VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0103-8
Math. Slovaca **63** (2013), No. 2, 381–395

PERIODIC SOLUTION TO p-LAPLACIAN NEUTRAL LIÉNARD TYPE EQUATION WITH VARIABLE PARAMETER

Bo Du

(*Communicated by Michal Fečkan*)

ABSTRACT. Using Mawhin's continuation theorem we obtain some existence results of periodic solutions for a type of p-Laplacian neutral Liénard equation

$$(\varphi_p((x(t) - c(t)x(t - \tau))'))' + f(x(t))x'(t) + g(x(t - \gamma(t))) = e(t).$$

It is worth noting that $c(t)$ is no longer a constant which is different from the corresponding ones of past work.

1. Introduction

This paper is devoted to investigating p-Laplacian neutral Liénard equation with variable parameter of the form:

$$(\varphi_p((x(t) - c(t)x(t - \tau))'))' + f(x(t))x'(t) + g(x(t - \gamma(t))) = e(t), \tag{1.1}$$

where $\varphi_p\colon \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$, $p > 1$; $f, g \in C(\mathbb{R}, \mathbb{R})$; c, γ, e are continuous T-periodic functions defined on $\mathbb{R}$ with $|c(t)| \neq 1$ and $\int_0^T e(s)\,\mathrm{d}s = 0$; τ is a given constant. When $p = 2$ and $c(t)$ is a constant c, equation (1.1) is a classic neutral Liénard equation. For this case, the authors [1] studied the following neutral equation

2010 Mathematics Subject Classification: Primary 34B15, 34B13.
Keywords: variable parameter, neutral, Mawhin's continuation theorem.
This work was supported by NSF of Jiangsu education office (11KJB110002), Postdoctoral Foundation of Jiangsu (1102096C) and Postdoctoral Foundation of China (2012M511296).

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(u(t)-cu(t-\tau))=f(u(t))u'(t)+\alpha(t)g(u(t))+\Sigma_{j=1}^{n}g(u(t-\gamma_j(t)))+p(t)$$

and obtained some existence results of periodic solutions for the above equation. When $p\neq 2$ and $c(t)$ is a constant c, Zhu and Lu [2] considered p-Laplacian neutral equation

$$(\varphi_p[(x(t)-cx(t-\sigma))'])'=f(x(t))x'(t)+\Sigma_{j=1}^{n}\beta_j(t)g(x(t-\gamma_j(t)))+p(t).$$

In 2009, when $c(t)$ is a T-periodic continuous function, we obtained the properties of difference operator A ($A\colon C_T\to C_T$, $[Ax](t)=x(t)-cx(t-\tau)$, for all $t\in\mathbb{R}$), see [3]. Based on the above work, in this paper we will study equation (1.1) and obtain the existence of periodic solutions by using Mawhin's continuation theorem. To the best of our knowledge, there is no paper to study the existence of periodic solutions to equation (1.1). The reasons for it lie in the following two aspects. The first is that the operator $\varphi_p(u)=|u|^{p-2}u$, $p\neq 2$ is no longer linear, so Mawhin's continuation theorem can not been used directly and verifying L-compact for nonlinear operator N is difficult; the second is that conditions of Mawhin's continuation theorem are not easy to verify when $c(t)$ is variable . Our research enriches the contents of neutral equations and generalizes informed results.

2. Main lemmas

Let

$$c_0=\max_{t\in[0,T]}|c(t)|,\quad \sigma=\min_{t\in[0,T]}|c(t)|,\quad c_1=\max_{t\in[0,T]}|c'(t)|,$$

$$C_T=\big\{x:\ x\in C(\mathbb{R},\mathbb{R})\ \&\ (\forall t\in\mathbb{R})\big(x(t+T)\equiv x(t)\big)\big\}$$

with the norm

$$|\varphi|_0=\max_{t\in[0,T]}|\varphi(t)|,\qquad \text{for all}\quad \varphi\in C_T$$

and

$$C_T^1=\big\{x:\ x\in C^1(\mathbb{R},\mathbb{R})\ \&\ (\forall t\in\mathbb{R})\big(x(t+T)\equiv x(t)\big)\big\}$$

with the norm

$$\|\varphi\|=\max_{t\in[0,T]}\{|\varphi|_0,|\varphi'|_0\},\qquad \text{for all}\quad \varphi\in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces. Define linear operator:

$$A\colon C_T\to C_T,\qquad [Ax](t)=x(t)-c(t)x(t-\tau),\qquad \text{for all}\quad t\in\mathbb{R}.$$

Lemma 2.1. ([3]) *If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j} c(t-(i-1)\tau) f(t-j\tau), \ \ c_0 < 1, \quad \text{for all } f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t+j\tau+\tau), \ \ \sigma > 1, \quad \text{for all } f \in C_T. \end{cases}$$

(2)

$$\int\limits_0^T |[A^{-1}f](t)| \, \mathrm{d}t \leq \begin{cases} \frac{1}{1-c_0} \int\limits_0^T |f(t)| \, \mathrm{d}t, \ \ c_0 < 1, \quad \text{for all } f \in C_T, \\ \frac{1}{\sigma-1} \int\limits_0^T |f(t)| \, \mathrm{d}t, \ \ \sigma > 1, \quad \text{for all } f \in C_T. \end{cases}$$

Lemma 2.2. ([4]) *Suppose that X and Y are two Banach spaces, and $L\colon D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N\colon \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(1) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, for all $\lambda \in (0,1)$,*
(2) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$,*
(3) *$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,*

where $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to obtain the existence of T-periodic solutions of the equation (1.1), we take $(Ax_1)'(t) = \varphi_q(x_2(t))$. From $(Ax_1')(t) = (Ax_1)'(t) + c'(t)x_1(t-\tau)$, we have

$$x_1'(t) = A^{-1}[\varphi_q(x_2(t)) + c'(t)x_1(t-\tau)],$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Hence we rewrite the equation (1.1) in the form of the two-dimensional differential system

$$\begin{cases} (Ax_1)'(t) = \varphi_q(x_2(t)), \\ x_2'(t) = -f(x_1(t))A^{-1}[\varphi_q(x_2(t)) + c'(t)x_1(t-\tau)] - g(x_1(t-\gamma(t))) + e(t), \end{cases} \tag{2.1}$$

Obviously if $x(t) = \big(x_1(t), x_2(t)\big)^T$ is a T-periodic solution to system (2.1), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, in order to prove that equation (1.1) has a T-periodic solution, it suffices to show that system (2.1) has a T-periodic solution. Now we set

$$X = \big\{x = (x_1(\cdot), x_2(\cdot))^T \in C(\mathbb{R}, \mathbb{R}^2) :\ x(t+T) \equiv x(t)\big\}$$

with the norm $\|x\| = \max\big\{|x_1|_0, |x_2|_0\big\}$. Equipped with the above norm $\|\cdot\|$, X is Banach space. Meanwhile, let

$$L\colon D(L) \subset X \to X, \qquad Lx = \begin{pmatrix} (Ax_1)' \\ x_2' \end{pmatrix}, \tag{2.2}$$

$N\colon X \longrightarrow X$,

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))A^{-1}[\varphi_q(x_2(t)) + c'(t)x_1(t-\tau)] - g(x_1(t-\gamma(t))) + e(t) \end{pmatrix}, \tag{2.3}$$

where $D(L) = \big\{x :\ x \in C^1(\mathbb{R}, \mathbb{R}^2)\ \&\ x(t+T) = x(t)\big\}$. We get

$$\operatorname{Im} L = \left\{ y :\ y \in X,\ \int_0^T y(s)\,\mathrm{d}s = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Since for all $x \in \operatorname{Ker} L$, $(x_1(t) - c(t)x_1(t-\tau))' = 0$, then

$$x_1(t) - c(t)x_1(t-\tau) = 1. \tag{2.4}$$

Let $\phi(t)$ be the unique T-periodic solution of (2.4), then $\phi(t) \neq 0$ and

$$\operatorname{Ker} L = \left\{ \begin{pmatrix} a\phi(t) \\ a \end{pmatrix},\ a \in \mathbb{R} \right\}.$$

Obviously, $\operatorname{Im} L$ is a closed in X and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$. Hence L is a Fredholm operator with index zero. Define continuous projectors P, Q

$$P\colon X \to \operatorname{Ker} L, \qquad (Px)(t) = \begin{pmatrix} \dfrac{\int_0^T x_1(t)\phi(t)\,\mathrm{d}t}{\int_0^T \phi^2(t)\,\mathrm{d}t}\phi(t) \\ \frac{1}{T}\int_0^T x_2(t)\,\mathrm{d}t \end{pmatrix}$$

and

$$Q\colon X \to X/\operatorname{Im} L, \qquad Qy = \begin{pmatrix} \frac{1}{T}\int_0^T y_1(t)\,\mathrm{d}t \\ \frac{1}{T}\int_0^T y_2(t)\,\mathrm{d}t \end{pmatrix}.$$

Hence

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L.$$

Let

$$L_P = L|_{D(L)\cap \operatorname{Ker} P}\colon D(L)\cap \operatorname{Ker} P \to \operatorname{Im} L,$$

then

$$L_P^{-1} = K_p\colon \operatorname{Im} L \to D(L)\cap \operatorname{Ker} P.$$

Since $\operatorname{Im} L \subset C_T$ and $D(L)\cap \operatorname{Ker} P \subset C_T^1$, so K_p is an embedding operator. Hence K_p is a completely operator in $\operatorname{Im} L$. By the definitions of Q and N, it knows that $QN(\bar\Omega)$ is bounded on $\bar\Omega$, here Ω is a bounded open set on X. Hence nonlinear operator N is L-compact on $\overline{\Omega}$.

3. Main results

For the sake of convenience, we list the following conditions.

(H$_1$) There is a constant $d > 0$ such that

$$xg(x) < 0, \qquad \text{whenever} \quad |x| > d.$$

(H$_2$) There is a constant r such that

$$\limsup_{x\to-\infty} \frac{|g(x)|}{|x|^{p-1}} \le r \in [0,\infty).$$

(H$_3$)

$$f_M = \sup_{x\in\mathbb{R}} |f(x)| < +\infty.$$

Theorem 3.1. *Suppose that Assumptions* (H$_1$)–(H$_3$) *hold* $\int\limits_0^T \phi^2(t)\,\mathrm{d}t \neq 0$. *Then equation* (1.1) *has at least one T-periodic solution, if $p = 2$ and*

$$\max\Big\{\tfrac{c_1T}{1-c_0}, \tfrac{c_0 f_M T + 8(1+c_0)rT^2}{(1-c_0-c_1T)^2}\Big\} < 1 \qquad \textit{for} \quad c_0 < \tfrac12,$$

$$\max\Big\{\tfrac{c_1T}{\sigma-1}, \tfrac{c_0 f_M T + 8(1+c_0)rT^2}{(\sigma-1-c_1T)^2}\Big\} < 1 \qquad \textit{for} \quad \sigma > 1;$$

or $p > 2$ and

$$\max\Big\{\tfrac{c_1T}{1-c_0}, \tfrac{2^{p+1}(1+c_0)rT^p}{(1-c_0-c_1T)^p}\Big\} < 1 \qquad \textit{for} \quad c_0 < \tfrac12,$$

$$\max\Big\{\tfrac{c_1T}{\sigma-1}, \tfrac{2^{p+1}(1+c_0)rT^p}{(\sigma-1-c_1T)^p}\Big\} < 1 \qquad \textit{for} \quad \sigma > 1.$$

P r o o f. Consider the following operator equation:

$$Lx = \lambda Nx, \qquad \lambda \in (0,1),$$

where L and N are are defined by (2.2) and (2.3), respectively. Let

$$\Omega_1 = \big\{x:\ x \in D(L),\ \ Lx = \lambda Nx,\ \ \lambda\in(0,1)\big\}.$$

If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega_1$, then x must satisfy

$$\begin{cases} (Ax_1)'(t) = \lambda\varphi_q(x_2(t)), \\ x_2'(t) = -\lambda f(x_1(t))A^{-1}\big[\varphi_q(x_2(t)) + c'(t)x_1(t-\tau)\big] - \lambda g\big(x_1(t-\gamma(t))\big) + \lambda e(t). \end{cases} \tag{3.1}$$

From the first equation of (3.1), we get $x_2(t) = \varphi_p\big(\frac{1}{\lambda}(Ax_1)'(t)\big)$, combining with the second equation of (3.1) yields

$$\begin{aligned} \big(\varphi_p((Ax_1)'(t))\big)' + \lambda^p f(x_1(t))A^{-1}\big[\varphi_q(x_2(t)) + c'(t)x_1(t-\tau)\big] \\ + \lambda^p g\big(x_1(t-\gamma(t))\big) = \lambda^p e(t), \end{aligned}$$

i.e.

$$(\varphi_p((Ax_1)'(t)))' + \lambda^p f(x_1(t))x_1'(t) + \lambda^p g(x_1(t-\gamma(t))) = \lambda^p e(t). \tag{3.2}$$

Integrating both sides of (3.2) over [0,T], we get

$$\int_0^T g(x_1(t-\gamma(t)))\,\mathrm{d}t = 0. \tag{3.3}$$

From integral mean value theorem and (3.3), we know that there exists a constant $t_1 \in [0,T]$ such that

$$g(x_1(t_1-\gamma(t_1))) = 0.$$

Assumption (H_1) implies

$$|x_1(t_1-\gamma(t_1))| \le d.$$

Because $x_1(t)$ is a T-periodic function, then there exists a constant $\xi \in [0,T]$ satisfying $t_1 - \gamma(t_1) = \xi + kT$, $k \in Z$, then we have $|x_1(\xi)| \le d$. Hence we get

$$|x_1|_0 = \max_{t\in[0,T]}\left|x_1(\xi) + \int_\xi^t x_1'(s)\,\mathrm{d}s\right| \le |x_1(\xi)| + \int_0^T |x_1'(s)|\,\mathrm{d}s \le d + \int_0^T |x_1'(s)|\,\mathrm{d}s. \tag{3.4}$$

Multiplying the two sides of equation (3.2) by $(Ax_1)(t)$ and integrating them over $[0,T]$, combining with (H_3), we have

$$\begin{aligned}\int_0^T |(Ax_1)'(t)|^p \,\mathrm{d}t &= \lambda^p \int_0^T f(x_1(t))x_1'(t)[x_1(t) - c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &\quad + \lambda^p \int_0^T g(x_1(t-\gamma(t)))[x_1(t) - c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &\quad - \lambda^p \int_0^T e(t)[x_1(t) - c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &= -\lambda^p \int_0^T f(x_1(t))x_1'(t)c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &\quad + \lambda^p \int_0^T g(x_1(t-\gamma(t)))[x_1(t) - c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &\quad - \lambda^p \int_0^T e(t)[x_1(t) - c(t)x_1(t-\tau)]\,\mathrm{d}t \\ &\le c_0|x_1|_0 f_M \int_0^T |x_1'(t)|\,\mathrm{d}t \\ &\quad + (1+c_0)|x_1|_0 \int_0^T |g(x_1(t-\gamma(t)))|\,\mathrm{d}t + (1+c_0)|x_1|_0 T|e|_0 \end{aligned} \tag{3.5}$$

Let

$$\begin{aligned} E_1 &= \big\{t \in [0,T] : \ x_1(t-\gamma(t)) < -\rho\big\}, \\ E_2 &= \big\{t \in [0,T] : \ |x_1(t-\gamma(t))| \le \rho\big\}, \\ E_3 &= \big\{t \in [0,T] : \ x_1(t-\gamma(t)) > \rho\big\}, \end{aligned}$$

where $\rho > d > 0$ is a given constant. Integrating the two sides of (3.2) on $[0,T]$, we get

$$\int_0^T g(x_1(t-\gamma(t)))\,\mathrm{d}t = 0.$$

Therefore, using (H_1) and (H_2), we obtain

$$\begin{aligned}\int\limits_{E_3} |g(x_1(t-\gamma(t)))|\,\mathrm{d}t &= -\int\limits_{E_3} g(x_1(t-\gamma(t)))\,\mathrm{d}t \\ &= \int\limits_{E_1\cup E_2} g(x_1(t-\gamma(t)))\,\mathrm{d}t \\ &\le \int\limits_{E_1\cup E_2} |g(x_1(t-\gamma(t)))|\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

Since $\frac{c_0 f_M T+8(1+c_0)rT^2}{(1-c_0-c_1T)^2}<1$ and $\frac{2^{p+1}(1+c_0)rT^p}{(1-c_0-c_1T)^p}<1$, there exists a constant $\varepsilon>0$ such that

$$\frac{c_0 f_M T+8(1+c_0)(r+\varepsilon)T^2}{(1-c_0-c_1T)^2}<1 \quad \text{and} \quad \frac{2^{p+1}(1+c_0)(r+\varepsilon)T^p}{(1-c_0-c_1T)^p}<1. \tag{3.7}$$

For such ε, by assumption (H_2), there exists a constant $\rho>0$ such that

$$|g(u)|\le (r+\varepsilon)|u|^{p-1} \qquad \text{for} \quad u<-\rho. \tag{3.8}$$

From (3.6) and (3.8), we get

$$\begin{aligned}\int\limits_0^T |g(x_1(t-\gamma(t)))|\,\mathrm{d}t &= \int\limits_{E_1\cup E_2\cup E_3} |g(x_1(t-\gamma(t)))|\,\mathrm{d}t \\ &\le 2\int\limits_{E_1\cup E_2} |g(x_1(t-\gamma(t)))|\,\mathrm{d}t \\ &\le 2(r+\varepsilon)T|x_1|_0^{p-1}+2Tg_\rho,\end{aligned} \tag{3.9}$$

where $g_\rho = \max_{t\in E_2} |g(x_1(t-\gamma(t)))|$. From (3.4), (3.5) and (3.9), we have

$$\begin{aligned}
&\int_0^T |(Ax_1)'(t)|^p \,\mathrm{d}t \\
&\le c_0|x_1|_0 f_M \int_0^T |x_1'(t)|\,\mathrm{d}t + 2(1+c_0)(r+\varepsilon)T|x_1|_0^p \\
&\quad + 2Tg_\rho(1+c_0)|x_1|_0 + (1+c_0)|x_1|_0 T|e|_0 \\
&\le \big[c_0 f_M d + 2Tg_\rho(1+c_0) + (1+c_0)T|e|_0\big] \int_0^T |x_1'(t)|\,\mathrm{d}t + c_0 f_M \Bigg(\int_0^T |x_1'(t)|\,\mathrm{d}t\Bigg)^2 \\
&\quad + 2(1+c_0)(r+\varepsilon)T\Bigg(d + \int_0^T |x_1'(t)|\,\mathrm{d}t\Bigg)^p + \big[2Tg_\rho(1+c_0) + (1+c_0)T|e|_0\big]d.
\end{aligned} \tag{3.10}$$

From $[Ax_1](t) = x_1(t) - c(t)x_1(t-\tau)$, for all $x_1 \in C_T^1$, we have

$$(Ax_1')(t) = (Ax_1)'(t) + c'(t)x_1(t-\tau),$$

then from Lemma 2.1 and (3.4), if $c_0 < \frac{1}{2}$ we have

$$\begin{aligned}
\int_0^T |x_1'(t)|\,\mathrm{d}t = \int_0^T |(A^{-1}Ax_1')(t)|\,\mathrm{d}t &\le \int_0^T \frac{|(Ax_1')(t)|}{1-c_0}\,\mathrm{d}t \\
&= \int_0^T \frac{|(Ax_1)'(t) + c'(t)x_1(t-\tau)|}{1-c_0}\,\mathrm{d}t \\
&\le \int_0^T \frac{|(Ax_1)'(t)|}{1-c_0}\,\mathrm{d}t + \frac{c_1T}{1-c_0}\Bigg(d + \int_0^T |x_1'(t)|\,\mathrm{d}t\Bigg).
\end{aligned}$$

In view of $\frac{c_1T}{1-c_0} < 1$, we have

$$\begin{aligned}
\int_0^T |x_1'(t)|\,\mathrm{d}t &\le \int_0^T \frac{|(Ax_1)'(t)|}{1-c_0-c_1T}\,\mathrm{d}t + \frac{c_1Td}{1-c_0-c_1T} \\
&\le \frac{T^{\frac{1}{q}}}{1-c_0-c_1T}\Bigg(\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t\Bigg)^{\frac{1}{p}} + \frac{c_1Td}{1-c_0-c_1T}.
\end{aligned} \tag{3.11}$$

Hence we have

$$
\begin{aligned}
&\left(\int_0^T |x_1'(t)|\,\mathrm{d}t\right)^2 \\
&\le \left(\int_0^T \frac{|(Ax_1)'(t)|}{1-c_0-c_1T}\,\mathrm{d}t + \frac{c_1Td}{1-c_0-c_1T}\right)^2 \\
&= \frac{\left(\int_0^T |(Ax_1)'(t)|\,\mathrm{d}t\right)^2}{(1-c_0-c_1T)^2} + \frac{2c_1Td\int_0^T |(Ax_1)'(t)|\,\mathrm{d}t}{(1-c_0-c_1T)^2} + \frac{c_1^2T^2d^2}{(1-c_0-c_1T)^2} \\
&\le \frac{T^{\frac{2}{q}}}{(1-c_0-c_1T)^2}\left(\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{2}{p}} \\
&\quad + \frac{2c_1dT^{2-\frac{1}{p}}}{(1-c_0-c_1T)^2}\left(\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \frac{c_1^2T^2d^2}{(1-c_0-c_1T)^2}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\left(d+\int_0^T |x_1'(t)|\,\mathrm{d}t\right)^p &\le \left(\int_0^T \frac{|(Ax_1)'(t)|}{1-c_0-c_1T}\,\mathrm{d}t + \frac{d-dc_0}{1-c_0-c_1T}\right)^p \\
&\le \frac{2^p\left(\int_0^T |(Ax_1)'(t)|\,\mathrm{d}t\right)^p}{(1-c_0-c_1T)^p} + \frac{2^p(d-dc_0)^p}{(1-c_0-c_1T)^P} \\
&\le \frac{2^pT^{\frac{p}{q}}}{(1-c_0-c_1T)^p}\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t + \frac{2^p(d-dc_0)^p}{(1-c_0-c_1T)^P},
\end{aligned}
\tag{3.13}
$$

here we use the inequality

$$(a+b)^l \le 2^l(a^l+b^l), \qquad \text{for all} \quad a>0,\ \ b>0,\ \ l>0.$$

From (3.10)–(3.13), we get

$$\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t \le \left[c_0f_Md + 2Tg_\rho(1+c_0) + (1+c_0)T|e|_0\right] \times$$

$$\begin{aligned}
&\times\left(\frac{T^{\frac{1}{q}}}{1-c_0-c_1T}\left(\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\frac{c_1Td}{1-c_0-c_1T}\right)\\
&+c_0f_M\left(\frac{T^{\frac{2}{q}}}{(1-c_0-c_1T)^2}\left(\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{2}{p}}\right.\\
&\left.+\frac{2c_1dT^{2-\frac{1}{p}}}{(1-c_0-c_1T)^2}\left(\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\frac{c_1^2T^2d^2}{(1-c_0-c_1T)^2}\right)\\
&+2(1+c_0)(r+\varepsilon)T\left(\frac{2^pT^{\frac{p}{q}}}{(1-c_0-c_1T)^p}\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t+\frac{2^p(d-dc_0)^p}{(1-c_0-c_1T)^P}\right)\\
&+\big[2Tg_\rho(1+c_0)+(1+c_0)T|e|_0\big]d\\
=&\left(\frac{[c_0f_Md+2Tg_\rho(1+c_0)+(1+c_0)T|e|_0]T^{\frac{1}{q}}}{1-c_0-c_1T}+\frac{2c_0f_Mc_1dT^{2-\frac{1}{p}}}{(1-c_0-c_1T)^2}\right)\\
&\times\left(\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\frac{c_0f_MT^{\frac{2}{q}}}{(1-c_0-c_1T)^2}\left(\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\right)^{\frac{2}{p}}\\
&+\frac{2^{p+1}(1+c_0)(r+\varepsilon)T^p}{(1-c_0-c_1T)^p}\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\\
&+\frac{\big[c_0f_Md+2Tg_\rho(1+c_0)+(1+c_0)T|e|_0\big]c_1Td}{1-c_0-c_1T}+\frac{c_0f_Mc_1^2T^2d^2}{(1-c_0-c_1T)^2}\\
&+\frac{2^{p+1}(1+c_0)(r+\varepsilon)T(d-dc_0)^p}{(1-c_0-c_1T)^P}+\big[2Tg_\rho(1+c_0)+(1+c_0)T|e|_0\big]d.
\end{aligned}\tag{3.14}$$

If $p=2$, by (3.7) and (3.14), there is a constant $M_1>0$ such that

$$\int_0^T|(Ax_1)'(t)|^2\,\mathrm{d}t\le M_1.$$

If $p>2$, by (3.7) and (3.14), there is a constant $N_1>0$ such that

$$\int_0^T|(Ax_1)'(t)|^p\,\mathrm{d}t\le N_1.$$

Take $M_2 = \max\{M_1, N_1\}$. It follows from (3.11) that

$$\int_0^T |x_1'(t)|\,\mathrm{d}t \le \frac{T^{\frac{1}{q}} M_2^{\frac{1}{p}}}{1-c_0-c_1T} + \frac{c_1TD}{1-c_0-c_1T} := M_3.$$

By (3.4) we get

$$|x_1|_0 \le d + M_3.$$

If $\sigma > 1$, from the conditions of Theorem 3.1, similar to the above proof, we also obtain that there exists a constant $M_4 > 0$

$$|x_1|_0 \le M_4.$$

Then we have

$$|x_1|_0 \le \max\{d + M_3, M_4\} := \widetilde{M}.$$

In view of the first equation of (3.1) we have $\int_0^T |x_2(t)|^{q-2}x_2(t)\,\mathrm{d}t = 0$. From integral mean value theorem, there exists a constant $\eta \in [0, T]$ such that $x_2(\eta) = 0$. Hence $|x_2|_0 \le \int_0^T |x_2'(t)|\,\mathrm{d}t$. By the second equation of (3.1) we get

$$\begin{aligned}\int_0^T |x_2'(t)|\,\mathrm{d}t &\le \int_0^T |f(x_1(t))||x_1'(t)|\,\mathrm{d}t + \int_0^T |g(x_1(t-\gamma(t)))|\,\mathrm{d}t + \int_0^T |e(t)|\,\mathrm{d}t \\ &\le f_M M_3 + Tg_{\widetilde{M}} + T|e|_0,\end{aligned}$$

where $g_{\widetilde{M}} = \max\limits_{|u|<\widetilde{M}} |g(u)|$. So we obtain

$$|x_2|_0 \le f_M M_3 + g_{\widetilde{M}} + T|e|_0 := \widehat{M}.$$

We have proved that if $x = (x_1, x_2)^T \in D(L)$, $Lx = \lambda Nx$, $\lambda \in (0, 1)$, then $|x_1|_0 \le \widetilde{M}$ and $|x_2|_0 \le \widehat{M}$. Let $M = \max\{\widetilde{M}, \widehat{M}\}$ and $\Omega = \{x = (x_1, x_2)^T \in X : |x_1|_0 \le M,\ |x_2|_0 \le M\}$. Then $M > d$ and the condition (1) of Lemma 2.2 is satisfied. Moreover, for any $x = (x_1, x_2)^T \in X$, we have

$$QNx = \begin{pmatrix} \frac{1}{T}\int_0^T \varphi_q(x_2(t))\,\mathrm{d}t \\ -\frac{1}{T}\int_0^T g(x_1(t-\gamma(t)))\,\mathrm{d}t \end{pmatrix}.$$

Since $\operatorname{Ker} L = (a\phi(t), a)^T$, where $a \in \mathbb{R}$, if $QNx = 0$ for some $x = (x_1, x_2)^T \in \partial\Omega \cap \operatorname{Ker} L$, then $x_2 \equiv 0$, $x_1 = a\phi(t)$, and

$$\int_0^T g(a\phi(t))\,\mathrm{d}t = 0. \tag{3.15}$$

When $c_0 < \frac{1}{2}$, we have

$$\begin{aligned}\phi(t) = A^{-1}(1) &= 1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)\\ &\geq 1 - \sum_{j=1}^{\infty}\prod_{i=1}^{j} c_0\\ &= 1 - \frac{c_0}{1-c_0}\\ &= \frac{1-2c_0}{1-c_0} := \delta_1 > 0.\end{aligned}$$

Then we have

$$a \leq \frac{d}{\delta_1}.$$

Otherwise, for all $t \in [0,T]$, $a\phi(t) > d$, from assumption (H_1), we have

$$\int_0^T g(a\phi(t))\,\mathrm{d}t < 0$$

which is contradiction to (3.15). When $\sigma > 1$, we have

$$\begin{aligned}\phi(t) = A^{-1}(1) &= -\frac{1}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}\\ &\leq -\frac{1}{\sigma} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{\sigma}\\ &= -\frac{1}{\sigma-1} := \delta_2 < 0.\end{aligned}$$

Then we have

$$a \leq -\frac{d}{\delta_2}.$$

Otherwise, for all $t \in [0,T]$, $a\phi(t) < -d$, from assumption (H_1), we have

$$\int_0^T g(a\phi(t))\,\mathrm{d}t > 0$$

which is contradiction to (3.15). One has $|x_1|_0 = \max\{\frac{d}{\delta_1}, -\frac{d}{\delta_2}\}|\phi|_0 = M \leq d$, which is a contradiction. So $QNx \neq 0$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$ and thus the condition (2) of Lemma 2.2 is satisfied. It remains to verify the condition (3) of Lemma 2.2. In order to prove it, let

$$J\colon\ \operatorname{Im} Q \to \operatorname{Ker} L, \qquad J(x_1,x_2)^T = (x_1,x_2)^T,$$

and $H(x,\mu)=\mu x+(1-\mu)JQNx$ for $(x,\mu)\in X\times[0,1]$. Then we have

$$H(x,\mu)=\begin{pmatrix}\mu x_1+\frac{(1-\mu)}{T}\int\limits_0^T\varphi_q(x_2(t))\,\mathrm{d}t\\ \mu x_2-\frac{(1-\mu)}{T}\int\limits_0^T g(x_1(t-\gamma(t)))\,\mathrm{d}t\end{pmatrix}.$$

It is not difficult to verify that, using (H_1), for any $x\in\partial\Omega\cap\operatorname{Ker}L$ and $\mu\in[0,1]$, we have $H(x,\mu)\neq 0$. Therefore,

$$\begin{aligned}\deg\{JQN,\Omega\cap\operatorname{Ker}L,0\}&=\deg\{H(\cdot,0),\Omega\cap\operatorname{Ker}L,0\}\\&=\deg\{H(\cdot,1),\Omega\cap\operatorname{Ker}L,0\}\\&=\deg\{I,\Omega\cap\operatorname{Ker}L,0\}\\&\neq 0.\end{aligned}$$

Therefore, by using Lemma 2.2, we see that the equation $Lx=Nx$ has a solution $x=(x_1,x_2)^T$ in $\bar{\Omega}$, i.e., the equation (1.1) has a T-periodic solution x_1. □

Remark 1. When $1<p<2$ or $\frac{1}{2}\le c_0<1$, there are no existence results of periodic solutions for equation (1.1). We hope that the authors are interested in doing further research on this issue.

As an application, we consider the following NFDE:

$$(\varphi_3((x(t)-(103-2\cos t)x(t-\tau))'))'+\Big(2+\frac{1}{5}\sin x\Big)x'+g(x(t-1/2\sin t))=\sin t, \tag{3.16}$$

where

$$f(x)=2+\frac{1}{5}\sin x$$

and

$$g(u)=\begin{cases}-u^2, & u>10,\\ -10u, & u\in[-10,10],\\ u^2, & u<-10.\end{cases}$$

Clearly, the Eq. (3.16) is a particular case of (1.1) in which

$$p=3,\quad c(t)=103-2\cos t,\quad \gamma(t)=\frac{1}{2}\sin t,\quad e(t)=\sin t.$$

Then we have $\sigma=101>1$, $c_0=105$, $c_1=2$, $T=2\pi$ and $r=1$, and thus

$$\frac{c_1T}{\sigma-1}=\frac{2\times 2\pi}{101-1}=0.1256<1$$

and

$$\frac{2^{p+1}(1+c_0)rT^p}{(\sigma-1-c_1T)^p}=\frac{2^4\times 106\times(2\pi)^3}{(100-4\pi)^3}\approx 0.6283<1.$$

Here assumptions (H_1) and (H_2) are satisfied. By using Theorem 3.1, we know that equation (3.16) has at least one 2π-periodic solution.

Acknowledgement

The author is very grateful to the referees for their helpful suggestions.

REFERENCES

[1] LU, S.—REN, J.—GE, W.: *Problems of periodic solutions for a kind of second order neutral functional differential equation*, Appl. Anal. **82** (2003), 411–426.
[2] ZHU, Y.—LU, S.: *Periodic solutions for p-Laplacian neutral functional differential equation with multiple deviating arguments*, J. Math. Anal. Appl. **336** (2007), 1357–1367.
[3] DU, B.—GUO, L.—GE, W.—LU, S.: *Periodic solutions for generalized Liénard neutral equation with variable parameter*, Nonlinear Anal. **70** (2009), 2387–2394.
[4] GAINES, R.—MAWHIN, J.: *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, 1977.

Received 30. 7. 2010
Accepted 14. 5. 2011

Department of Mathematics
Nanjing Normal University
210097 Nanjing
P. R. CHINA

Department of Mathematics
Huaiyin Normal University
Jiangsu
223300 Huaian
P. R. CHINA

E-mail: dubo7307@163.com