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ON ORDERED SEMIGROUPS WHICH ARE SEMILATTICES OF LEFT SIMPLE SEMIGROUPS

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ABSTRACT. It has been proved by Tôru Saitô that a semigroup S is a semilattice of left simple semigroups, that is, it is decomposable into left simple semigroups, if and only if the set of left ideals of S is a semilattice under the multiplication of subsets, and that this is equivalent to say that S is left regular and every left ideal of S is two-sided. Besides, S. Lajos has proved that a semigroup S is left regular and the left ideals of S are two-sided if and only if for any two left ideals L_1, L_2 of S, we have $L_1 \cap L_2 = L_1 L_2$. The present paper generalizes these results in case of ordered semigroups. Some additional information concerning the semigroups (without order) are also obtained.

1. Introduction and prerequisites

Decomposition of semigroups into left simple subsemigroups has been considered by Tôru Saitô who proved that a semigroup S is a semilattice of left simple semigroups if and only if S is left regular and every left ideal of S is two-sided. He moreover proved that this type of semigroups are the semigroups in which the set of left ideals is a semilattice under the multiplication of subsets [7]. On the other hand, S. Lajos has proved that a semigroup S is left regular and its left ideals are two-sided if and only if for every left ideals L_1, L_2 of S, we have $L_1 \cap L_2 = L_1 L_2$ [6]. The right analogue of the above results also hold, leading to characterization of semilattices of groups in terms of left and right ideals. The aim of the present paper is to examine the results given by Saitô in [7] in case of ordered semigroups, that is, for semigroups with an order relation, the multiplication being compatible with the ordering. Based on Theorem 6 of the present paper, the results in [7] can be generalized in case of ordered

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semigroups considering that each semigroup endowed with the equality relation $\leq := \{(x,y) \mid x = y\}$ is an ordered semigroup.

For an ordered semigroup $(S, \cdot, \leq)$ and a subset H of S, we denote by $(H]$ the subset of S defined by:

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset T of S is called a *subsemigroup* of S, if $T^2 \subseteq T$. A subsemigroup F of S is called a *filter* of S if

(1) $a, b \in S$, $ab \in F$ implies $a, b \in F$ and

(2) if $a \in F$ and $b \in S$, $b \geq a$, then $b \in F$.

For an element a of S we denote by $N(a)$ the filter of S generated by a and by $\mathcal{N}$ the relation on S defined by $\mathcal{N} := \{(a,b) \mid N(a) = N(b)\}$. A nonempty subset A of S is called *left* (resp. *right*) *ideal* of S if

(1) $SA \subseteq A$ (resp. $AS \subseteq A$) and

(2) if $a \in A$ and $b \in S$, $b \leq a$, then $b \in A$, that is if $(A] = A$.

A is called an *ideal* of S if it is both a left and a right ideal of S. For an element a of S, we denote by $L(a)$ (resp. $R(a)$) the left (resp. right) ideal of S generated by a and by $I(a)$ the ideal of S generated by a. We have $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, and $I(a) = (a \cup Sa \cup aS \cup SaS]$. When it is convenient we use the usual notation $S^1 := S \cup \{1\}$, where 1 is an element not contained in S and $1a = a1 = a$ for every $a \in S^1$, and write $L(a) = (S^1a]$, $R(a) = (aS^1]$ and $I(a) = (aS^1a]$. A left (resp. right) ideal A of S is clearly a subsemigroups of S i.e. $A^2 \subseteq A$. We denote by $\mathcal{L}$ and $\mathcal{I}$ the relations on S defined by $\mathcal{L} := \{(a,b) \in S \mid L(a) = L(b)\}$, $\mathcal{I} := \{(x,y) \mid I(x) = I(y)\}$, respectively. An ordered semigroup $(S, \cdot, \leq)$ is called *left* (resp. *right*) *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$). One can easily see that S is left (resp. right) regular if and only if $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for every $a \in S$. An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$, equivalently if $a \in (Sa^2S]$ for every $a \in S$. An equivalence relation σ on S is called *congruence* if $(a,b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. If σ is a semilattice congruence of S, then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$. Recall that if σ is a complete semilattice congruence on S then, since $a \leq a$, we have $(a, a^2) \in \sigma$. So a complete semilattice congruence on S can be also defined as a congruence σ on S such that $(ab, ba) \in \sigma$ and $a \leq b$ implies $(a, ab) \in \sigma$ for every $a, b \in S$. An ordered semigroup S is called *left simple* if S is the only left ideal of S, that is, if L is a left ideal of S, then $L = S$. An ordered semigroup S is called a *semilattice of left simple semigroups* (resp. *complete semilattice of left simple semigroups*) if there exists a semilattice congruence (resp. complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left simple subsemigroup of S for every $x \in S$. An ordered semigroup S is a semilattice of left simple semigroups if and only if there exists a semilattice Y and a family $\{S_\alpha \mid \alpha \in Y\}$ of left simple subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset$ for every $\alpha, \beta \in Y$, $\alpha \neq \beta$.

(2) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$.

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for every $\alpha, \beta \in Y$.

In ordered semigroups the semilattice congruences are defined exactly as in semigroups (without order) so the two definitions are equivalent.

An ordered semigroup is a complete semilattice of left simple semigroups if and only in addition to (1), (2) and (3) above, we have the following:

(4) $S_\beta \cap (S_\alpha] \neq \emptyset$ implies $\beta = \alpha\beta$ (cf. [4]).

A subsemigroup T of an ordered semigroup $(S, \cdot, \leq)$ is called *left simple* if the set T with the multiplication $\cdot$ and the order $\leq$ of S is a left simple semigroup. That is, for every left ideal A of $(T, \cdot, \leq)$, we have $A = T$.

Instead of saying "ordered semigroups which are semilattices of left simple semigroups" we could briefly say "semilattices of left simple ordered semigroups".

2. Main results

Lemma 1. *An ordered semigroup $(S, \cdot, \leq)$ is left simple if and only if $S = (Sa]$ for every $a \in S$.*

Proof.

$\Longrightarrow$: Let S be left simple and $a \in S$. The set $(Sa]$ is a left ideal of S. Indeed, $(Sa]$ is a nonempty subset of S, $S(Sa] = (S](Sa] \subseteq (S^2a] \subseteq (Sa]$ and $((Sa]] = (Sa]$. Since S is left simple, we have $(Sa] = S$.

$\Longleftarrow$: Let L be a left ideal of S. Take an element $b \in L$ $(L \neq \emptyset)$. By hypothesis, we have $S = (Sb] \subseteq (SL] \subseteq (L] = L$, so $L = S$. $\square$

Lemma 2. *An ordered semigroup $(S, \cdot, \leq)$ is left simple if and only if for every $a, b \in S$ there exists $x \in S$ such that $b \leq xa$.*

Lemma 3. *If S is a left (resp. right) regular ordered semigroup, then it is intra-regular.*

Proof. Let S be left regular and $a \in S$. Then $a \in (Sa^2] \subseteq (S(Sa^2]a] = (S(Sa^2)a] \subseteq (Sa^2S]$, thus S is intra-regular. $\square$

Lemma 4. (cf. [5], [3]) *An ordered semigroup S is intra-regular if and only if $\mathcal{N} = \mathcal{I}$.*

Lemma 5. (cf. [1], [2]) *If S is an ordered semigroup, then the relation $\mathcal{N}$ is a complete semilattice congruence of S.*

Theorem 6. *Let $(S, \cdot, \leq)$ be an ordered semigroup. The following are equivalent:*

(1) *S is a semilattice of left simple semigroups.*

(2) *If L_1, L_2 and L are left ideals of S, then $(L_1L_2] = (L_2L_1]$ and $(L^2] = L$.*

(3) *If L_1, L_2 are left ideals of S, then $L_1 \cap L_2 = (L_1L_2]$.*

(4) *$(aS] \subseteq (Sa] = L(a) = \big(L(a)^2\big]$ for every $a \in S$.*

(5) *S is left regular and every left ideal of S is two-sided.*

(6) *$\mathcal{N} = \mathcal{I} = \mathcal{L}$.*

(7) *$\mathcal{L}$ is a complete semilattice congruence on S.*

(8) *S is a complete semilattice of left simple semigroups.*

Proof.

(1) $\Longrightarrow$ (2). Let σ be a semilattice congruence on S such that $(x)_\sigma$ is a left simple subsemigroup of S for every $x \in S$. Let L_1, L_2 be left ideals of S and $c \in (L_1L_2]$. Then $c \leq ab$ for some $a \in L_1$, $b \in L_2$. Since σ is a semilattice congruence on S, we have $(ab, ba) \in \sigma$, i.e. $(ab)_\sigma = (ba)_\sigma$, so $ba, ab \in (ba)_\sigma$. Since $(ab)_\sigma$ is a left simple subsemigroup of S, by Lemma 2, there exists $x \in S$ such that $ab \leq x(ba)$. Then we have $c \leq (xb)a \in (SL_2)L_1 \subseteq L_2L_1$, and $c \in (L_2L_1]$. Similarly we get $(L_2L_1] \subseteq (L_1L_2]$, so $(L_1L_2] = (L_2L_1]$. Let now L be a left ideal of S and let $c \in S$. Since $c \in (c)_\sigma$ and $(c)_\sigma$ is a subsemigroup of S, we have $c^2 \in (c)_\sigma$. Since $c^2, c \in (c)_\sigma$ and $(c)_\sigma$ is left simple, by Lemma 2, there exists $x \in (c)_\sigma$ such that $c \leq xc^2$. Since $xc^2 = (xc)c \in (SL)L \subseteq L^2$, we have $c \in (L^2]$. On the other hand, we have $(L^2] \subseteq (SL] \subseteq L$, so $(L^2] = L$.

(2) $\Longrightarrow$ (3). Let L_1, L_2 be left ideals of S. Since $L_1L_2 \subseteq SL_2 \subseteq L_2$ and $L_2L_1 \subseteq SL_1 \subseteq L_1$, we have $(L_1L_2] \subseteq (L_2] = L_2$ and $(L_2L_1] \subseteq (L_1] = L_1$. By (2), $(L_1L_2] = (L_2L_1]$. Thus we have $(L_1L_2] \subseteq L_1 \cap L_2$. Since $(L_1L_2] \neq \emptyset$, we have $L_1 \cap L_2 \neq \emptyset$. Then $L_1 \cap L_2$ is a left ideal of S and, by (2), $L_1 \cap L_2 = ((L_1 \cap L_2)(L_1 \cap L_2)] \subseteq (L_1L_2]$. Thus we have $L_1 \cap L_2 = (L_1L_2]$.

(3) $\Longrightarrow$ (4). Let $a \in S$. Then, by (3), we have $L(a) = (L(a)L(a)] := (L(a)^2]$,

$$\begin{aligned} L(a) &= S \cap L(a) = (SL(a)] = (S(a \cup Sa]] = (S(a \cup Sa)] \\ &= (Sa \cup S^2a] = (Sa], \end{aligned}$$

and

$$(aS] \subseteq (L(a)S] = L(a) \cap S = L(a) = (Sa].$$

(4) $\Longrightarrow$ (5). Let $a \in S$. By (4), we have

$$\begin{aligned} a \in L(a) &= (L(a)L(a)] = ((Sa](Sa]] = ((Sa)(Sa)] = (S(aS)a] \\ &= (S(aS]a] \subseteq (S(Sa]a] = (S(Sa)a] \subseteq (Sa^2], \end{aligned}$$

so $a \in (Sa^2]$, and S is left regular. Let now L be a left ideal of S. Then $LS \subseteq L$. Indeed, if $a \in L$, $s \in S$ then, by (4), $as \in aS \subseteq (aS] \subseteq (Sa] \subseteq (SL] \subseteq (L] = L$, so $as \in L$.

(5) $\Longrightarrow$ (6). Since S is left regular, by Lemma 3, S is intra-regular. Then, by Lemma 4, we have $\mathcal{N} = \mathcal{I}$. Moreover, we have $I(a) = L(a)$ for every $a \in S$. In fact, let $a \in S$. Since $L(a)$ is a left ideal of S, by hypothesis, $L(a)$ is an ideal of S. Since $L(a)$ is an ideal of S containing a and $I(a)$ is the ideal of S generated by a, we have $I(a) \subseteq L(a)$. On the other hand, $L(a) = (a \cup Sa] \subseteq (a \cup Sa \cup aS \cup SaS] = I(a)$, thus $L(a) = I(a)$. Now $\mathcal{I} = \mathcal{L}$. In fact,

$$(a, b) \in \mathcal{I} \iff I(a) = I(b) \iff L(a) = L(b) \iff (a, b) \in \mathcal{L}.$$

Hence we obtain $\mathcal{N} = \mathcal{I} = \mathcal{L}$.

(6) $\Longrightarrow$ (7). By Lemma 5, $\mathcal{N}$ is a complete semilattice congruence on S. Then, by (6), $\mathcal{L}$ is a complete semilattice congruence on S, as well.

(7) $\Longrightarrow$ (8). Let $a \in S$. Since $\mathcal{L}$ is a complete semilattice congruence on S, it is enough to prove that $(a)_{\mathcal{L}}$ is a left simple subsemigroup of S. In fact: First of all, since $\mathcal{L}$ is a semilattice congruence on S, $(a)_{\mathcal{L}}$ is a subsemigroup of S.

Let now $b, c \in (a)_{\mathcal{L}}$. Then there exists $y \in (a)_{\mathcal{L}}$ such that $c \leq yb$. Indeed: Since $b \in (a)_{\mathcal{L}}$ and $(a)_{\mathcal{L}}$ is a subsemigroup of S, we have $b^3 \in (a)_{\mathcal{L}}$, so $(b^3, a) \in \mathcal{L}$. Since $c \in (a)_{\mathcal{L}}$, we have $(a, c) \in \mathcal{L}$, thus $(b^3, c) \in \mathcal{L}$ and $c \in L(c) = L(b^3) = (S^1 b^3]$. Then there exists $u \in S^1$ such that $c \leq ub^3$. We put $x := ub$ and we have

$$c \leq ub^3 = (ub)b^2 = xb^2 = (xb)b.$$

Moreover, $xb \in (a)_{\mathcal{L}}$. Indeed: Since $\mathcal{L}$ is a complete semilattice congruence on S and $c \leq xb^2$, we have $(c, cxb^2) \in \mathcal{L}$. Since $\mathcal{L}$ is a semilattice congruence on S, we have $(cx, xc) \in \mathcal{L}$ and $(cxb^2, xcb^2) \in \mathcal{L}$. Then we have $(c, xcb^2) \in \mathcal{L}$. Since $(cb^2, b^2c) \in \mathcal{L}$, we have $(xcb^2, xb^2c) \in \mathcal{L}$. Then we have $(c, xb^2c) \in \mathcal{L}$. Since $c, b \in (a)_{\mathcal{L}}$, we get $(c, b) \in \mathcal{L}$ and $(b^2c, b^3) \in \mathcal{L}$. Then, since $(b^3, c) \in \mathcal{L}$, we have $(b^2c, c) \in \mathcal{L}$. Since $(c, b) \in \mathcal{L}$, we have $(b^2c, b) \in \mathcal{L}$ and $(xb^2c, xb) \in \mathcal{L}$. Then, since $(c, xb^2c) \in \mathcal{L}$, we obtain $(c, xb) \in \mathcal{L}$ and $xb \in (c)_{\mathcal{L}} = (a)_{\mathcal{L}}$.

(8) $\Longrightarrow$ (1). This is obvious. □

Remark 7. The right analogue of Theorem 6 also holds.

By Theorem 6, we have the following

Corollary 8. ([7: Theorem]) *For a semigroup $(S, \cdot)$ the following conditions are equivalent:*

(1) *S is a semilattice of left simple semigroups.*

(2) *The set of all left ideals of S is a semilattice under the multiplication of subsets.*

(3) *$L_1 \cap L_2 = L_1 L_2$ for every left ideals L_1 and L_2 of S.*

(4) *S is left regular and every left ideal of S is two-sided.*

Proof.

(1) $\Longrightarrow$ (2). Let σ be a semilattice congruence on $(S, \cdot)$ such that $(x)_\sigma$ is a left simple subsemigroup of $(S, \cdot)$ for every $x \in S$ and let L_1, L_2 and L be left ideals of $(S, \cdot)$. We endow $(S, \cdot)$ with the equality relation $\leq := \{(x, y) \mid x = y\}$. Then

$(S, \cdot, \leq)$ is an ordered semigroup, σ is a semilattice congruence on $(S, \cdot, \leq)$ and $((x)_\sigma, \cdot, \leq)$ is a left simple subsemigroup of $(S, \cdot, \leq)$ for every $x \in (S, \cdot, \leq)$. Moreover, the sets L_1, L_2 and L are left ideals of $(S, \cdot, \leq)$. By Theorem 6 (1) $\Longrightarrow$ (2), we have $(L_1L_2] = (L_2L_1]$ and $(L^2] = L$. On the other hand, $(L_1L_2] = L_1L_2$. Indeed, if $x \in (L_1L_2]$, then $x \leq yz$ for some $y \in L_1$, $z \in L_2$. Since $(x, yz) \in \leq$, we have $x = yz$, thus $x \in L_1L_2$. Similarly we have $(L_2L_1] = L_2L_1$ and $(L^2] = L$ and (2) holds.

The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) can be proved in a similar way. □

Corollary 9. ([7: Corollary]) *A semigroup S is a semilattice of groups if and only for any two left (resp. right) ideals L_1, L_2 (resp. R_1, R_2) of S, we have $L_1 \cap L_2 = L_1L_2$ and $R_1 \cap R_2 = R_1R_2$.*

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