition 4.2. *Let R be a ring, then $R[x]$ is weakly reflexive if and only if $R[x; x^{-1}]$ is weakly reflexive.*

Proof. It suffices to show the necessity. Suppose that $f(x), g(x) \in R[x; x^{-1}]$ with $f(x)h(x)g(x) = 0$ for all $h(x) \in R[x; x^{-1}]$. Then there exists $s \in N$ such that $f_1(x) = f(x)x^s, g_1(x) = g(x)x^s$ and $h_1(x) = h(x)x^s \in R[x]$. Since $R[x]$ is weakly reflexive and $f_1(x)h_1g_1(x) = 0$ by the hypothesis, there exists $n \in N$ such that $(g_1(x)h_1f_1(x))^n = 0$. Then we have $(g(x)h(x)f(x))^n = (x^{-3s})^n(g_1(x)h_1(x)f_1(x))^n = 0$. This shows that $R[x; x^{-1}]$ is weakly reflexive. $\square$

Since every completely reflexive ring is semicommutative [7: Lemma 1.4], we may conjecture that weakly reflexive rings may be semicommutative. But the following example erases the possibility.

Example 4.3. Let F be a division ring and we consider the 2-by-2 upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. It is clear that R is not a semicommutative ring, but R is weakly reflexive by Proposition 4.1.

The next example shows that there exists a weakly reflexive ring which is not completely reflexive. Thus the class of weakly reflexive rings stand as a nontrivial generalization of completely reflexive rings and reflexive rings.

Example 4.4. Let R be a reduced ring and let

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R \right\}.$$

Let $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $C = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix}$ be in S with $ABC = 0$ for all $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$, then $a_1a_2a_3 = 0$ and $c_1c_2c_3 = 0$. Since every reduced ring is weakly reflexive, there exist $n_1, n_2 \in \mathbb{N}$ such that $(a_3a_2a_1)^{n_1} = (c_3c_2c_1)^{n_2} = 0$. It is easy to see that

$$(CBA)^{\max\{n_1,n_2\}+3} = 0.$$

On the other hand, it is straightforward to verify that S is not completely reflexive.

According to [10], a ring R is called weak Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then a_ib_j is a nilpotent element of R for each i, j. Semicommutative rings are weak Armendariz rings [10: Corollary 3.4].

It was shown in [13: Example 2] that if R is a semicommutative ring, then $R[x]$ need not be semicommutative (and hence need not be completely reflexive). However, we have the following

Proposition 4.5. *Let R be a semicommutative ring. Then $R[x]$ is weakly reflexive.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} c_jx^j \in R[x]$ with $f(x)h(x)g(x) = 0$ for all $h(x) = \sum_{k=0}^{p} b_kx^k \in R[x]$. Since semicommutative rings are weak Armendariz, there exists $n_{ij} \in N$ such that $(a_ib_kc_j)^{n_{ij}} = 0$ for any i and j, and hence $c_jb_ka_i \in \mathrm{nil}(R)$ by the semicommutativity of R. Note that

$$g(x)h(x)f(x) = \left(\sum_{j=0}^{n} c_jx^j\right)\left(\sum_{k=0}^{p} b_kx^k\right)\left(\sum_{i=0}^{m} a_ix^i\right) = \sum_{t=0}^{m+n+p}\left(\sum_{i+j+k=t} c_jb_ka_i\right)x^t.$$

We can see that $\sum_{i+j+k=t} c_jb_ka_i \in \mathrm{nil}(R)$ for any t by [10: Lemma 3.1]. It follows from [10: Lemma 3.7] that $g(x)h(x)f(x) \in \mathrm{nil}(R[x])$. This shows that $R[x]$ is weakly reflexive. □

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