VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0102-9
Math. Slovaca **63** (2013), No. 2, 349–380

# GRAPHIC REPRESENTATION OF MV-ALGEBRA PASTINGS

Ferdinand Chovanec

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We deal with a construction of some difference posets via a method of a pasting of MV-algebras. We generalize Greechie diagrams used in MV-algebra pastings. We give necessary and sufficient conditions under which the resulting pasting of an admissible system MV-algebras is a lattice-ordered D-poset.



## 1. Introduction

A method of a construction of quantum logics (orthomodular posets and orthomodular lattices) making use of the pasting of Boolean algebras was originally suggested by Greechie in 1971 [9]. Such quantum logics are called *Greechie logics.* In Greechie logics Boolean algebras generate blocks with the intersection of each pair of blocks containing at most one atom. One of useful tools in order to construct interesting orthomodular posets and orthomodular lattices is Greechie's Loop Lemma which gives the necessary and sufficient conditions under which a Greechie logic is lattice-ordered. In addition, Greechie's pasting technique allowed to prove the existence of an orthomodular lattice admitting no state.

The method of the pasting of Boolean algebras has been later generalized by many authors, above all by Dichtl [5], Navara and Rogalewicz [15], Navara [13]. In [5], Dichtl has succeeded in obtaining characterizations of orthomodular posets and orthomodular lattices under assumptions more general than those of the Greechie's Loop Lemma. In [13], Navara formulated sufficient and necessary

---

2010 Mathematics Subject Classification: Primary 81P10, 03G12; Secondary 06D35.
Keywords: D-poset, D-lattice, MV-algebra, block, pasting, Greechie diagram, Loop Lemma.
This work has been supported by the Slovak Research and Development Agency under the contract VEGA-2/0046/11 and VEGA-2/0059/12.

conditions under which a pasting of a family of Boolean algebras is an orthoalgebra. For more details we refer the reader to Navara [14] and the references given there.

In the early nineties, Kôpka and Chovanec [12] introduced a new algebraic structure called a *difference poset* (*D-poset* for short). In this structure difference of comparable elements is a primary notion. Independently, Foulis and Bennett [7] introduced an essentially equivalent structure called an *effect algebra* with a partial addition as a primary operation. The notion of a D-poset (an effect algebra) generalizes orthoalgebras (and hence orthomodular posets), MV-algebras (including Boolean algebras) as well as the system of Hilbert-space effects which plays an important role in the theory of so-called unsharp quantum measurements.

Short time after D-posets (effect algebras) were discovered, the attempts have arisen to generalize the method of the pasting in order to construct miscellaneous examples of difference posets and in order to study difference lattices admitting no states (probability measures). These efforts were successful only after Riečanová [17] proved that every lattice-ordered effect algebra (D-lattice) is a set-theoretical union of maximal sub-D-lattices of pairwise compatible elements, i.e. maximal sub-MV-algebras.

A method of a construction of difference lattices by means of an MV-algebra pasting was originally suggested in [4]. Thereafter many authors have tried to use another pasting techniques in order to construct various types of difference posets (effect algebras) (see [18], [11], [14], [19]).

In this paper, we give some re-formulations of the basic notions introduced in [4] and generalize Greechie diagrams used on a graphical representation of Greechie logics. Finally we present some sufficient conditions under which a pasting of an admissible system of MV-algebras is a lattice-ordered D-poset.

## 2. Basic definitions and facts

In this section, we summarize some necessary definitions and facts about D-posets. For more details we refer to [3] or [6].

Let $\mathcal{P}$ be a bounded partially ordered set with the least element $0_{\mathcal{P}}$ and the greatest one $1_{\mathcal{P}}$. Let $\ominus$ be a partial binary difference operation on $\mathcal{P}$ such that there is $b \ominus a$ in $\mathcal{P}$ if and only if $a \leq b$ and the following axioms hold.

(D1) $a \ominus 0_{\mathcal{P}} = a$ for any $a \in \mathcal{P}$.

(D2) $a \leq b \leq c$ implies $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The structure $(\mathcal{P}, \le, \ominus, 0_\mathcal{P}, 1_\mathcal{P})$ is called a *difference poset* (a *D-poset*). For the simplicity of the notation, we write $\mathcal{P}$ instead of $(\mathcal{P}, \le, \ominus, 0_\mathcal{P}, 1_\mathcal{P})$.

A lattice-ordered D-poset is called a *D-lattice*.

*Example* 1. Let $(\mathcal{P}, \le, {}', 0_\mathcal{P}, 1_\mathcal{P})$ be an orthomodular poset (an orthomodular lattice, resp.), where $'$ is an orthocomplementation. For $a, b \in \mathcal{P}$ such that $a \le b$ we define $b \ominus a$ as follows

$$b \ominus a = b \wedge a'.$$

Then $(\mathcal{P}, \le, \ominus, 0_\mathcal{P}, 1_\mathcal{P})$ is a D-poset (a D-lattice).

A D-lattice $\mathcal{P}$ is said to be *σ-complete* if for any countable sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{P}$ the least upper bound $\bigvee\limits_{n=1}^{\infty} a_n$ and the greatest lower bound $\bigwedge\limits_{n=1}^{\infty} a_n$ exist in $\mathcal{P}$.

A non-zero element $a$ of a D-poset $\mathcal{P}$ is called an *atom* if the inequality $b \le a$ entails either $b = 0_\mathcal{P}$ or $b = a$. A D-poset $\mathcal{P}$ is said to be *atomic* if for any non-zero element $b \in \mathcal{P}$ there is an atom $a \in \mathcal{P}$ such that $a \le b$.

For any element $a$ in a D-poset, the element $1_\mathcal{P} \ominus a$ is called the *orthosupplement* of $a$ and is denoted by $a^\perp$. The unary operation $\perp \colon a \mapsto a^\perp$ is an involution ($(a^\perp)^\perp = a$) and order reversing ($a \le b$ implies $b^\perp \le a^\perp$).

The set $[a, b] = \{x \in \mathcal{P} : \ a \le x \le b\}$ is called an *interval* in a D-poset $\mathcal{P}$.

In every D-poset, a partial operation $\oplus$ (an *orthosummation*) can be defined as follows.

$$a \oplus b = (a^\perp \ominus b)^\perp, \qquad \text{for} \quad a \leqq b^\perp.$$

It is easy to see that

(i) $a \oplus a^\perp = 1_\mathcal{P}$,

(ii) $a \oplus 0_\mathcal{P} = a$,

(iii) $a \oplus b = b \oplus a \quad \text{if} \quad a \le b^\perp$.

An additive counterpart to a D-poset is an *effect algebra* introduced by Foulis and Bennett in [7]. Although D-posets and effect algebras are essentially equivalent structures, D-posets seem preferable when we want to emphasize the primary role of the difference operation.

A D-poset (an effect algebra) $\mathcal{P}$ with the property $a \le a^\perp$ implies $a = 0_\mathcal{P}$ is called an *orthoalgebra* [8]. Orthomodular posets and lattices are special types of orthoalgebras.

Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in a D-poset $\mathcal{P}$. We define

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \qquad n \ge 3,$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus\ a_{n-1}) \oplus a_n$ exist in $\mathcal{P}$. We say that a finite system $F = \{a_1, a_2, \ldots, a_n\}$ is $\oplus$-*orthogonal*, if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists in $\mathcal{P}$ and then we write

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n = \bigoplus_{i=1}^{n} a_i.$$

For every $a \in \mathcal{P}$ and positive integer $n$ we define $0a = 0_{\mathcal{P}}$ and $(n+1)a = na \oplus a$, if all involved elements exist. The greatest $n$ such that $na$ exists is called the *isotropic index* of $a$ and denoted $\tau(a)$. If $na$ exists for every integer $n$ then $\tau(a) = \infty$. A D-poset $\mathcal{P}$ is called *Archimedean* if $\tau(a) < \infty$ for every non-zero element $a \in \mathcal{P}$. Every $\sigma$-complete D-poset is Archimedean.

A finite set of atoms $\{a_1, a_2, \ldots, a_n\}$ of an Archimedean D-poset $\mathcal{P}$ is called a *finite atomic decomposition of the unity*, if the set $\{\tau(a_1)a_1, \tau(a_2)a_2, \ldots$ $\ldots, \tau(a_n)a_n\}$ is $\oplus$-orthogonal and

$$\bigoplus_{i=1}^{n} \tau(a_i)a_i = 1_{\mathcal{P}}.$$

Elements $a$ and $b$ from a D-poset $\mathcal{P}$ are called *compatible* and we write $a \leftrightarrow b$, if there are $c, d \in \mathcal{P}$, $c \le a \le d$ and $c \le b \le d$ such that $d \ominus a = b \ominus c$.

If $a \leftrightarrow b$ then the elements of the set $\{a, b, a^{\perp}, b^{\perp}\}$ are mutually compatible. In a D-lattice, $a \leftrightarrow b$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$.

It is well known that an orthomodular lattice of pairwise mutually compatible elements forms a Boolean algebra. According to [2], a D-lattice of pairwise mutually compatible elements forms an MV-algebra (introduced by Chang in [1]), therefore, MV-algebras play a similar role in difference posets as Boolean algebras do in orthomodular structures.

Let $\mathcal{P}$ be a D-poset. A subset $\mathcal{Q} \subseteq \mathcal{P}$ is called a *sub-D-poset* of $\mathcal{P}$ if $1_{\mathcal{P}} \in \mathcal{Q}$ and $b \ominus a \in \mathcal{Q}$ for every $a, b \in \mathcal{Q}$ such that $a \le b$.

If a sub-D-poset $\mathcal{Q}$ of a D-poset $\mathcal{P}$ is organized as a Boolean algebra (an MV-algebra) with respect to the order and the difference defined in $\mathcal{P}$, we call $\mathcal{Q}$ a Boolean subalgebra (a sub-MV-algebra) of $\mathcal{P}$.

Let $\mathcal{P} = (\mathcal{P}, \le_{\mathcal{P}}, \ominus_{\mathcal{P}}, 0_{\mathcal{P}}, 1_{\mathcal{P}})$ and $\mathcal{T} = (\mathcal{T}, \le_{\mathcal{T}}, \ominus_{\mathcal{T}}, 0_{\mathcal{T}}, 1_{\mathcal{T}})$ be D-posets. A mapping $w\colon \mathcal{P} \to \mathcal{T}$ is called a *D-morphism* of $\mathcal{P}$ into $\mathcal{T}$ if the following conditions are satisfied.

(DH1) $w(1_{\mathcal{P}}) = 1_{\mathcal{T}}$.

(DH2) If $a, b \in \mathcal{P}$, $b \le_{\mathcal{P}} a$, then $w(a) \le_{\mathcal{T}} w(b)$.

(DH3) If $a, b \in \mathcal{P}$, $b \le_{\mathcal{P}} a$, then $w(b \ominus_{\mathcal{P}} a) = w(b) \ominus_{\mathcal{T}} w(a)$.

If, moreover, $w$ is bijective and its inverse is also a D-morphism, then $w$ is called an isomorphism and we say that $\mathcal{P}$ and $\mathcal{T}$ are isomorphic.

It is known (see [16]) that compatible events of a quantum mechanical system belong to some classical subsystem. It means that from the algebraic point of view compatible elements of a quantum logic (an orthomodular poset) belong to a Boolean subalgebra of this logic. A maximal Boolean subalgebra of a quantum logic (an orthoalgebra) is called a *block*. In D-posets, a more general definition of a block has been used.

**Definition 2.** A block in a D-poset $\mathcal{P}$ is a maximal sub-MV-algebra of $\mathcal{P}$.

Note that if $\mathcal{A}$ is a maximal sub-MV-algebra of an orthoalgebra $\mathcal{P}$ then $\mathcal{A}$ is simultaneously a maximal Boolean subalgebra of $\mathcal{P}$.

If $\{a_1, a_2, \dots, a_n\}$ is a finite atomic decomposition of the unity of an Archimedean D-lattice $\mathcal{P}$, then the set

$$\mathcal{B} = \Big\{x \in \mathcal{P} : \ x = \bigoplus_{i=1}^{n} \alpha_i a_i, \ \ 0 \le \alpha_i \le \tau(a_i)\Big\}$$

is a block in $\mathcal{P}$.

In [13], Navara showed that every orthoalgebra is the union of its blocks. A similar result for D-lattices was achieved by Riečanová in [17]. This outcome evoked a question how we could construct a D-poset from a given collection of MV-algebras. The first attempts to solve this problem appeared in [4], but later it has been revealed that some notions require a revision, especially the definition of an admissible system MV-algebras for a pasting.

## 3. Construction of an MV-algebra pasting

In this section, we will deal with the problem of constructions of difference posets by the method of an MV-algebra pasting. At first we have to bring an answer to the question: What do we mean by an MV-algebra pasting? The answer consists in the following definition.

**Definition 3.** Let $\mathcal{S} = \{\mathcal{A}_t : \ t \in T\}$ be a countable system of atomic $\sigma$-complete MV-algebras. By an *MV-algebra pasting* of the system $\mathcal{S}$ we understand a construction of a difference poset $\mathcal{P}$ such that the following conditions are fulfilled.

(P1) There is a system $\mathcal{S}^* = \{\mathcal{A}_t^* : \ t \in T\}$ of maximal sub-MV-algebras (blocks) of $\mathcal{P}$.

(P2) There is a bijection $\psi$ from $\mathcal{S}$ onto $\mathcal{S}^*$ such that the MV-algebras $\mathcal{A}_t$ and blocks $\mathcal{A}_t^* = \psi(\mathcal{A}_t)$ are isomorphic for every $t \in T$.

(P3) $\mathcal{P} = \bigcup\limits_{t \in T} \mathcal{A}_t^*$.

We denote the set of all atoms of an MV-algebra $\mathcal{A}$ by $\mathrm{At}(\mathcal{A})$ and the cardinality of a set A by $|A|$.

**Definition 4.** Let $\mathcal{A}$ and $\mathcal{B}$ be different atomic $\sigma$-complete MV-algebras. Let $A$ and $B$ be finite sets of atoms such that $A \subset \mathrm{At}(\mathcal{A})$, $B \subset \mathrm{At}(\mathcal{B})$ and $|A| = |B|$. We say that the sets $A$ and $B$ are *isotropically equivalent*, and write $A \sim_\tau B$, if there is a bijection $\varphi\colon A \to B$ such that $\tau(a) = \tau(\varphi(a))$ for every $a \in A$.

Note that the relation $\sim_\tau$ is symmetric and transitive, and moreover, $A \sim_\tau B$ whenever $A = \emptyset$ and $B = \emptyset$.

Let $\mathcal{S} = \{\mathcal{A}_t :\ t \in T\}$ be a countable system of atomic $\sigma$-complete MV-algebras. We choose exactly one pair $(A, B)$ of isotropically equivalent sets of atoms from every couple $(\mathcal{A}_t, \mathcal{A}_s)$ $(t \neq s)$ of MV-algebras of the system $\mathcal{S}$ and one bijection $\varphi_{ts}$ such that $B = \varphi_{ts}(A)$ and $\tau(\varphi_{ts}(a)) = \tau(a)$ for any $a \in A$. Let us denote such a choice by $\mathcal{U}$. Thus

$$\mathcal{U} = \big\{((A,B),\varphi_{ts}) :\ A \subset \mathrm{At}(\mathcal{A}_t),\ B \subset \mathrm{At}(\mathcal{A}_s),\ B = \varphi_{ts}(A),\ \tau(\varphi_{ts}(a)) = \tau(a)\big\}.$$

We demand, in addition, that the choice $\mathcal{U}$ has the following properties.

(U1) If $((A,B),\varphi_{ts}) \in \mathcal{U}$ then $\big((B,A),\varphi_{ts}^{-1}\big) \in \mathcal{U}$, where $\varphi_{ts}^{-1}$ is the inverse map of $\varphi_{ts}$.

(U2) If $A \subset \mathrm{At}(\mathcal{A}_t)$, $B \subset \mathrm{At}(\mathcal{A}_s)$, $C \subset \mathrm{At}(\mathcal{A}_r)$ such that $((A,B),\varphi_{ts}) \in \mathcal{U}$ and $((B,C),\varphi_{sr}) \in \mathcal{U}$, then $((A,C),\varphi_{tr}) \in \mathcal{U}$, where $\varphi_{tr} = \varphi_{sr} \circ \varphi_{ts}$ (i.e. $\varphi_{tr}(a) = \varphi_{sr}(\varphi_{ts}(a))$ for all $a \in A$).

**Definition 5.** Let $\mathcal{S} = \{\mathcal{A}_t :\ t \in T\}$ be a countable system of atomic $\sigma$-complete MV-algebras and $\mathcal{U}$ be a choice of pairs of isotropically equivalent sets of atoms (described above). A couple $(\mathcal{S},\mathcal{U})$ is called the *admissible system* of MV-algebras (for a pasting), if the following conditions hold for arbitrary MV-algebras $\mathcal{A}_t$, $\mathcal{A}_s$, $\mathcal{A}_r \in \mathcal{S}$.

(AS1) If $((A,B),\varphi_{ts}) \in \mathcal{U}$ such that $A \subset \mathrm{At}(\mathcal{A}_t)$ and $B \subset \mathrm{At}(\mathcal{A}_s)$, then $\mathrm{At}(\mathcal{A}_t) \setminus A \neq \emptyset$ and $\mathrm{At}(\mathcal{A}_s) \setminus B \neq \emptyset$. Moreover, if $\mathrm{At}(\mathcal{A}_t) \setminus A = \{a\}$, resp. $\mathrm{At}(\mathcal{A}_s) \setminus B = \{b\}$, then $\tau(a) > 1$, resp. $\tau(b) > 1$.

(AS2) If $A \subset \mathrm{At}(\mathcal{A}_t)$, $B \subset \mathrm{At}(\mathcal{A}_s)$, $C \subset \mathrm{At}(\mathcal{A}_r)$ such that $((A,B),\varphi_{ts}) \in \mathcal{U}$ and $((\mathrm{At}(\mathcal{A}_t) \setminus A, C),\varphi_{tr}) \in \mathcal{U}$ then there are an MV-algebra $\mathcal{A}_p \in \mathcal{S}$, a subset $D \subset \mathrm{At}(\mathcal{A}_p)$ and bijections $\varphi_{sp}$, $\varphi_{rp}$ such that $((\mathrm{At}(\mathcal{A}_s) \setminus B, D),\varphi_{sp}) \in \mathcal{U}$ and $((\mathrm{At}(\mathcal{A}_r) \setminus C, \mathrm{At}(\mathcal{A}_p) \setminus D),\varphi_{rp}) \in \mathcal{U}$.

Note that every countable system $\mathcal{S}$ of atomic $\sigma$-complete MV-algebras is admissible with respect to a choice $\mathcal{U}_0$ of pairs of empty sets.

Let $(\mathcal{S},\mathcal{U})$ be an admissible system of MV-algebras, where $\mathcal{S} = \{\mathcal{A}_t :\ t \in T\}$. We define a relation $\sim$ on $\bigcup\limits_{t \in T} \mathcal{A}_t$ in the following way.

(1) If $((\emptyset,\emptyset),\varphi_{ts})\in\mathcal{U}$, then $0_{\mathcal{A}_t}\sim 0_{\mathcal{A}_s}$ and $1_{\mathcal{A}_t}\sim 1_{\mathcal{A}_s}$.

(2) If $x,y\in\mathcal{A}_t$, then $x\sim y$ if and only if $x=y$.

(3) Let $x\in(\mathcal{A}_t,\leq_t,\ominus_t,0_{\mathcal{A}_t},1_{\mathcal{A}_t})$, $y\in(\mathcal{A}_s,\leq_s,\ominus_s,0_{\mathcal{A}_s},1_{\mathcal{A}_s})$, $((A,B),\varphi_{ts})\in\mathcal{U}$, $A=\{a_1,a_2,\dots,a_n\}$, $n\geq 1$. Then $x\sim y$ if there are $\alpha_i\in\{0,1,2,\dots,\tau(a_i)\}$ for $i=1,2,\dots,n$, such that either

$$x=\alpha_1a_1\oplus_t\alpha_2a_2\oplus_t\cdots\oplus_t\alpha_na_n=\bigoplus_{i=1}^{n}{}_t\,\alpha_ia_i$$

and

$$y=\alpha_1\varphi_{ts}(a_1)\oplus_s\alpha_2\varphi_{ts}(a_2)\oplus_s\cdots\oplus_s\alpha_n\varphi_{ts}(a_n)=\bigoplus_{i=1}^{n}{}_s\,\alpha_i\varphi_{ts}(a_i),$$

or

$$x^{\perp_t}=\bigoplus_{i=1}^{n}{}_t\,\alpha_ia_i\qquad\text{and}\qquad y^{\perp_s}=\bigoplus_{i=1}^{n}{}_s\,\alpha_i\varphi_{ts}(a_i).$$

(4) Let $x\in\mathcal{A}_t$, $y\in\mathcal{A}_s$, $t\neq s$, $A=\{a_1,\dots,a_n\}\subset\mathrm{At}(\mathcal{A}_t)$, $n\geq 1$, $\alpha_i\in\{0,\dots,\tau(a_i)\}$ for $i=1,2,\dots,n$, $B=\{b_1,\dots,b_m\}\subset\mathrm{At}(\mathcal{A}_s)$, $m\geq 1$, $\beta_j\in\{0,1,\dots,\tau(b_j)\}$ for $j=1,2,\dots,m$, such that

$$x=\bigoplus_{i=1}^{n}{}_t\,\alpha_ia_i,\qquad y=\bigoplus_{j=1}^{m}{}_s\,\beta_ib_i.$$

Let $(A,X)\notin\mathcal{U}$ and $(Y,B)\notin\mathcal{U}$ for every $X\subset\mathrm{At}(\mathcal{A}_s)$ and $Y\subset\mathrm{At}(\mathcal{A}_t)$. Let $\mathcal{A}_r,\mathcal{A}_q\in\mathcal{S}$, $C\subset\mathrm{At}(\mathcal{A}_r)$, $D\subset\mathrm{At}(\mathcal{A}_q)$, such that $((A,C),\varphi_{tr})\in\mathcal{U}$, $((\mathrm{At}(\mathcal{A}_r)\setminus C,\mathrm{At}(\mathcal{A}_s)\setminus B),\varphi_{rs})\in\mathcal{U}$, $((\mathrm{At}(\mathcal{A}_t)\setminus A,\mathrm{At}(\mathcal{A}_q)\setminus D),\varphi_{tq})\in\mathcal{U}$, $((B,D),\varphi_{sq})\in\mathcal{U}$ and $\mathrm{At}(\mathcal{A}_r)\setminus C=\{e_1,\dots,e_{n_1}\}$, $n_1\geq 1$, $\mathrm{At}(\mathcal{A}_q)\setminus D=\{f_1,\dots,f_{n_2}\}$, $n_2\geq 1$. Let

$$c=\bigoplus_{i=1}^{n}{}_r\,\alpha_i\varphi_{tr}(a_i)\qquad\text{and}\qquad d=\bigoplus_{j=1}^{m}{}_q\,\beta_j\varphi_{sq}(b_j)$$

($x\sim c$ and $y\sim d$ in the sense of (3)). If there are $\gamma_k\in\{0,1,\dots,\tau(e_k)\}$ for $k=1,\dots,n_1$, and $\delta_l\in\{0,1,\dots,\tau(f_l)\}$ for $l=1,\dots,n_2$, such that

$$c^{\perp_r}=\bigoplus_{k=1}^{n_1}{}_r\,\gamma_ke_k,\quad y^{\perp_s}=\bigoplus_{k=1}^{n_1}{}_s\,\gamma_k\varphi_{rs}(e_k)\qquad(c\sim y\ \text{in the sense of (3)})$$

and

$$d^{\perp_q}=\bigoplus_{l=1}^{n_2}{}_q\,\delta_lf_l,\quad x^{\perp_t}=\bigoplus_{l=1}^{n_2}{}_t\,\delta_l\varphi_{qt}(f_l)\qquad(d\sim x\ \text{in the sense of (3)}),$$

then $x\sim y$.

Note that $x\sim y$ if and only if $x^{\perp_t}\sim y^{\perp_s}$. In addition, $0_{\mathcal{A}_t}\sim 0_{\mathcal{A}_s}$ and $1_{\mathcal{A}_t}\sim 1_{\mathcal{A}_s}$ whenever $((A,B),\varphi_{ts})\in\mathcal{U}$.

The relation $\sim$ is reflexive, symmetric and transitive, so it is an equivalence on $\bigcup\limits_{t\in T} \mathcal{A}_t$.

Let $[x]$ be an equivalence class determined by $x$ and let $\mathcal{P}$ be the quotient set, i.e.

$$[x] = \Big\{y \in \bigcup_{t\in T} \mathcal{A}_t : \ y \sim x\Big\} \qquad \text{and} \qquad \mathcal{P} = \Big\{[x] : \ x \in \bigcup_{t\in T} \mathcal{A}_t\Big\}.$$

If we denote $\mathcal{A}_t^* = \{[x] : \ x \in \mathcal{A}_t\}$ then $\mathcal{P} = \bigcup\limits_{t\in T} \mathcal{A}_t^*$.

A partial ordering $\leq$ and a difference $\ominus$ on $\mathcal{P}$ are defined as follows.

$[x] \leq [y]$ if and only if there is an MV-algebra $(\mathcal{A}_t, \leq_t, \ominus_t, 0_{\mathcal{A}_t}, 1_{\mathcal{A}_t}) \in \mathcal{S}$ and elements $u, v \in \mathcal{A}_t$ such that $u \in [x]$, $v \in [y]$ and $u \leq_t v$. In this case we define

$$[y] \ominus [x] := [v \ominus_t u].$$

We prove that the relation $\leq$ and the partial operation $\ominus$ are independent of the choice of representatives.

Let $((A,B), \varphi_{ts}) \in \mathcal{U}$, $A = \{a_1, \dots, a_n\} \subset \mathrm{At}(\mathcal{A}_t)$, $B = \varphi_{ts}(A)$. Suppose that $u_1, v_1 \in \mathcal{A}_t$ and $u_2, v_2 \in \mathcal{A}_s$ such that $u_1 \sim u_2$ and $v_1 \sim v_2$. We show that $u_1 \leq_t v_1$ if and only if $u_2 \leq_s v_2$, and moreover, $v_1 \ominus_t u_1 \sim v_2 \ominus_s u_2$. The inequality $u_1 \leq_t v_1$ yields only three possibilities.

(i) If $v_1 = \bigoplus\limits_{i=1}^{n}{}_t\, \alpha_i a_i$ then necessarily $u_1 = \bigoplus\limits_{i=1}^{n}{}_t\, \beta_i a_i$, where $0 \leq \beta_i \leq \alpha_i \leq \tau(a_i)$ for every $i = 1, 2, \dots, n$. Since $u_1 \sim u_2$ and $v_1 \sim v_2$, it follows that $u_2 = \bigoplus\limits_{i=1}^{n}{}_s\, \beta_i \varphi_{ts}(a_i)$, $v_2 = \bigoplus\limits_{i=1}^{n}{}_s\, \alpha_i \varphi_{ts}(a_i)$ and thus $u_2 \leq_s v_2$. In addition,

$$v_1 \ominus_t u_1 = \bigoplus_{i=1}^{n}{}_t\, (\alpha_i - \beta_i) a_i \qquad \text{and} \qquad v_2 \ominus_s u_2 = \bigoplus_{i=1}^{n}{}_s\, (\alpha_i - \beta_i) \varphi_{ts}(a_i),$$

so, $v_1 \ominus_t u_1 \sim v_2 \ominus_s u_2$, therefore $[v_2 \ominus_s u_2] = [v_1 \ominus_t u_1] = [y] \ominus [x]$.

(ii) If $u_1 = \Big(\bigoplus\limits_{i=1}^{n}{}_t\, \beta_i a_i\Big)^{\perp_t}$ then $v_1 = \Big(\bigoplus\limits_{i=1}^{n}{}_t\, \alpha_i a_i\Big)^{\perp_t}$, where $0 \leq \alpha_i \leq \beta_i \leq \tau(a_i)$ for every $i = 1, 2, \dots, n$. Then using of duality and (i) we immediately obtain

$$u_2 = \left(\bigoplus_{i=1}^{n}{}_s\, \beta_i \varphi_{ts}(a_i)\right)^{\perp_s} \leq_s \left(\bigoplus_{i=1}^{n}{}_s\, \alpha_i \varphi_{ts}(a_i)\right)^{\perp_s} = v_2$$

and also $v_1 \ominus_t u_1 \sim v_2 \ominus_s u_2$.

(iii) If $u_1 = \bigoplus_{i=1}^{n}{}_t\, \beta_i a_i$ and $v_1 = \left(\bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i\right)^{\perp_t}$, then $\beta_i + \alpha_i \leq \tau(a_i)$ for every $i = 1, 2, \ldots, n$, because the inequality $\bigoplus_{i=1}^{n}{}_t\, \beta_i a_i \leq_t \left(\bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i\right)^{\perp_t}$ implies that

$$\left(\bigoplus_{i=1}^{n}{}_t\, \beta_i a_i\right) \oplus_t \left(\bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i\right) = \bigoplus_{i=1}^{n}{}_t\, (\beta_i + \alpha_i) a_i \leq_t \bigoplus_{i=1}^{n}{}_t\, \tau(a_i) a_i.$$

Apparently $u_2 = \bigoplus_{i=1}^{n}{}_s\, \beta_i \varphi_{ts}(a_i)$ and $v_2 = \left(\bigoplus_{i=1}^{n}{}_s\, \alpha_i \varphi_{ts}(a_i)\right)^{\perp_s}$. As

$$\begin{aligned} 1_{\mathcal{A}_s} \quad \geq_s \quad & \bigoplus_{i=1}^{n}{}_s\, \tau(a_i)\varphi_{ts}(a_i) \geq_s \bigoplus_{i=1}^{n}{}_s\, (\beta_i + \alpha_i)\varphi_{ts}(a_i) \\ = \quad & \left(\bigoplus_{i=1}^{n}{}_s\, \beta_i \varphi_{ts}(a_i)\right) \oplus_s \left(\bigoplus_{i=1}^{n}{}_s\, \alpha_i \varphi_{ts}(a_i)\right), \end{aligned}$$

we get $u_2 = \bigoplus_{i=1}^{n}{}_s\, \beta_i \varphi_{ts}(a_i) \leq_s \left(\bigoplus_{i=1}^{n}{}_s\, \alpha_i \varphi_{ts}(a_i)\right)^{\perp_s} = v_2$. We have

$$(v_1 \ominus_t u_1)^{\perp_t} = v_1^{\perp_t} \oplus_t u_1 = \left(\bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i\right) \oplus_t \left(\bigoplus_{i=1}^{n}{}_t\, \beta_i a_i\right) = \bigoplus_{i=1}^{n}{}_t\, (\alpha_i + \beta_i) a_i,$$

and similarly

$$(v_2 \ominus_s u_2)^{\perp_s} = v_2^{\perp_s} \oplus_s u_2 = \bigoplus_{i=1}^{n}{}_s\, (\alpha_i + \beta_i)\varphi_{ts}(a_i),$$

which gives $(v_1 \ominus_t u_1)^{\perp_t} \sim (v_2 \ominus_s u_2)^{\perp_s}$ and consequently $v_1 \ominus_t u_1 \sim v_2 \ominus_s u_2$.

Note that the case $u_1 = \left(\bigoplus_{i=1}^{n}{}_t\, \beta_i a_i\right)^{\perp_t} \leq_t \bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i = v_1$ is impossible. Indeed, if we denote $v = \bigoplus_{i=1}^{n}{}_t\, \tau(a_i) a_i$, then

$$v^{\perp_t} = \left(\bigoplus_{i=1}^{n}{}_t\, \tau(a_i) a_i\right)^{\perp_t} \leq_t \left(\bigoplus_{i=1}^{n}{}_t\, \beta_i a_i\right)^{\perp_t} \leq_t \bigoplus_{i=1}^{n}{}_t\, \alpha_i a_i \leq_t \bigoplus_{i=1}^{n}{}_t\, \tau(a_i) a_i = v,$$

hence $v^{\perp_t} = v \wedge_t v^{\perp_t} = 0_{\mathcal{A}_t}$, which gives $v = 1_{\mathcal{A}_t}$, a contradiction. In the same manner it can be proved that $u_2 \leq_s v_2$ implies $u_1 \leq_t v_1$.

For this reason $\mathcal{P}$ is a partially ordered set with the greatest element $[1_{\mathcal{A}_t}]$ (denoted by $1_{\mathcal{P}}$) and the least element $[0_{\mathcal{A}_t}]$ (denoted by $0_{\mathcal{P}}$). The partial operation $\ominus$ satisfies the axioms (D1) and (D2), so $(\mathcal{P}, \leq, 1_{\mathcal{P}}, 0_{\mathcal{P}}, \ominus)$ is a D-poset. Moreover, $\mathcal{P}$ is an MV-algebra pasting in the sense of Definition 3.

We note that the Greechie logic is a specific case of an MV-algebra pasting described above. Contrary to the Greechie logic, the intersection of blocks in an MV-algebra pasting may contain more than one atom.

If $(\mathcal{S}, \mathcal{U}_0)$ is an admissible system of MV-algebras with respect to a choice of pairs of empty sets, then such a pasting is called the 0-1-pasting or the horizontal sum of MV-algebras. Every 0-1-pasting of MV-algebras is a lattice-ordered D-poset, specifically, the 0-1-pasting of an admissible system of Boolean algebras creates an orthomodular lattice.

## 4. Graphical diagrams of MV-algebra pastings

A very useful tool for a graphical representation of finite partially ordered sets (posets) and Greechie logics are Hasse and Greechie diagrams. A Hasse diagram of a finite poset $\mathcal{P}$ is an oriented graph (digraph) where objects, called *vertices*, are elements of $\mathcal{P}$ and *edges* correspond to the covering relation. Vertices are usually drawn as points or small black circles and edges as lines going from lower to higher covering elements, i. e. edges are upward directed. A Greechie diagram of a Greechie logic $\mathcal{L}$ is a hypergraph where vertices are atoms of $\mathcal{L}$ and edges correspond to blocks (maximal Boolean sub-algebras) in $\mathcal{L}$. Vertices are drawn as points or small black circles and edges as smooth lines connecting atoms belonging to a block. Greechie diagrams are basically condensed Hasse diagrams.

A Greechie diagram of a finite Boolean algebra enables to reconstruct this algebra. If a Boolean algebra $\mathcal{A}$ contains $n$ atoms (the Greechie diagram of $\mathcal{A}$ consists of $n$ vertices lying on one line), then $\mathcal{A}$ is isomorfic to the power set of a set with $n$ elements and $|\mathcal{A}| = 2^n$.

If $\mathrm{At}(\mathcal{M}) = \{a_1, a_2, \dots, a_n\}$ is a set of all atoms of a finite MV-algebra $\mathcal{M}$, then $\mathcal{M}$ is uniquely determined, up to isomorphism, by isotropic indices of its atoms and in this case we write $\mathcal{M} = \mathcal{M}(\tau(a_1), \tau(a_2), \dots, \tau(a_n))$. The cardinality of the MV-algebra $\mathcal{M}(\tau(a_1), \tau(a_2), \dots, \tau(a_n))$ is equal to the following expression

$$|\mathcal{M}(\tau(a_1), \tau(a_2), \dots, \tau(a_n))| = (\tau(a_1) + 1)\,(\tau(a_2) + 1) \cdots (\tau(a_n) + 1)\,.$$

Let us assume MV-algebras $\mathcal{A}(2,3)$ and $\mathcal{B}(3,2)$. Then $|\mathcal{A}(2,3)| = |\mathcal{B}(3,2)| = 12$ and it is obvious that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

For a graphical representing of MV-algebra pastings we use also Greechie diagrams. In this case we denote a vertex $a$ of a Greechie diagram in the form $a(\tau(a))$, where $a$ is an atom and $\tau(a)$ is its isotropic index. Then a finite MV-algebra is uniquely determined by its Greechie diagram.

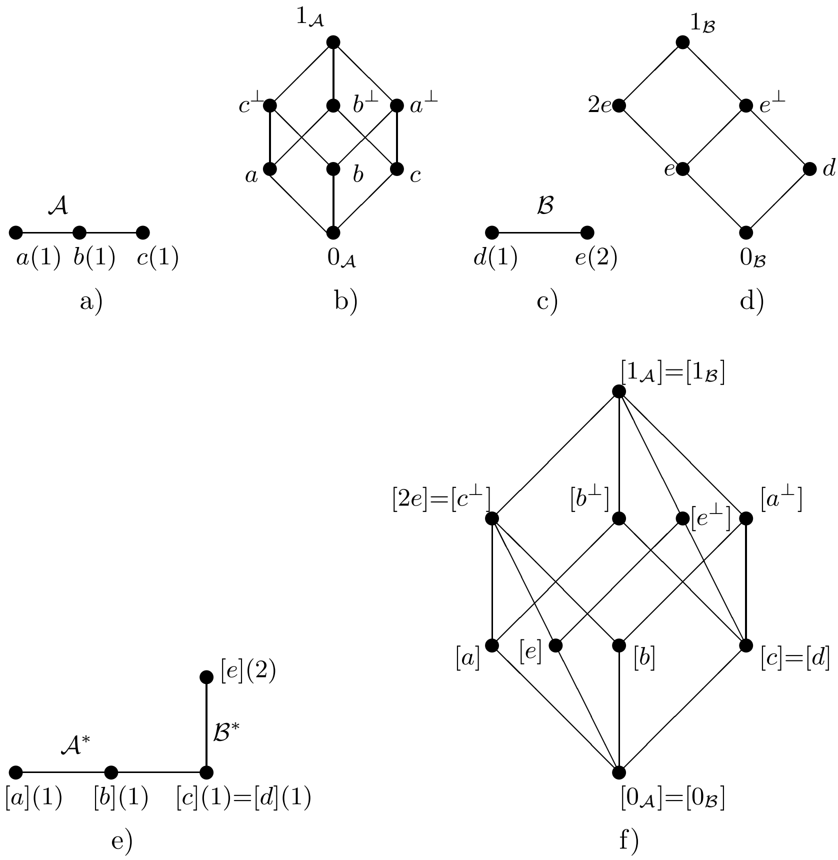


FIGURE 1. Greechie and Haase diagrams (Example 6)

*Example* 6. Let $\mathcal{A} = \{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1)\}$ and $\mathcal{B} = \{(0,0),(1,0),(0,1),(1,1),(2,0),(2,1)\}$. If an ordering and a difference of comparable elements are defined by coordinates, then $\mathcal{A}$ becomes a Boolean algebra and $\mathcal{B}$ becomes an MV-algebra. Greechie diagrams of $\mathcal{A}$ and $\mathcal{B}$ are displayed in Fig. 1a) and 1c), respectively. Their Hasse diagrams are displayed in Fig. 1b) and 1d). Let us put $0_{\mathcal{A}} = (0,0,0)$, $a = (1,0,0)$, $b = (0,1,0)$, $c = (0,0,1)$, $1_{\mathcal{A}} = (1,1,1)$, $0_{\mathcal{B}} = (0,0)$, $d = (1,0)$, $e = (0,1)$, $1_{\mathcal{B}} = (2,1)$. Then $\mathrm{At}(\mathcal{A}) = \{a,b,c\}$ and $\mathrm{At}(\mathcal{B}) = \{d,e\}$. We see that $\tau(c) = 1 = \tau(d)$, therefore, if we put

$$\mathcal{U} = \big\{((\{c\},\{d\}),\varphi)\,,\big((\{d\},\{c\}),\varphi^{-1}\big)\big\},$$

where $\varphi(c) = d$ and $\varphi^{-1}(d) = c$, then $\{c\} \sim_{\tau} \{d\}$ and hence $c \sim d$, $c^{\perp} \sim d^{\perp} = 2e$, $0_{\mathcal{A}} \sim 0_{\mathcal{B}}$, $1_{\mathcal{A}} \sim 1_{\mathcal{B}}$. This gives that $[c] = [d]$, $[c^{\perp}] = [2e]$, $[0_{\mathcal{A}}] = [0_{\mathcal{B}}]$, $[1_{\mathcal{A}}] = [1_{\mathcal{B}}]$ and in the remaining cases $x \in [y]$ if and only if $x = y$. The structures $\mathcal{A}$ and

$\mathcal{B}$ create an admissible system of MV-algebras with respect to the choice $\mathcal{U}$, so their pasting $\mathcal{P}$ exists and $\mathcal{P} = \{[x] : x \in \mathcal{A} \cup \mathcal{B}\}$. The pasting $\mathcal{P}$ is presented by the Greechie diagram in Fig. 1e) and by the Hasse diagram in Fig. 1f). Evidently $\mathcal{P}$ is a D-lattice.

Greechie diagrams are useful only in the case if the intersection of blocks contains a small number of atoms. Otherwise we suggest to use so-called *cluster Greechie diagrams.*

**Definition 7.** Let $\mathcal{P} = \bigcup\limits_{t \in T} \mathcal{A}_t^*$ be an MV-algebra pasting of an admissible system $(\mathcal{S}, \mathcal{U})$, where $\mathcal{S} = \{\mathcal{A}_t : t \in T\}$ is a countable system of atomic $\sigma$-complete MV-algebras and $\mathcal{U}$ is a choice of pairs of isotropically equivalent sets of atoms. A cluster Greechie diagram (a CG-diagram for short) is a hypergraph $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ (the set of vertices) is a system of pairwise disjoint subsets of $\mathrm{At}(\mathcal{P})$ such that $\bigcup \mathcal{V} = \mathrm{At}(\mathcal{P})$ and $\mathcal{E}$ (the set of edges) is a system of sets of atoms of individual blocks in $\mathcal{P}$, i.e. $\mathcal{E} = \{\mathrm{At}(\mathcal{A}_t^*) : t \in T\}$.

Vertices of a CG-diagram are drawn as small circles and edges as smooth lines connecting all sets of atoms belonging to a block.

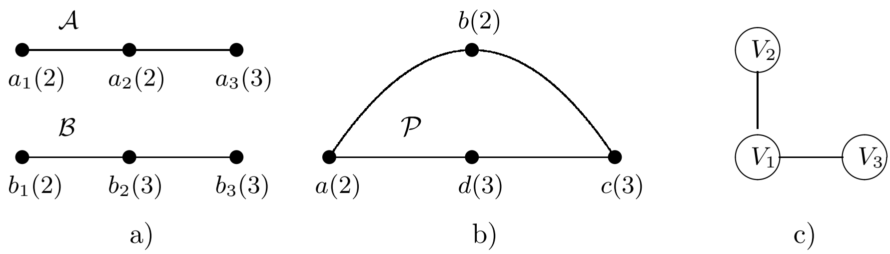


Figure 2. Greechie and CG-diagrams (Example 8)

*Example* 8. Let $\mathcal{A} = \mathcal{A}(2,2,3)$ and $\mathcal{B} = \mathcal{B}(2,3,3)$ be MV-algebras such that $\mathrm{At}(\mathcal{A}) = \{a_1, a_2, a_3\}$, $\mathrm{At}(\mathcal{B}) = \{b_1, b_2, b_3\}$ (see Fig. 2a)). Then $\tau(a_1) = 2$, $\tau(a_2) = 2$, $\tau(a_3) = 3$, $\tau(b_1) = 2$, $\tau(b_2) = 3$, $\tau(b_3) = 3$. If we put $A = \{a_1, a_3\}$, $B = \{b_1, b_3\}$ and define a bijection $\varphi\colon A \to B$ such that $\varphi(a_1) = b_1$ and $\varphi(a_3) = b_3$, then $A \sim_\tau B$, so the MV-algebras $\mathcal{A}$ and $\mathcal{B}$ create an admissible system with respect to a choice $\mathcal{U} = \{((A,B),\varphi), ((B,A),\varphi^{-1})\}$. We obtain the following equivalence classes: $[0_\mathcal{A}] = \{0_\mathcal{A}, 0_\mathcal{B}\} = [0_\mathcal{B}]$, $[1_\mathcal{A}] = \{1_\mathcal{A}, 1_\mathcal{B}\} = [1_\mathcal{B}]$, $[a_1] = \{a_1, b_1\} = [b_1]$, $[a_2] = \{a_2\}$, $[a_3] = \{a_3, b_3\} = [b_3]$, $[b_2] = \{b_2\}$, $[2a_1] = \{2a_1, 2b_1\} = [2b_1]$, $[2b_2] = \{2b_2\}$. Because $(2a_2)^\perp = (2a_1) \oplus (3a_3)$ and $(3b_2)^\perp = (2b_1) \oplus (3b_3)$, we get that $(2a_2)^\perp \sim (3b_2)^\perp$ and hence $(2a_2) \sim (3b_2)$, so $[2a_2] = [3b_2]$. Then $\mathcal{P} = \{[x] : x \in \mathcal{A} \cup \mathcal{B}\}$ is a pasting of the MV-algebras $\mathcal{A}$ and $\mathcal{B}$. For the simplicity we denote $a = [a_1] = [b_1]$, $b = [a_2]$, $c = [a_3] = [b_3]$,

$d = [b_2]$ and we put $V_1 = \{a, c\}$, $V_2 = \{b\}$, $V_3 = \{d\}$. The Greechie diagram of the MV-algebra pasting $\mathcal{P}$ is displayed in Fig. 2b) and the CG-diagram of $\mathcal{P}$ in Fig. 2c).

In the whole following text we will identify an equivalence class $[x]$ with the element $x$ determining this class.

## 5. Loops in MV-algebra pastings

In this section we give conditions under which a pasting of an admissible system of MV-algebras is a lattice-ordered D-poset (a D-lattice). We follow Dichtl ideas (cf. [5]), but we present them in a different way.

**Lemma 9.** *Let $\mathcal{P}$ be a pasting of an admissible system of MV-algebras. Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be different blocks of $\mathcal{P}$ with countable sets of atoms* $\mathrm{At}(\mathcal{A}^*)$ *and* $\mathrm{At}(\mathcal{B}^*)$*, respectively. Let* $\mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*)$ *be a finite non-empty set of atoms,*

$$\mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*) = \{v_1, v_2, \dots, v_n\}, \qquad n \geq 1.$$

*Then the following statements are true.*

(i) *$\mathcal{A}^* \cap \mathcal{B}^*$ is a sub-MV-algebra of $\mathcal{P}$.*

(ii) $\mathrm{At}(\mathcal{A}^* \cap \mathcal{B}^*) = \{v_1, v_2, \dots, v_n, v^\perp\}$,
*where* $v = \bigoplus_{i=1}^{n} \tau(v_i)v_i$ *and, moreover,* $\tau(v^\perp) = 1$.

(iii) *$\mathcal{A}^* \cap \mathcal{B}^* = [0_\mathcal{P}, v] \cup [v^\perp, 1_\mathcal{P}]$ and $[0_\mathcal{P}, v] \cap [v^\perp, 1_\mathcal{P}] = \emptyset$.*

Proof.

(i) It is straightforward.

(ii) Let $x \in \mathcal{A}^* \cap \mathcal{B}^*$ and $x \leq v_j$ for some $j \in \{1, 2, \dots, n\}$. Because $x \in \mathcal{A}^*$ and $v_j$ is an atom in $\mathcal{A}^*$, either $x = 0_\mathcal{P}$ or $x = v_j$, which means that $v_j$ is an atom in $\mathcal{A}^* \cap \mathcal{B}^*$ and so $\{v_1, v_2, \dots, v_n\} \subset \mathrm{At}(\mathcal{A}^* \cap \mathcal{B}^*)$. Denote

$$A = \mathrm{At}(\mathcal{A}^*) \setminus (\mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*)) = \{a_t :\ t \in T\}, \qquad a = \bigoplus_{t \in T} \tau(a_t)a_t,$$

$$B = \mathrm{At}(\mathcal{B}^*) \setminus (\mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*)) = \{b_s :\ s \in S\}, \qquad b = \bigoplus_{s \in S} \tau(b_s)b_s,$$

where $T, S$ are countable index sets. Then $\mathrm{At}(\mathcal{A}^*) = \{v_1, v_2, \dots, v_n\} \cup A$, $\mathrm{At}(\mathcal{B}^*) = \{v_1, v_2, \dots, v_n\} \cup B$, $v \oplus a = 1_\mathcal{P} = v \oplus b$ and hence $a = v^\perp = b$.

We prove that $v \wedge v^{\perp} = 0_{\mathcal{P}}$. Let $w \in \mathcal{P}$ such that $w \leq v$ and $w \leq v^{\perp}$. Suppose that $w > 0_{\mathcal{P}}$. Because $a_t \wedge v = 0_{\mathcal{P}}$ and $b_s \wedge v = 0_{\mathcal{P}}$ for every $t \in T$ and $s \in S$, there is an atom $v_j$ ($j \in \{1, 2, \ldots, n\}$) such that $v_j \leq w$. We have

$$\tau(v_j)v_j \leq v \leq w^{\perp} \leq v_j^{\perp},$$

which contradicts the isotropic index of $v_j$, therefore $w = 0_{\mathcal{P}}$.

Suppose that $x \in \mathcal{A}^* \cap \mathcal{B}^*$. Because $x \in \mathcal{A}^*$, the element $x$ can be expressed in the form $x = x_1 \oplus x_2$, where

$$x_1 = \bigoplus_{i=1}^{n} \alpha_i v_i, \qquad \alpha_i \in \big\{0, 1, 2, \ldots, \tau(v_i)\big\}, \ \ i = 1, 2, \ldots, n,$$

$$x_2 = \bigoplus_{t \in T} \beta_t a_t, \qquad \beta_t \in \big\{0, 1, 2, \ldots, \tau(a_t)\big\}, \ \ t \in T.$$

The element $x$ belongs simultaneously to the block $\mathcal{B}^*$ and this is possible if and only if either $x_2 = 0_{\mathcal{P}}$ or $x_2 = v^{\perp}$. Indeed, supposing $x_2 > 0_{\mathcal{P}}$ and $x_2 \neq v^{\perp}$ we have $x_2 \in \mathcal{B}^*$, $x_2 < b$, $b \ominus x_2 \in \mathcal{B}^*$, $b \ominus x_2 > 0_{\mathcal{P}}$, which implies that there is an atom $c \in \mathcal{B}^*$ such that $c \leq b \ominus x_2$. There are two possibilities: either $c = v_j$ for some $j \in \{0, 1, \ldots, n\}$ or $c = b_{s_0}$ for some $s_0 \in S$. The first possibility gives $v_j \leq b \ominus x_2 \leq b$ and hence

$$v_j = v_j \wedge b \leq v \wedge v^{\perp} = 0_{\mathcal{P}}.$$

If $b_{s_0} \leq b \ominus x_2$ then $b_{s_0} \leq x_2^{\perp} \leq a_{t_0}^{\perp}$ for $t_0 \in T$ such that $\beta_{t_0} \geq 1$, therefore $b_{s_0} \leftrightarrow a_{t_0}$. It means that $a_{t_0}, b_{s_0} \in \mathcal{A}^* \cap \mathcal{B}^*$ and thus $a_{t_0}, b_{s_0} \in \mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*)$, which is in contradiction with the definition of sets $A$ and $B$. Thus the element $x$ can be expressed in the form

$$x = \left(\bigoplus_{i=1}^{n} \alpha_i v_i\right) \oplus k v^{\perp}, \qquad k \in \{0, 1\}. \tag{1}$$

If $k = 0$ then $x \in [0_{\mathcal{P}}, v]$, and $k = 1$ implies that $x \in [v^{\perp}, 1_{\mathcal{P}}]$.

In order to prove that $v^{\perp}$ is an atom in $\mathcal{A}^* \cap \mathcal{B}^*$, we assume that there is $x \in \mathcal{A}^* \cap \mathcal{B}^*$ such that $x \leq v^{\perp}$. As above, the element $x$ can be expressed in the form (1). If $k = 0$ then $x \leq v$ and consequently $x \leq v \wedge v^{\perp} = 0_{\mathcal{P}}$. If $k = 1$ then $x \geq v^{\perp}$, so $x = v^{\perp}$, which proves that $v^{\perp}$ is an atom in $\mathcal{A}^* \cap \mathcal{B}^*$.

The statement $\tau(v^{\perp}) = 1$ follows immediately from the equality $v \wedge v^{\perp} = 0_{\mathcal{P}}$.

(iii) In accordance with the above mentioned results, we can write

$$\begin{aligned}\mathcal{A}^* \cap \mathcal{B}^* &= \Big\{x = \Big(\bigoplus_{i=1}^{n} \alpha_i v_i\Big) \oplus k v^{\perp}, \ \alpha_i \in \big\{0, 1, 2, \ldots, \tau(v_i)\big\}, \ \ k \in \{0, 1\}\Big\} \\ &= \Big\{x = \bigoplus_{i=1}^{n} \alpha_i v_i\Big\} \cup \Big\{x = \Big(\bigoplus_{i=1}^{n} \alpha_i v_i\Big) \oplus v^{\perp}\Big\} = [0_{\mathcal{P}}, v] \cup [v^{\perp}, 1_{\mathcal{P}}].\end{aligned}$$

Suppose that there is $w \in [0_{\mathcal{P}}, v] \cap [v^{\perp}, 1_{\mathcal{P}}]$. Then $v^{\perp} \le w$ and $w \le v$, hence $v^{\perp} \le v$, which gives $v^{\perp} = v^{\perp} \wedge v = 0_{\mathcal{P}}$, a contradiction. □

**Corollary 10.** *If the assumptions of the previous* Lemma 9 *are fulfilled, then*

$$|\mathcal{A}^* \cap \mathcal{B}^*| = 2(\tau(v_1)+1)(\tau(v_2)+1)\cdots(\tau(v_n)+1).$$

**Theorem 11.** *Let $\mathcal{P}$ be an MV-algebra pasting of an admissible system $(\mathcal{S},\mathcal{U})$, where $\mathcal{S} = \{\mathcal{A},\mathcal{B}\}$. Then $\mathcal{P} = \mathcal{A}^* \cup \mathcal{B}^*$ is a D-lattice.*

Proof. From the construction of a pasting of an admissible system of MV-algebras we know that $\mathcal{P}$ is a D-poset. We prove that $\mathcal{P}$ is a lattice. Let us denote

$$V = \mathrm{At}(\mathcal{A}^*) \cap \mathrm{At}(\mathcal{B}^*) = \{v_1, v_2, \dots, v_n\}, \qquad v = \bigoplus_{i=1}^{n} \tau(v_i)v_i, \quad n \ge 1,$$

$$A = \mathrm{At}(\mathcal{A}^*) \setminus V = \{a_t :\ t \in T\}, \qquad a = \bigoplus_{t\in T} \tau(a_t)a_t,$$

$$B = \mathrm{At}(\mathcal{B}^*) \setminus V = \{b_s :\ s \in S\}, \qquad b = \bigoplus_{s\in S} \tau(b_s)b_s,$$

where $T$ and $S$ are countable index sets. Then $\mathrm{At}(\mathcal{A}^*) = A \cup V$, $\mathrm{At}(\mathcal{B}^*) = B \cup V$ and $a = v^{\perp} = b$. Let $x \in \mathcal{A}^*$, $x = x_1 \oplus x_2$ and $y \in \mathcal{B}^*$, $y = y_1 \oplus y_2$ such that

$$x_1 = \bigoplus_{i=1}^{n} \alpha_{1i}v_i, \quad x_2 = \bigoplus_{t\in T} \alpha_{2t}a_t, \quad y_1 = \bigoplus_{i=1}^{n} \beta_{1i}v_i, \quad y_2 = \bigoplus_{s\in S} \beta_{2s}b_s,$$

where $\alpha_{1i}, \beta_{1i} \in \{0,1,\dots,\tau(v_i)\}$ for $i = 1,2,\dots,n$, $\alpha_{2t} \in \{0,1,\dots,\tau(a_t)\}$ for $t \in T$, $\beta_{2s} \in \{0,1,\dots,\tau(b_s)\}$ for $s \in S$. Then $x_1 \le v$, $x_2 \le a$, $y_1 \le v$, $y_2 \le b$. It is obvious that $x_1 \oplus x_2$ is the supremum of $x_1$ and $x_2$ in the block $\mathcal{A}^*$, i.e. $x_1 \oplus x_2 = x_1 \vee_{\mathcal{A}^*} x_2$. Similarly $y_1 \oplus y_2 = y_1 \vee_{\mathcal{B}^*} y_2$. Observe that the supremum of $x_1$ and $y_1$ exists in $\mathcal{P}$, because $x_1 \vee y_1 = \bigoplus_{i=1}^{n} \gamma_i v_i$, where $\gamma_i = \max\{\alpha_{1i}, \beta_{1i}\}$, $i = 1,2,\dots,n$. Let us denote $c = x_1 \vee y_1$. Then $c \le v = b^{\perp} \le y_2^{\perp}$ and $c \wedge y_2 = 0_{\mathcal{P}}$. If $x_2 = 0_{\mathcal{P}}$ then $x, y \in \mathcal{B}^*$, so

$$c \oplus y_2 = c \vee y_2 = (x_1 \vee y_1) \vee y_2 = x_1 \vee (y_1 \vee y_2) = x \vee y.$$

Similarly $x \vee y$ exists in $\mathcal{P}$ if $y_2 = 0_{\mathcal{P}}$.

Now let $x_2 \ne 0_{\mathcal{P}}$ and $y_2 \ne 0_{\mathcal{P}}$. We prove that $x_1 \oplus x_2 = x_1 \vee x_2$ and $y_1 \oplus y_2 = y_1 \vee y_2$. Let $z \in \mathcal{P}$ such that $x_1 \le z$ and $x_2 \le z$. Then necessarily $z \in \mathcal{A}^*$ ($z \in \mathcal{B}^*$ if and only if $z \in \mathcal{A}^* \cap \mathcal{B}^*$), therefore $x_1 \oplus x_2 = x_1 \vee_{\mathcal{A}^*} x_2 \le z$, which gives that $x_1 \oplus x_2 = x_1 \vee x_2$. In a similar manner we obtain $y_1 \oplus y_2 = y_1 \vee y_2$. Further we have $x_1 \le c \le v^{\perp} \oplus c$ and $x_2 \le v^{\perp} \le v^{\perp} \oplus c$, therefore $x \le v^{\perp} \oplus c$, and also $y \le v^{\perp} \oplus c$. We prove that $v^{\perp} \oplus c$ is the supremum of $x$ and $y$. Suppose that $w \in \mathcal{P}$ such that $x \le w$ and $y \le w$. Then $w \in \mathcal{A}^* \cap \mathcal{B}^* = [0_{\mathcal{P}}, v] \cup [v^{\perp}, 1_{\mathcal{P}}]$.

There are two possibilities: either $w \in [0_{\mathcal{P}}, v]$ or $w \in [v^{\perp}, 1_{\mathcal{P}}]$. If $w \in [0_{\mathcal{P}}, v]$ then $w \le v$ and $a = v^{\perp} \le w^{\perp} \le x^{\perp} \le x_2^{\perp}$. Inasmuch as $x_2 \ne 0_{\mathcal{P}}$, there is an atom $a_{t_0}$ for some $t_0 \in T$ such that $a_{t_0} \le x_2$. Then

$$\tau(a_{t_0})a_{t_0} \le a = v^{\perp} \le x_2^{\perp} \le a_{t_0}^{\perp},$$

which contradicts the isotropic index of the atom $a_{t_0}$. Then necessarily $w \in [v^{\perp}, 1_{\mathcal{P}}]$. Since $c \le v$ and $c \wedge v^{\perp} \le v \wedge v^{\perp} = 0_{\mathcal{P}}$, it follows that

$$v^{\perp} \oplus c = v^{\perp} \vee_{\mathcal{A}^* \cap \mathcal{B}^*} c \le w,$$

which proves that $v^{\perp} \oplus c = x \vee y$. Especially, if $x_1 = 0_{\mathcal{P}} = y_1$ then $x_2 \vee y_2 = v^{\perp}$. Moreover, if $V = \emptyset$, i.e. $\mathcal{P}$ is the 0-1-pasting, then $x \vee y = x_2 \vee y_2 = 1_{\mathcal{P}}$. □

**Definition 12.** Let $\mathcal{P} = \bigcup_{t \in T} \mathcal{A}_t^*$ be an MV-algebra pasting of an admissible system $(\mathcal{S}, \mathcal{U})$, where $\mathcal{S} = \{\mathcal{A}_t :\ t \in T\}$ is a countable system of atomic $\sigma$-complete MV-algebras and $\mathcal{U}$ is a choice of pairs of isotropically equivalent sets of atoms. Let $\mathcal{A}_0^*, \mathcal{A}_1^*, \dots, \mathcal{A}_{n-1}^*$ be a finite system of $n$ mutually different blocks of $\mathcal{P}$, $n \ge 3$. For every $i = 0, 1, \dots, n-1$ we define the index set

$$K_i = \{0, 1, 2, \dots, n-1\} \setminus \{i, i+1\} \pmod n$$

and we put

$$V_{i+1} = \mathrm{At}(\mathcal{A}_i^*) \cap \mathrm{At}(\mathcal{A}_{i+1}^*) \setminus \bigcup_{k \in K_i} \mathrm{At}(\mathcal{A}_k^*) \ \text{(indices} \mod n),$$

$$W = \bigcap_{i=0}^{n-1} \mathrm{At}(\mathcal{A}_i^*).$$

(1) The system of blocks $\mathcal{A}_0^*, \mathcal{A}_1^*, \dots, \mathcal{A}_{n-1}^*$ is said to be a *loop of order $n$* (*$n$-loop* for short), if the following conditions are fulfilled.
   - (L1) $V_i \ne \emptyset$ for every $i = 0, 1, \dots, n-1$.
   - (L2) For every $i = 0, 1, \dots, n-1$ and $j \notin \{i, i+1, i+2\}$

$$\mathrm{At}(\mathcal{A}_{i+1}^*) \cap \mathrm{At}(\mathcal{A}_j^*) \setminus \bigcap_{k \in K_{ij}} \mathrm{At}(\mathcal{A}_k^*) = \emptyset \ \text{(indices} \mod n),$$

     where $K_{ij} = \{0, 1, 2, \dots, n-1\} \setminus \{i+1, j\} \pmod n$.

(2) Atoms belonging to the sets $V_i$ $(i = 0, 1, \dots, n-1)$ are called *nodal vertices*, and atoms belonging to the set $W$ are called *central nodal vertices* of the $n$-loop.

(3) A 4-loop is called an *astroid* if
$\mathrm{At}(\mathcal{A}_i^*) = V_i \cup V_{i+1} \cup W$ for every $i = 0, 1, 2, 3 \pmod 4$.

Comments on Definition 12:

(a) The sets $V_i$ $(i = 0, 1, \dots, n-1)$ and $W$ are finite.

(b) $V_i \cap V_j = \emptyset$ for every $i \neq j$.

(c) $V_i \cap \mathrm{At}(\mathcal{A}^*_{i+1}) = \emptyset$ for every $i = 0, 1, \dots, n$ (mod $n$).

(d) It follows from the condition (L2) that either $\mathrm{At}(\mathcal{A}^*_{i+1}) \cap \mathrm{At}(\mathcal{A}^*_j) = \emptyset$ (and then $W = \emptyset$), or $\mathrm{At}(\mathcal{A}^*_{i+1}) \cap \mathrm{At}(\mathcal{A}^*_j) \neq \emptyset$, which gives that $\mathrm{At}(\mathcal{A}^*_{i+1}) \cap \mathrm{At}(\mathcal{A}^*_j) = W$ for every $i = 0, 1, \dots, n-1$ and $j \notin \{i, i+1, i+2\}$ (mod $n$).

(e) An astroid is generated only by nodal vertices (including central nodal vertices).

Fig. 3a) displays an astroid with the least number of atoms and with the empty set of central nodal vertices. Fig. 3b) displays an astroid with the only one central nodal vertex ($W = \{c\}$). A CG-diagram of an astroid is visible in Fig. 3c).

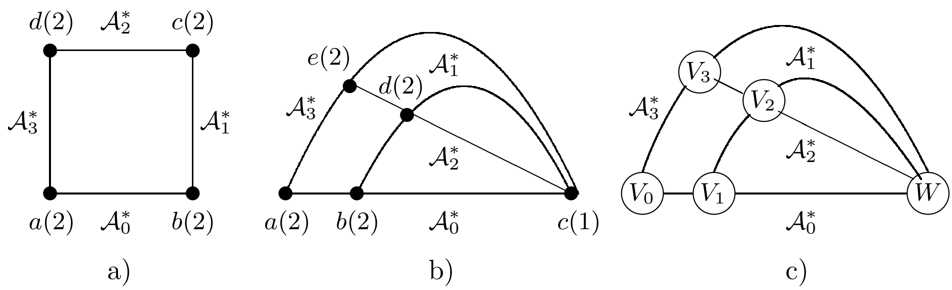


FIGURE 3. Astroids

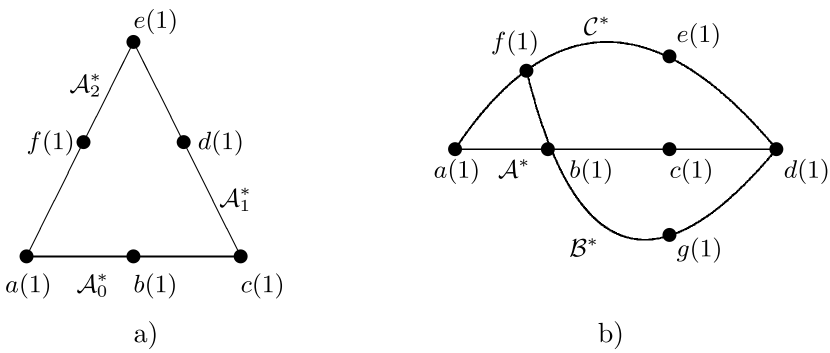


FIGURE 4. Loops of order 3

The Greechie diagram in Fig. 4a) (called Wright triangle) represents a pasting $\mathcal{P}$ of three Boolean algebras, $\mathcal{P} = \bigcup_{i=0}^{2} \mathcal{A}_i^*$, where $\mathrm{At}(\mathcal{A}_0^*) = \{a, b, c\}$, $\mathrm{At}(\mathcal{A}_1^*) = \{c, d, e\}$, $\mathrm{At}(\mathcal{A}_2^*) = \{e, f, a\}$. The pasting $\mathcal{P}$ is an orthoalgebra which is not an orthomodular poset ([9]). The atoms $a, c, e$ are nodal vertices, $V_0 = \{a\}$, $V_1 = \{c\}$, $V_2 = \{e\}$ and $W = \emptyset$. It is easily visible that the blocks $\mathcal{A}_0^*$, $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ generate a 3-loop.

In Fig. 4b) we see the Greechie diagram of a pasting $\mathcal{Q} = \mathcal{A}^* \cup \mathcal{B}^* \cup \mathcal{C}^*$ of Boolean algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$, where $\mathrm{At}(\mathcal{A}^*) = \{a, b, c, d\}$, $\mathrm{At}(\mathcal{B}^*) = \{b, d, f, g\}$ and $\mathrm{At}(\mathcal{C}^*) = \{a, d, e, f\}$, $V_0 = \{a\}$, $V_1 = \{b\}$, $V_2 = \{f\}$ and $W = \{d\}$. There is no question that the blocks $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{C}^*$ generate a 3-loop.

**LEMMA 13.** *Let $\mathcal{A}_0^*, \mathcal{A}_1^*, \dots, \mathcal{A}_{n-1}^*$ be an n-loop in an MV-algebra pasting $\mathcal{P}$ of an admissible system $(\mathcal{S}, \mathcal{U})$. If $x \in \mathcal{A}_i^*$ and $y \in \mathcal{A}_j^*$ $(i, j \in \{0, 1, \dots, n-1\})$ are elements generated by atoms that are not nodal vertices of the n-loop, then $x \vee y$ exists in $\mathcal{P}$.*

Proof. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{i\alpha_i}\}$ $(i = 0, 1, 2, \dots, n-1)$ be the sets of nodal vertices and $W = \{w_1, w_1, \dots, w_k\}$ be the set of central nodal vertices of the $n$-loop $\mathcal{A}_0^*$, $\mathcal{A}_1^*$, …, $\mathcal{A}_{n-1}^*$. Let us denote

$$A_i = \mathrm{At}(\mathcal{A}_i^*) \smallsetminus (V_i \cup V_{i+1} \cup W) = \{a_{it} :\ t \in T_i\}, \qquad a_i = \bigoplus_{t \in T_i} \tau(a_{it}) a_{it},$$
$$v_i = \bigoplus_{j=1}^{\alpha_i} \tau(v_{ij}) v_{ij}, \qquad w = \bigoplus_{s=1}^{k} \tau(w_s) w_s,$$

where $T_i$ are countable index sets, $i = 0, 1, 2, \dots, n-1$.

Let $x \in \mathcal{A}_i^*$ and $y \in \mathcal{A}_m^*$ for some $i, m \in \{0, 1, 2, \dots, n-1\}$, such that $x > 0_{\mathcal{P}}$, $x = \bigoplus_{t \in T_i} \alpha_t a_{it}$, $\alpha_t \in \{0, 1, \dots, \tau(a_{it})\}$, and $y > 0_{\mathcal{P}}$, $y = \bigoplus_{s \in T_m} \beta_s a_{ms} > 0_{\mathcal{P}}$, $\beta_s \in \{0, 1, \dots, \tau(a_{ms})\}$. Then $x, y$ are generated by atoms that are not nodal vertices of the $n$-loop and, moreover, $x \le a_i$, $y \le a_m$. Obviously $x \vee y$ exists in $\mathcal{P}$ if $i = m$. Suppose that $m = i + 1 \pmod n$. We have

$$v_i \oplus a_i \oplus v_{i+1} \oplus w = 1_{\mathcal{P}} = v_{i+1} \oplus a_{i+1} \oplus v_{i+2} \oplus w,$$

and hence

$$v_i \oplus a_i = (v_{i+1} \oplus w)^{\perp} = a_{i+1} \oplus v_{i+2},$$

which gives $x \le a_i \le (v_{i+1} \oplus w)^{\perp}$ and $y \le a_{i+1} \le (v_{i+1} \oplus w)^{\perp}$. We prove that $(v_{i+1} \oplus w)^{\perp}$ is the supremum of $x$ and $y$ in $\mathcal{P}$. Suppose that $z \in \mathcal{P}$ such that $x, y \le z$. Then

$$z \in \mathcal{A}_i^* \cap \mathcal{A}_{i+1}^* = [0_{\mathcal{P}}, v_{i+1} \oplus w] \cup [(v_{i+1} \oplus w)^{\perp}, 1_{\mathcal{P}}].$$

There are two possibilities: either $z \in [0_{\mathcal{P}}, v_{i+1} \oplus w]$ or $z \in [(v_{i+1} \oplus w)^{\perp}, 1_{\mathcal{P}}]$. In the first case $z \le v_{i+1} \oplus w$ and there is an atom $a_{it_0}$, $t_0 \in T_i$, such that $a_{it_0} \le x$. Then

$$a_{it_0} \le x \le z \le v_{i+1} \oplus w = (v_i \oplus a_i)^{\perp} \le a_i^{\perp} \le (\tau(a_{it_0})a_{it_0})^{\perp},$$

which contradicts the isotropic index of the atom $a_{it_0}$. For that reason the only eventuality $z \in [(v_{i+1} \oplus w)^{\perp}, 1_{\mathcal{P}}]$ is possible. This gives that $(v_{i+1} \oplus w)^{\perp} \le z$, therefore $(v_{i+1} \oplus w)^{\perp} = x \vee y$.

Now suppose that $x \in \mathcal{A}_i^*$ and $y \in \mathcal{A}_m^*$ for $m \notin \{i-1, i, i+1\} \pmod n$. In the event that $\mathrm{At}(\mathcal{A}_i^*) \cap \mathrm{At}(\mathcal{A}_m^*) = \emptyset$ then $\mathcal{A}_i^* \cap \mathcal{A}_m^* = \{0_{\mathcal{P}}, 1_{\mathcal{P}}\}$, therefore $x \vee y = 1_{\mathcal{P}}$. If $\mathrm{At}(\mathcal{A}_i^*) \cap \mathrm{At}(\mathcal{A}_m^*) \ne \emptyset$ then $\mathrm{At}(\mathcal{A}_i^*) \cap \mathrm{At}(\mathcal{A}_m^*) = W$ and $x \vee y = w^{\perp}$. □

We give some sufficient conditions under which a pasting of an admissible system of MV-algebras is not a lattice-ordered D-poset.

**Theorem 14.** *Let $\mathcal{P} = \bigcup_{i=0}^{2} \mathcal{A}_i^*$ be an MV-algebra pasting of an admissible system $(\mathcal{S}, \mathcal{U})$, where $\mathcal{S} = \{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2\}$. If blocks $\mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*$ form a* 3-*loop then a D-poset $\mathcal{P}$ is not lattice-ordered.*

Proof. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{i\alpha_i}\}$ $(i = 0, 1, 2)$ be the sets of nodal vertices and $W = \{w_1, w_2, \dots, w_k\}$ be the set of central nodal vertices of the 3-loop $\mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*$. Let us put

$$A_i = \mathrm{At}(\mathcal{A}_i^*) \setminus (V_i \cup V_{i+1} \cup W) = \{a_{it} :\ t \in T_i\}, \qquad a_i = \bigoplus_{t \in T_i} \tau(a_{it})a_{it},$$

$$v_i = \bigoplus_{j=1}^{\alpha_i} \tau(v_{ij})v_{ij}, \quad w = \bigoplus_{s=1}^{k} \tau(w_s)w_s, \qquad i = 0, 1, 2 \pmod 3,$$

where $T_i$ $(i = 0, 1, 2)$ are countable index sets.

Obviously $v_i \ne 0_{\mathcal{P}}$, $a_i \ne 0_{\mathcal{P}}$, $(w \oplus a_i)^{\perp} \in \mathcal{A}_i^*$ and $(w \oplus v_{i+2})^{\perp} \in \mathcal{A}_{i+1}^* \cap \mathcal{A}_{i+2}^*$ for $i = 0, 1, 2 \pmod 3$. The elements $(w \oplus a_i)^{\perp}$ and $(w \oplus v_{i+2})^{\perp}$ are two different minimal upper bounds of $v_i$ and $v_{i+1}$ for $i = 0, 1, 2 \pmod 3$ and it is easily verifiable that there is no block containing a smaller common upper bound, so the supremum of $v_i$ and $v_{i+1}$ does not exist in $\mathcal{P}$ for every $i = 0, 1, 2 \pmod 3$. For the sake of completeness, we note that also the supremum of $a_i$ and $v_{i+2}$ does not exist in $\mathcal{P}$ for every $i = 0, 1, 2 \pmod 3$. □

The following corollary of Theorem 14 is in accordance with Greechie's Loop Lemma [9] (cf. also [14]).

**Corollary 15.** *Let* $\mathcal{P} = \bigcup_{i=0}^{2} \mathcal{A}_i^*$ *be a pasting of an admissible system of three Boolean algebras. If blocks* $\mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*$ *(Boolean subalgebras of* $\mathcal{P}$*) form a 3-loop then* $\mathcal{P}$ *is an orthoalgebra that is not an orthomodular poset.*

Proof. The pasting $\mathcal{P}$ is a regular D-poset, this means that if $a \in \mathcal{P}$ and $a \le a^\perp$ then $a = 0_\mathcal{P}$. It suffices to prove that there are two orthogonal elements of $\mathcal{P}$ such that their supremum does not exist in $\mathcal{P}$. We use the same notation as in the proof of Theorem 14. It is obvious that the elements $v_i$ and $v_{i+1}$ are orthogonal, i.e. $v_i \le v_{i+1}^\perp$, but the supremum of $v_i$ and $v_{i+1}$ does not exist in $\mathcal{P}$ for every $i = 0, 1, 2 \pmod 3$. □

**Theorem 16.** *Let* $\mathcal{P} = \bigcup_{i=0}^{3} \mathcal{A}_i^*$ *be an MV-algebra pasting of an admissible system* $(\mathcal{S}, \mathcal{U})$*, where* $\mathcal{S} = \{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$*. If the blocks* $\mathcal{A}_i^*$ $(i = 0, 1, 2, 3)$ *form a 4-loop that is not an astroid, then* $\mathcal{P}$ *is not a lattice-ordered D-poset.*

Proof. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{i\alpha_i}\}$ $(i = 0, 1, 2, 3)$ be the sets of nodal vertices and $W = \{w_1, w_2, \dots, w_k\}$ be the set of central nodal vertices of the 4-loop $\mathcal{A}_0^*$, $\mathcal{A}_1^*$, $\mathcal{A}_2^*$, $\mathcal{A}_3^*$. Let us denote

$$A_i = \mathrm{At}(\mathcal{A}_i^*) \setminus (V_i \cup V_{i+1} \cup W) = \{a_{it} : \ t \in T_i\}, \qquad a_i = \bigoplus_{t \in T_i} \tau(a_{it}) a_{it},$$

$$v_i = \bigoplus_{j=1}^{\alpha_i} \tau(v_{ij}) v_{ij}, \quad w = \bigoplus_{s=1}^{k} \tau(w_s) w_s, \qquad i = 0, 1, 2, 3 \pmod 4,$$

where $T_i$ $(i = 0, 1, 2, 3)$ are countable index sets.

Note that $A_i \ne \emptyset$ because this 4-loop is not an astroid, so $a_i \ne 0_\mathcal{P}$, $v_i \ne 0_\mathcal{P}$ and $v_i \oplus a_i = (v_{i+1} \oplus w)^\perp = v_{i+2} \oplus a_{i+1}$ for every $i \in \{0, 1, 2, 3\} \pmod 4$. Suppose that $\mathcal{P}$ is a lattice. Then $v_i \vee v_{i+2} \le (v_{i+1} \oplus w)^\perp$, $v_i \vee v_{i+2} \le (v_{i+3} \oplus w)^\perp$ and

$$\begin{aligned}(v_{i+1} \oplus w)^\perp \ominus (v_i \vee v_{i+2}) &= \big((v_{i+1} \oplus w)^\perp \ominus v_i\big) \wedge \big((v_{i+1} \oplus w)^\perp \ominus v_{i+2}\big) \\ &= ((v_i \oplus a_i) \ominus v_i) \wedge ((v_{i+2} \oplus a_{i+1}) \ominus v_{i+2}) \\ &= a_i \wedge a_{i+1} = 0_\mathcal{P},\end{aligned}$$

hence $v_i \vee v_{i+2} = (v_{i+1} \oplus w)^\perp$. Likewise $v_i \vee v_{i+2} = (v_{i+3} \oplus w)^\perp$, so $v_{i+1} = v_{i+3}$ for $i \in \{0, 1, 2, 3\} \pmod 4$. Then $v_{i+1} = v_{i+1} \vee v_{i+3} = (v_{i+2} \oplus w)^\perp = v_{i+1} \oplus a_{i+1}$, which gives $a_{i+1} = 0_\mathcal{P}$, a contradiction. We proved that the supremum of $v_{i+1}$ and $v_{i+3}$ does not exist in $\mathcal{P}$ for $i = 0, 1 \pmod 4$. □

Note that the supremum of orthogonal elements exists in a 4-loop. Moreover, if all blocks forming a 4-loop are Boolean subalgebras, then this pasting is an orthomodular poset.

Now we give sufficient conditions for a pasting of an admissible system of MV-algebras containing a 3-loop or a 4-loop, respectively, to be a lattice-ordered D-poset. These conditions are inspired by Dichtl's results from the construction of orthomodular lattices as the pasting of a pasted family of finite Boolean algebras, that were published in [5]. First we prove the following very useful lemma.

**Lemma 17.** *Let $\mathcal{P}$ be a D-lattice and $a, b, c \in \mathcal{P}$. If $a \leq c^{\perp}$, $b \leq c^{\perp}$, $a \wedge b = 0_{\mathcal{P}}$, then $(c \oplus a) \wedge (c \oplus b) = c$.*

Proof.

$$(c \oplus a) \wedge (c \oplus b) = (c^{\perp} \ominus a)^{\perp} \wedge (c^{\perp} \ominus b)^{\perp} = \big((c^{\perp} \ominus a) \vee (c^{\perp} \ominus b)\big)^{\perp}$$
$$= \big((c^{\perp} \ominus (a \wedge b)\big)^{\perp} = (c^{\perp} \ominus 0_{\mathcal{P}})^{\perp} = c. \qquad \square$$

**Theorem 18.** *Let $\mathcal{P} = \mathcal{A}_0^* \cup \mathcal{A}_1^* \cup \mathcal{A}_2^* \cup \mathcal{B}^*$ be a pasting of an admissible system $(\mathcal{S}, \mathcal{U})$, where $\mathcal{S} = \{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}\}$ and the blocks $\mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*$ form a 3-loop. If the block $\mathcal{B}^*$ contains all nodal vertices of the 3-loop, then the pasting $\mathcal{P}$ is a lattice-ordered D-poset.*

Proof. Let $V_i, A_i, W, v_i, a_i, w$ $(i = 0, 1, 2)$ be defined as in the proof of Theorem 14 and $B = \mathrm{At}(\mathcal{B}^*) \setminus (V_0 \cup V_1 \cup V_2 \cup W) = \{b_s : s \in S\}$, $b = \bigoplus_{s \in S} \tau(b_s) b_s$, where $S$ is a countable index set. The situation where $W = \emptyset$ is illustrated in Fig. 5a).

We have

$$v_i \oplus a_i \oplus v_{i+1} \oplus w = 1_{\mathcal{P}} = v_i \oplus v_{i+1} \oplus v_{i+2} \oplus b \oplus w,$$

hence $a_i = (v_i \oplus v_{i+1} \oplus w)^{\perp} = v_{i+2} \oplus b$. Then $v_{i+2} \leq a_i$ and $b \leq a_i$, thus

$$v_{i+2} \vee a_i = a_i = b \vee a_i$$

for every $i = 0, 1, 2 \pmod 3$.

Let $z_i, o_i, u \in \mathcal{A}_i^*$ and $b_0 \in \mathcal{B}^*$ such that $z_i \leq v_i$, $o_i \leq a_i$ for $i = 0, 1, 2$, $u \leq w$, $b_0 \leq b$. Then $z_i \leq v_i \oplus v_{i+1} \oplus w$ and $z_{i+1} \leq v_i \oplus v_{i+1} \oplus w$, therefore

$$z_i, z_{i+1} \in \mathcal{A}_i^* \cap \mathcal{B}^* = [0_{\mathcal{P}}, v_i \oplus v_{i+1} \oplus w] \cup [(v_i \oplus v_{i+1} \oplus w)^{\perp}, 1_{\mathcal{P}}].$$

We prove that $z_i \oplus z_{i+1}$ is the supremum of $z_i$ and $z_{i+1}$ in $\mathcal{P}$. Let $z \in \mathcal{P}$ such that $z_i \leq z$ and $z_{i+1} \leq z$. Then

$$\begin{aligned} z^{\perp} &\leq z_i^{\perp} = ((v_i \ominus z_i) \oplus v_{i+1} \oplus a_i \oplus w) \in \mathcal{A}_i^* \cap \mathcal{B}^*, \\ z^{\perp} &\leq z_{i+1}^{\perp} = v_i \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w \in \mathcal{A}_i^* \cap \mathcal{B}^*. \end{aligned}$$

Using Lemma 17 and the fact that $z_i \wedge z_{i+1} = 0_{\mathcal{P}}$, we obtain

$$\begin{aligned} z^{\perp} \leq\ & ((v_i \ominus z_i) \oplus v_{i+1} \oplus a_i \oplus w) \wedge_{\mathcal{A}_i^* \cap \mathcal{B}^*} (v_i \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w) \\ =\ & (((v_i \ominus z_i) \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w) \oplus z_{i+1}) \\ & \wedge_{\mathcal{A}_i^* \cap \mathcal{B}^*} (((v_i \ominus z_i) \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w) \oplus z_i) \\ =\ & (v_i \ominus z_i) \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w, \end{aligned}$$

and thus

$$z \geq ((v_i \ominus z_i) \oplus (v_{i+1} \ominus z_{i+1}) \oplus a_i \oplus w)^{\perp} = z_i \oplus z_{i+1},$$

which gives that $z_i \oplus z_{i+1} = z_i \vee z_{i+1}$. In the same way it can be proved that $z_i \oplus u = z_i \vee u$ and $(z_i \oplus z_{i+1}) \oplus u = (z_i \vee z_{i+1}) \vee u$. Similarly, the element $(z_0 \oplus z_1) \oplus z_2$ exists in $\mathcal{P}$ and it is the least upper bound of $z_0 \oplus z_1$ and $z_2$ in the block $\mathcal{B}^*$. If $z \in \mathcal{P}$ such that $z_0 \oplus z_1 \leq z$ and $z_2 \leq z$ then necessarily $z \in \mathcal{B}^*$ and consequently $(z_0 \oplus z_1) \oplus z_2 \leq z$, which implies that

$$(z_0 \oplus z_1) \oplus z_2 = (z_0 \oplus z_1) \vee z_2 = (z_0 \vee z_1) \vee z_2.$$

In a similar manner, $z_0 \oplus z_1 \oplus z_2 \oplus u = z_0 \vee z_1 \vee z_2 \vee u$. Thus we have shown that the supremum of the elements generated by nodal vertices exists in $\mathcal{P}$.

If $o_i > 0_{\mathcal{P}}$ and $o_{i+1} > 0_{\mathcal{P}}$, resp. $o_i > 0_{\mathcal{P}}$ and $o_{i+2} > 0_{\mathcal{P}}$, then according to Lemma 13, $o_i \vee o_{i+1} = (v_{i+1} \oplus w)^{\perp}$, resp. $o_i \vee o_{i+2} = (v_i \oplus w)^{\perp}$ for every $i = 0, 1, 2$ (mod 3). Suppose that $o_0, o_1, o_2$ are nonzero elements. Then $o_0 \vee o_1 = (v_1 \oplus w)^{\perp}$, $o_1 \vee o_2 = (v_2 \oplus w)^{\perp}$ and in addition

$$o_0 \vee o_1,\ o_1 \vee o_2 \in \mathcal{A}_1^* \cap \mathcal{B}^* = [0_{\mathcal{P}}, v_1 \oplus v_2 \oplus w] \cup [(v_1 \oplus v_2 \oplus w)^{\perp}, 1_{\mathcal{P}}].$$

We prove that $w^{\perp}$ is the least upper bound of $o_0, o_1, o_2$ in $\mathcal{P}$. Let $z \in \mathcal{P}$ such that $o_0 \leq z$, $o_1 \leq z$ and $o_2 \leq z$. Because $z \geq o_0 \vee o_1 = (v_1 \oplus w)^{\perp}$ and $z \geq o_1 \vee o_2 = (v_2 \oplus w)^{\perp}$, then using Lemma 17 and the fact that $v_1 \wedge v_2 = 0_{\mathcal{P}}$, we get

$$z^{\perp} \leq (v_1 \oplus w) \wedge_{\mathcal{A}_1^* \cap \mathcal{B}^*} (v_2 \oplus w) = w,$$

and hence $w^{\perp} \leq z$. We have proved that $w^{\perp} = (o_0 \vee o_1) \vee (o_1 \vee o_2) = o_0 \vee o_1 \vee o_2$. Recall that $W = \emptyset$ implies $o_0 \vee o_1 \vee o_2 = 1_{\mathcal{P}}$.

It can be proved in a routine manner that $o_i \vee z_i = o_i \oplus z_i$, $o_i \vee z_{i+1} = o_i \oplus z_{i+1}$ and, moreover, if $o_i > 0_{\mathcal{P}}$, $b_0 > 0_{\mathcal{P}}$ and $z_{i+2} > 0_{\mathcal{P}}$, then

$$o_i \vee z_{i+2} = (v_i \oplus v_{i+1} \oplus w)^{\perp} = o_i \vee b_0$$

for every $i = 0, 1, 2 \pmod 3$.

Let $x \in \mathcal{A}_i^*$, $x = x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w$, where $x_{v_i} \leq v_i$, $x_{a_i} \leq a_i$, $x_{v_{i+1}} \leq v_{i+1}$, $x_w \leq w$, $i \in \{0, 1, 2\}$ (mod 3). We show that $x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w$ is the supremum of elements $x_{v_i}, x_{a_i}, x_{v_{i+1}}, x_w$ in $\mathcal{P}$. As above, we have $x_{v_i} \oplus x_{v_{i+1}} \oplus x_w$

$= x_{v_i} \vee x_{v_{i+1}} \vee x_w$. Let $z \in \mathcal{P}$ such that $x_{v_i} \leq z$, $x_{a_i} \leq z$, $x_{v_{i+1}} \leq z$ and $x_w \leq z$. Then

$$\begin{aligned} z &\geq x_{v_i} \vee x_{v_{i+1}} \vee x_w = x_{v_i} \oplus x_{v_{i+1}} \oplus x_w \\ &= \big((v_i \ominus x_{v_i}) \oplus a_i \oplus (v_{i+1} \ominus x_{v_{i+1}}) \oplus (w \ominus x_w)\big)^{\perp} \in \mathcal{A}_i^*, \\ z &\geq x_{a_i} = (v_i \oplus (a_i \ominus x_{a_i}) \oplus v_{i+1} \oplus w)^{\perp} \in \mathcal{A}_i^*. \end{aligned}$$

Using Lemma 17 and the fact that

$$x_{a_i} \wedge (x_{v_i} \oplus x_{v_{i+1}} \oplus x_w) = x_{a_i} \wedge (x_{v_i} \vee x_{v_{i+1}} \vee x_w) \leq a_i \wedge a_i^{\perp} = 0_{\mathcal{P}},$$

we get

$$\begin{aligned} z^{\perp} \leq\ & \big(((v_i \ominus x_{v_i}) \oplus (a_i \ominus x_{a_i}) \oplus (v_{i+1} \ominus x_{v_{i+1}}) \oplus (w \ominus x_w)) \oplus x_{a_i}\big) \\ & \wedge_{\mathcal{A}_i^*} \big(((v_i \ominus x_{v_i}) \oplus (a_i \ominus x_{a_i}) \oplus (v_{i+1} \ominus x_{v_{i+1}}) \\ & \oplus (w \ominus x_w)) \oplus (x_{v_i} \oplus x_{v_{i+1}} \oplus x_w)\big) \\ =\ & \big(((v_i \ominus x_{v_i}) \oplus (a_i \ominus x_{a_i}) \oplus (v_{i+1} \ominus x_{v_{i+1}}) \oplus (w \ominus x_w))\big), \end{aligned}$$

and thus

$$\begin{aligned} z &\geq \big(((v_i \ominus x_{v_i}) \oplus (a_i \ominus x_{a_i}) \oplus (v_{i+1} \ominus x_{v_{i+1}}) \oplus (w \ominus x_w))\big)^{\perp} \\ &= x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w, \end{aligned}$$

so, $x = x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w = x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} \vee x_w$.

Let $y \in \mathcal{B}^*$, $y = y_{v_i} \oplus y_{v_{i+1}} \oplus y_{v_{i+2}} \oplus y_w \oplus y_b$, $y_{v_i} \leq v_i$, $y_{v_{i+1}} \leq v_{i+1}$, $y_{v_{i+2}} \leq v_{i+2}$, $y_w \leq w$, $y_b \leq b$. Similarly as above, $y = y_{v_i} \vee y_{v_{i+1}} \vee y_{v_{i+2}} \vee y_w \vee y_b$.

If $x_{a_i} = 0_{\mathcal{P}}$ ($y_{v_{i+2}} \vee y_b = 0_{\mathcal{P}}$) then $x \in \mathcal{B}^*$ ($y \in \mathcal{A}_i^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_{\mathcal{P}}$ and $y_{v_{i+2}} \vee y_b > 0_{\mathcal{P}}$ then $x_{a_i} \vee (y_{v_{i+2}} \vee y_b) = (v_i \oplus v_{i+1} \oplus w)^{\perp} = a_i$ $= v_{i+2} \oplus b$. Without doubt the element $(x_{v_i} \vee y_{v_i}) \vee (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)$ exists in $\mathcal{P}$and we denote it by $d$. In addition, $d \in \mathcal{A}_i^* \cap \mathcal{B}^*$, $a_i \oplus d = a_i \vee d$, $x \leq a_i \oplus d$ and $y \leq a_i \oplus d$. Let $z \in \mathcal{P}$ such that $x \leq z$ and $y \leq z$. Then $z \geq x_{a_i} \vee y_{v_{i+2}} \vee y_b = a_i$ and also $z \geq d$, which gives $a_i \oplus d = a_i \vee d \leq z$, therefore $x \vee y = a_i \oplus d = (v_i \oplus v_{i+1} \oplus w)^{\perp} \oplus d$.

Now let $y \in \mathcal{A}_{i+1}^*$, $y = y_{v_{i+2}} \oplus y_{a_{i+1}} \oplus y_{v_{i+1}} \oplus y_w$, where $y_{v_{i+2}} \leq v_{i+2}$, $y_{a_{i+1}} \leq a_{i+1}$, $y_{v_{i+1}} \leq v_{i+1}$, $y_w \leq w$. Then certainly $y = y_{v_{i+2}} \vee y_{a_{i+1}} \vee y_{v_{i+1}} \vee y_w$.

If $x_{a_i} = 0_{\mathcal{P}}$ or $y_{a_{i+1}} = 0_{\mathcal{P}}$ then $x \in \mathcal{B}^*$ or $y \in \mathcal{B}^*$, so $x \vee y$ exists in $\mathcal{P}$. Suppose that $x_{a_i} > 0_{\mathcal{P}}$ and $y_{a_{i+1}} > 0_{\mathcal{P}}$. Then $x_{a_i} \vee y_{a_{i+1}} = (v_{i+1} \oplus w)^{\perp}$ and, moreover,

$$\begin{aligned} x_{v_i} \vee y_{v_{i+2}} &\leq v_i \vee v_{i+2} = v_i \oplus v_{i+2} = (v_{i+1} \oplus w \oplus b)^{\perp} \leq (v_{i+1} \oplus w)^{\perp} \\ &= x_{a_i} \vee y_{a_{i+1}}, \end{aligned}$$

therefore, $(x_{v_i} \vee y_{v_{i+2}}) \vee (x_{a_i} \vee y_{a_{i+1}}) = x_{a_i} \vee y_{a_{i+1}} = (v_{i+1} \oplus w)^{\perp}$. It is clear that the element $(x_{v_{i+1}} \vee y_{v_{i+1}}) \oplus (x_w \vee y_w)$ exists in $\mathcal{P}$ and we denote it by $c$.

Then $c \leq v_{i+1} \oplus w$ and we prove that $(v_{i+1} \oplus w)^{\perp} \oplus c$ is the supremum of $x$ and $y$. Observe that

$$\begin{aligned}(v_{i+1} \oplus w)^{\perp} \oplus c &= v_i \oplus a_i \oplus \big((x_{v_{i+1}} \vee y_{v_{i+1}}) \oplus (x_w \vee y_w)\big) \\ &= v_i \vee a_i \vee (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w) = (v_{i+1} \oplus w)^{\perp} \vee c.\end{aligned}$$

We have

$$\begin{aligned} x_{v_i} \vee x_{a_i} &= x_{v_i} \oplus x_{a_i} \leq v_i \oplus a_i = (v_{i+1} \oplus w)^{\perp} \leq (v_{i+1} \oplus w)^{\perp} \oplus c, \\ x_{v_{i+1}} \vee x_w &\leq (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w) = (v_{i+1} \oplus w)^{\perp} \oplus c, \end{aligned}$$

so $x \leq (v_{i+1} \oplus w)^{\perp} \oplus c$, as well as $y \leq (v_{i+1} \oplus w)^{\perp} \oplus c$. Let $z \in \mathcal{P}$ such that $x \leq z$ and $y \leq z$. Then

$$z \geq (x_{v_i} \vee y_{v_{i+2}}) \vee (x_{a_i} \vee y_1) = (v_{i+1} \oplus w)^{\perp}, \qquad z \geq (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w) = c,$$

so $(v_{i+1} \oplus w)^{\perp} \oplus c = (v_{i+1} \oplus w)^{\perp} \vee c \leq z$, which implies that $(v_{i+1} \oplus w)^{\perp} \oplus c = x \vee y$.

We proved that any two elements of $\mathcal{P}$ have a supremum and thus the proof is complete. □

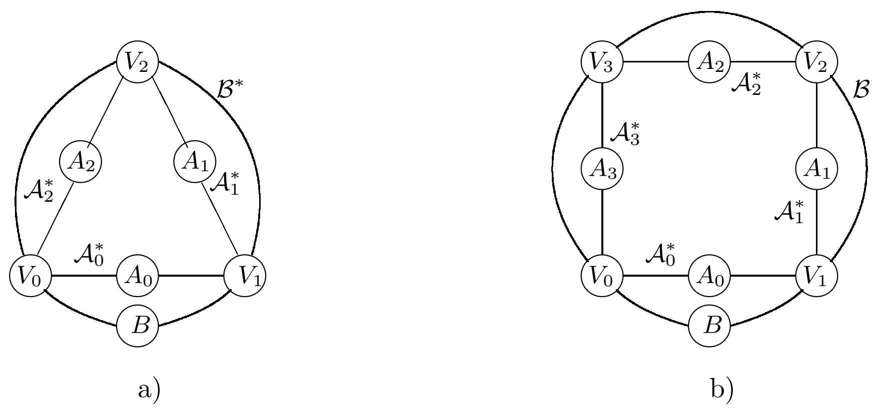


FIGURE 5. Lattice ordered D-posets containing: a) a 3-loop, b) a 4-loop.

**DEFINITION 19.** We say that a 3-loop is unbound in a pasting $\mathcal{P}$ of an admissible system of MV-algebras, if there is no block in $\mathcal{P}$ containing all its nodal vertices.

The following theorem generalizes Theorem 14.

**THEOREM 20.** *Every pasting of an admissible system of MV-algebras containing an unbound 3-loop is not a lattice-ordered D-poset.*

P r o o f. The proof may be made in the same manner as the proof of Theorem 14. □

**Theorem 21.** *Every astroid is a D-lattice.*

Proof. Let $\mathcal{P} = \bigcup_{i=0}^{3} \mathcal{A}_i^*$ be a pasting of an admissible system of MV-algebras such that blocks $\mathcal{A}_0^*, \mathcal{A}_1^*, \mathcal{A}_2^*, \mathcal{A}_3^*$ create an astroid. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{ik_i}\}$ be the sets of nodal vertices, $W = \{w_1, w_2, \dots, w_k\}$ be the set of central nodal vertices of the astroid, $w = \bigoplus_{s=1}^{k} \tau(w_s)w_s$, $v_i = \bigoplus_{j=1}^{k_i} \tau(v_{ij})v_{ij}$ for $i = 0, 1, 2, 3$. This situation is typified in Fig. 3c). The equalities

$$v_i \oplus v_{i+1} \oplus w = 1_{\mathcal{P}} = v_{i+1} \oplus v_{i+2} \oplus w$$

immediately give $v_i = v_{i+2}$ for $i = 0, 1$. Let $z_i \in \mathcal{A}_i^*$ such that $z_i \le v_i$ ($i = 0, 1, 2, 3$) and $u \le w$. Obviously $z_i \vee u = z_i \oplus u$, $z_i \vee z_{i+1} = z_i \oplus z_{i+1}$, $z_i \vee z_{i+1} \vee u = z_i \oplus z_{i+1} \oplus u$. We still need to determine the supremum of $z_i$ and $z_{i+2}$ for $i = 0, 1$. Suppose that $z_i > 0_{\mathcal{P}}$ and $z_{i+2} > 0_{\mathcal{P}}$. We have

$$z_i \le (v_{i+1} \oplus w)^{\perp} = (v_{i+3} \oplus w)^{\perp}, \qquad z_{i+2} \le (v_{i+1} \oplus w)^{\perp} = (v_{i+3} \oplus w)^{\perp},$$

thus $z_i \vee z_{i+2} = (v_{i+1} \oplus w)^{\perp} = (v_{i+3} \oplus w)^{\perp}$ for $i = 0, 1$.

Let $x \in \mathcal{A}_i^*$, $x = x_{v_i} \oplus x_{v_{i+1}} \oplus x_w$, such that $x_{v_i} \le v_i$, $x_{v_{i+1}} \le v_{i+1}$, $x_w \le w$ and $y \in \mathcal{A}_{i+1}^*$, $y = y_{v_{i+1}} \oplus y_{v_{i+2}} \oplus y_w$, where $y_{v_{i+1}} \le v_{i+1}$, $y_{v_{i+2}} \le v_{i+2}$, $y_w \le w$ for some $i \in \{0, 1, 2, 3\}$ (mod 4). Then $x = x_{v_i} \vee x_{v_{i+1}} \vee x_w$ and $y = y_{v_{i+1}} \vee y_{v_{i+2}} \vee y_w$.

If $x_{v_i} = 0_{\mathcal{P}}$ ($y_{v_{i+2}} = 0_{\mathcal{P}}$) then $x \in \mathcal{A}_{i+1}^*$ ($y \in \mathcal{A}_i^*$) and for that reason the supremum of $x, y$ exists in $\mathcal{P}$. If $x_{v_i} > 0_{\mathcal{P}}$ and $y_{v_{i+2}} > 0_{\mathcal{P}}$ then $x_{v_i} \vee y_{v_{i+2}} = (v_{i+1} \oplus w)^{\perp}$. It is clear that $(x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)$ exists in $\mathcal{P}$ and we denote it by $c$. In the same manner as in the proof of Theorem 18 it can be proved that $x \vee y = (v_{i+1} \oplus w)^{\perp} \oplus c$.

Let $y \in \mathcal{A}_{i+2}^*$, $y = y_{v_{i+2}} \oplus y_{v_{i+3}} \oplus y_w$, where $y_{v_{i+2}} \le v_{i+2}$, $y_{v_{i+3}} \le v_{i+3}$, $y_w \le w$. Then $y = y_{v_{i+2}} \vee y_{v_{i+3}} \vee y_w$ and there are the following possibilities.

(i) If $x_{v_i} = 0_{\mathcal{P}}$ and $y_{v_{i+3}} = 0_{\mathcal{P}}$ then $x \in \mathcal{A}_i^* \cap \mathcal{A}_{i+1}^*$, $y \in \mathcal{A}_{i+1}^* \cap \mathcal{A}_{i+2}^*$ and therefore

$$x \vee y = x_{v_{i+1}} \vee y_{v_{i+2}} \vee (x_w \vee y_w).$$

(ii) If $x_{v_i+1} = 0_{\mathcal{P}}$ and $y_{v_{i+2}} = 0_{\mathcal{P}}$ then $x \in \mathcal{A}_i^* \cap \mathcal{A}_{i+3}^*$, $y \in \mathcal{A}_{i+2}^* \cap \mathcal{A}_{i+3}^*$ and therefore

$$x \vee y = x_{v_i} \vee y_{v_{i+3}} \vee (x_w \vee y_w).$$

(iii) If $x_{v_i+1} = 0_{\mathcal{P}}$ (or $y_{v_{i+3}} = 0_{\mathcal{P}}$) and $x_{v_i} > 0_{\mathcal{P}}$, $y_{v_{i+2}} > 0_{\mathcal{P}}$, then

$$x \vee y = (v_{i+1} \oplus w)^{\perp} \vee y_{v_{i+3}} \vee (x_w \vee y_w) \ \left(x \vee y = (v_{i+1} \oplus w)^{\perp} \vee x_{v_{i+1}} \vee (x_w \vee y_w)\right).$$

(iv) If $x_{v_i} = 0_{\mathcal{P}}$ (or $y_{v_{i+2}} = 0_{\mathcal{P}}$) and $x_{v_{i+1}} > 0_{\mathcal{P}}$, $y_{v_{i+3}} > 0_{\mathcal{P}}$, then

$$x \vee y = (v_i \oplus w)^{\perp} \vee y_{v_{i+2}} \vee (x_w \vee y_w) \ \left(x \vee y = (v_i \oplus w)^{\perp} \vee x_{v_i} \vee (x_w \vee y_w)\right).$$

(v) If $x_{v_i} > 0_{\mathcal{P}}$, $x_{v_{i+1}} > 0_{\mathcal{P}}$, $y_{v_{i+2}} > 0_{\mathcal{P}}$, $y_{v_{i+3}} > 0_{\mathcal{P}}$, then

$$x \vee y = w^{\perp} \oplus (x_w \vee y_w).$$

In this case, $W = \emptyset$ implies $x \vee y = 1_{\mathcal{P}}$. □

**Theorem 22.** *Let $\mathcal{P} = \mathcal{A}_0^* \cup \mathcal{A}_1^* \cup \mathcal{A}_2^* \cup \mathcal{A}_3^* \cup \mathcal{B}^*$ be a pasting of an admissible system of five MV-algebras, where the blocks $\mathcal{A}_i^*$, $i = 0, 1, 2, 3$, form a 4-loop and the block $\mathcal{B}^*$ contains all nodal vertices of the 4-loop. Then $\mathcal{P}$ is a D-lattice.*

Proof. Let $V_i$ $(i = 0, 1, 2, 3)$ be the sets of nodal vertices and $W$ be the set of central nodal vertices of the 4-loop. Let us denote

$$B = \mathrm{At}(\mathcal{B}^*) \setminus (V_0 \cup V_1 \cup V_2 \cup V_3 \cup W) = \{b_s :\ s \in S\}, \qquad b = \bigoplus_{s \in S} \tau(b_s) b_s,$$

where $S$ is a countable index set. Since $B \neq \emptyset$, it follows that $b > 0_{\mathcal{P}}$. Fig. 5b) shows this situation for $W = \emptyset$. We have

$$v_i \oplus a_i \oplus v_{i+1} \oplus w = 1_{\mathcal{P}} = v_i \oplus v_{i+1} \oplus v_{i+2} \oplus v_{i+3} \oplus w \oplus b,$$

and hence $a_i = (v_i \oplus v_{i+1} \oplus w)^{\perp} = v_{i+2} \oplus v_{i+3} \oplus b$ for every $i = 0, 1, 2, 3 \pmod 4$. Apparently $v_i \oplus v_{i+2}$ is the least upper bound of $v_i$ and $v_{i+2}$ in the block $\mathcal{B}^*$. We claim that $v_i \oplus v_{i+2}$ is the supremum of $v_i$ and $v_{i+2}$ in $\mathcal{P}$. Indeed, if $z \in \mathcal{P}$ such that $v_i \le z$ and $v_{i+2} \le z$, then necessarily $z \in \mathcal{B}^*$, so $v_i \oplus v_{i+2} = v_i \vee_{\mathcal{B}^*} v_{i+2} \le z$ and thus $v_i \oplus v_{i+2}$ is also the least upper bound of $v_i$ and $v_{i+2}$ in $\mathcal{P}$.

Let $x \in \mathcal{A}_i^*$, $x = x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w$, $x_{v_i} \le v_i$, $x_{a_i} \le a_i$, $x_{v_{i+1}} \le v_{i+1}$, $x_w \le w$ for some $i \in \{0, 1, 2, 3\} \pmod 4$. Undoubtedly $x = x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} \vee x_w$.

Let $y \in \mathcal{B}^*$, $y = y_{v_i} \oplus y_{v_{i+1}} \oplus y_{v_{i+2}} \oplus y_{v_{i+3}} \oplus y_w$, $y_{v_i} \le v_i$, $y_{v_{i+1}} \le v_{i+1}$, $y_{v_{i+2}} \le v_{i+2}$, $y_{v_{i+3}} \le v_{i+3}$, $y_w \le w$. Then $y = y_{v_i} \vee y_{v_{i+1}} \vee y_{v_{i+2}} \vee y_{v_{i+3}} \vee y_w$.

If $x_{a_i} = 0_{\mathcal{P}}$ $(y_{v_{i+2}} \vee y_{v_{i+3}} \vee y_b = 0_{\mathcal{P}})$ then $x \in \mathcal{B}^*$ $(y \in \mathcal{A}_i^*)$, so $x \vee y$ exists in $\mathcal{P}$. Suppose that $x_{a_i} > 0_{\mathcal{P}}$ and $y_{v_{i+2}} \vee y_{v_{i+3}} \vee y_b > 0_{\mathcal{P}}$. Then

$$x_{a_i} \vee (y_{v_{i+2}} \vee y_{v_{i+3}} \vee y_b) = (v_i \oplus v_{i+1} \oplus w)^{\perp} = a_i$$

and by putting $d = (x_{v_i} \vee y_{v_i}) \vee (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)$ we get $x \vee y = a_i \oplus d$.

Let $y \in \mathcal{A}_{i+1}^*$, $y = y_{v_{i+1}} \oplus y_{a_{i+1}} \oplus y_{v_{i+2}} \oplus y_w$, $y_{v_{i+1}} \le v_{i+1}$, $y_{a_{i+1}} \le a_{i+1}$, $y_{v_{i+2}} \le v_{i+2}$, $y_w \le w$. Then $y = y_{v_{i+1}} \vee y_{a_{i+1}} \vee y_{v_{i+2}} \vee y_w$. If $x_{a_i} = 0_{\mathcal{P}}$ $(y_{a_{i+1}} = 0_{\mathcal{P}})$ then $x \in \mathcal{B}^*$ $(y \in \mathcal{B}^*)$, so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_{\mathcal{P}}$ and $y_{a_{i+1}} > 0_{\mathcal{P}}$ then $x_{a_i} \vee y_{a_{i+1}} = (v_{i+1} \oplus w)^{\perp}$ and $x \vee y = (v_{i+1} \oplus w)^{\perp} \oplus \big((x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)\big)$.

Let $y \in \mathcal{A}_{i+2}^*$, $y = y_{v_{i+3}} \oplus y_{a_{i+2}} \oplus y_{v_{i+2}} \oplus y_w$, $y_{v_{i+3}} \le v_{i+3}$, $y_{a_{i+2}} \le a_{i+2}$, $y_{v_{i+2}} \le v_{i+2}$, $y_w \le w$. Clearly $y = y_{v_{i+3}} \vee y_{a_{i+2}} \vee y_{v_{i+2}} \vee y_w$. As above, $x_{a_i} = 0_{\mathcal{P}}$ $(y_{a_{i+2}} = 0_{\mathcal{P}})$ gives that $x \in \mathcal{B}^*$ $(y \in \mathcal{B}^*)$, therefore $x \vee y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_{\mathcal{P}}$ and $y_{a_{i+2}} > 0_{\mathcal{P}}$ then $x_{a_i} \vee y_{a_{i+2}} = w^{\perp}$ and hence $x \vee y = w^{\perp} \oplus (x_w \vee y_w)$. □

**Theorem 23.** *Let $\mathcal{P} = \mathcal{A}_0^* \cup \mathcal{A}_1^* \cup \mathcal{A}_2^* \cup \mathcal{A}_3^* \cup \mathcal{C}_0^* \cup \mathcal{C}_1^*$ be a pasting of an admissible system of six MV-algebras, where the blocks $\mathcal{A}_i^*$, $i = 0,1,2,3$, form a 4-loop. Let $V_i$ $(i = 0,1,2,3)$ be the sets of nodal vertices and $W$ be the set of central nodal vertices of the 4-loop. If $V_i \cup V_{i+1} \cup V_{i+2} \cup W \subset \mathrm{At}(\mathcal{C}_0^*)$ and $V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup W \subset \mathrm{At}(\mathcal{C}_1^*)$ for some $i \in \{0,1,2,3\}$ (mod 4), then $\mathcal{P}$ is a D-lattice.*

P r o o f. Without loss of generality we assume that $V_0 \cup V_1 \cup V_2 \cup W \subset \mathrm{At}(\mathcal{C}_0^*)$ and $V_2 \cup V_3 \cup V_0 \cup W \subset \mathrm{At}(\mathcal{C}_1^*)$. Let us denote

$$C_0 = \mathrm{At}(\mathcal{C}_0^*) \setminus (V_0 \cup V_1 \cup V_2 \cup W) = \{c_{0t} :\ t \in T\}, \qquad c_0 = \bigoplus_{t\in T} \tau(c_{0t}) c_{0t},$$

$$C_1 = \mathrm{At}(\mathcal{C}_1^*) \setminus (V_2 \cup V_3 \cup V_0 \cup W) = \{c_{1s} :\ s \in S\}, \qquad c_1 = \bigoplus_{s\in S} \tau(c_{1s}) c_{1s},$$

where $T$ and $S$ is are countable index set. Since $C_j \neq \emptyset$, it follows that $c_j > 0_{\mathcal{P}}$ for $j = 1,2$. We have

$$v_0 \oplus v_1 \oplus c_0 \oplus v_2 \oplus w = 1_{\mathcal{P}} = v_0 \oplus c_1 \oplus v_3 \oplus v_2 \oplus w,$$

and hence $v_1 \oplus c_0 = (v_0 \oplus v_2 \oplus w)^\perp = v_3 \oplus c_1$. Denote $a = v_0 \oplus v_2 \oplus w$. We prove that $a^\perp$ is the supremum of $v_1$ and $v_3$ in $\mathcal{P}$. Let $z \in \mathcal{P}$ such that $v_1 \le z$ and $v_3 \le z$. Then $z \in \mathcal{C}_0^* \cap \mathcal{C}_1^* = [0_{\mathcal{P}}, a] \cup [a^\perp, 1_{\mathcal{P}}]$. If $z \in [0_{\mathcal{P}}, a]$ then

$$v_1 \le z \le a = v_0 \oplus v_2 \oplus w = (v_1 \oplus c_0)^\perp \le v_1^\perp,$$

which contradicts the isotropic index of the element $v_1$, because $\tau(v_1) = 1$. Thus necessarily $z \in [a^\perp, 1_{\mathcal{P}}]$, which gives $a^\perp \le z$. This proves that the supremum of $v_1$ and $v_3$ exists in $\mathcal{P}$, namely $v_1 \vee v_3 = a^\perp$. In the same way it can be shown that $c_0 \vee c_1 = a^\perp$.

Since $v_0 \oplus v_2$ is the least upper bound in the block $\mathcal{C}_0^*$ and also in the block $\mathcal{C}_1^*$, it is the least upper bound in $\mathcal{C}_0^* \cap \mathcal{C}_1^*$, too. If $z \in \mathcal{P}$ such that $v_0 \le z$ and $v_2 \le z$, then necessarily $z \in \mathcal{C}_0^* \cap \mathcal{C}_1^*$, therefore $v_0 \oplus v_2 = v_0 \vee_{\mathcal{C}_0^* \cap \mathcal{C}_1^*} v_2 \le z$ and thus $v_0 \oplus v_2$ is the suppremum of $v_0$ and $v_2$ in $\mathcal{P}$.

Let $x \in \mathcal{C}_0^*$, $y \in \mathcal{C}_1^*$, $x = x_{v_0} \oplus x_{v_1} \oplus x_{c_0} \oplus x_{v_2} \oplus x_w$, $y = y_{v_0} \oplus y_{c_1} \oplus y_{v_3} \oplus y_{v_2} \oplus y_w$, $x_{v_0} \le v_0$, $x_{v_1} \le v_1$, $x_{c_0} \le c_0$, $x_{v_2} \le v_2$, $x_w \le w$, $y_{v_0} \le v_0$, $y_{c_1} \le c_1$, $y_{v_2} \le v_2$, $y_{v_3} \le v_3$, $y_w \le w$. There is no doubt that $x = x_{v_0} \vee x_{v_1} \vee x_{c_0} \vee x_{v_2} \vee x_w$, $y = y_{v_0} \vee y_{c_1} \vee y_{v_3} \vee y_{v_2} \vee y_w$. If $x_{v_1} \vee x_{c_0} = 0_{\mathcal{P}}$ ($y_{c_1} \vee y_{v_3} = 0_{\mathcal{P}}$) then $x \in \mathcal{C}_1^*$ ($y \in \mathcal{C}_0^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{v_1} \vee x_{c_0} > 0_{\mathcal{P}}$ and $y_{c_1} \vee y_{v_3} > 0_{\mathcal{P}}$ then $(x_{v_1} \vee x_{c_0}) \vee (y_{v_3} \vee x_{c_1}) = a^\perp$ and

$$x \vee y = a^\perp \oplus \left((x_{v_0} \vee y_{v_0}) \vee (x_{v_2} \vee y_{v_2}) \vee (x_w \vee y_w)\right).$$

Let $x \in \mathcal{A}_0^*$, $x = x_{v_0} \oplus x_{a_0} \oplus x_{v_1} \oplus x_w$, $x_{v_0} \le v_0$, $x_{a_0} \le a_0$, $x_{v_1} \le v_1$, $x_w \le w$, and $y \in \mathcal{C}_0^*$, $y = y_{v_0} \oplus y_{v_1} \oplus y_{c_0} \oplus y_{v_2} \oplus y_w$, $y_{v_0} \le v_0$, $y_{v_1} \le v_1$, $y_{c_0} \le c_0$, $y_{v_2} \le v_2$, $y_w \le w$. Then $x = x_{v_0} \vee x_{a_0} \vee x_{v_1} \vee x_w$ and $y = y_{v_0} \vee y_{v_1} \vee y_{c_0} \vee y_{v_2} \vee y_w$. If

$x_{a_0} = 0_{\mathcal{P}}$ ($y_{c_0} \vee y_{v_2} = 0_{\mathcal{P}}$) then $x \in \mathcal{C}_0^*$ ($y \in \mathcal{A}_0^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_0} > 0_{\mathcal{P}}$ and $y_{c_0} \vee y_{v_2} > 0_{\mathcal{P}}$ then $x_{a_0} \vee (y_{c_0} \vee y_{v_2}) = (v_0 \oplus v_1 \oplus w)^{\perp}$ and hence

$$x \vee y = (v_0 \oplus v_1 \oplus w)^{\perp} \oplus ((x_{v_0} \vee y_{v_0}) \vee (x_{v_1} \vee y_{v_1}) \vee (x_w \vee y_w)) .$$

Let $x \in \mathcal{A}_0^*$ and $y \in \mathcal{C}_1^*$, where $x$ and $y$ are defined as above. If $x_{a_0} \vee x_{v_1} = 0_{\mathcal{P}}$ ($y_{v_2} \vee y_{v_3} \vee y_{c_1} = 0_{\mathcal{P}}$) then $x \in \mathcal{C}_1^*$ ($y \in \mathcal{A}_0^*$), therefore $x \vee y$ exists in $\mathcal{P}$. Provided that $x_{a_0} \vee x_{v_1} > 0_{\mathcal{P}}$ and $y_{v_2} \vee y_{v_3} \vee y_{c_1} > 0_{\mathcal{P}}$ we get $(x_{a_0} \vee x_{v_1}) \vee (y_{v_2} \vee y_{v_3} \vee y_{c_1}) = (v_0 \oplus w)^{\perp}$ and thus

$$x \vee y = (v_0 \oplus w)^{\perp} \oplus ((x_{v_0} \vee y_{v_0}) \vee (x_w \vee y_w)) .$$

Similarly it can be shown that $x \vee y$ exists in $\mathcal{P}$ if $x \in \mathcal{A}_i^*$ ($i = 1, 2, 3$) and $y \in \mathcal{C}_0^*$ or $y \in \mathcal{C}_1^*$.

Let $x \in \mathcal{A}_0^*$, $y \in \mathcal{A}_1^*$, $y = y_{v_{i+1}} \oplus y_{a_{i+1}} \oplus y_{v_{i+2}} \oplus y_w$, $y_{v_{i+1}} \le v_{i+1}$, $y_{a_{i+1}} \le a_{i+1}$, $y_{v_{i+2}} \le v_{i+2}$, $y_w \le w$. It is clear that $y = y_{v_{i+1}} \vee y_{a_{i+1}} \vee y_{v_{i+2}} \vee y_w$. If $x_{a_0} = 0_{\mathcal{P}}$ ($y_{a_1} = 0_{\mathcal{P}}$) then $x \in \mathcal{C}_0^*$ ($y \in \mathcal{C}_0^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_0} > 0_{\mathcal{P}}$ and $y_{a_1} > 0_{\mathcal{P}}$ then $(x_{v_0} \vee x_{a_0}) \vee (y_{v_2} \vee y_{a_1}) = (v_1 \oplus w)^{\perp}$ and therefore

$$x \vee y = (v_1 \oplus w)^{\perp} \oplus ((x_{v_1} \vee y_{v_1}) \vee (x_w \vee y_w)) .$$

In the same manner it can be shown that $x \vee y$ exists in $\mathcal{P}$ if $x \in \mathcal{A}_0^*$ and $y \in \mathcal{A}_3^*$.

Now suppose that $x \in \mathcal{A}_0^*$ and $y \in \mathcal{A}_2^*$, $y = y_{v_2} \oplus y_{a_2} \oplus y_{v_3} \oplus y_w$, where $y_{v_2} \le v_2$, $y_{a_2} \le a_2$, $y_{v_3} \le v_3$, $y_w \le w$. Then, of course, $y = y_{v_2} \vee y_{a_2} \vee y_{v_3} \vee y_w$. If $x_{a_0} = 0_{\mathcal{P}}$ ($y_{a_2} = 0_{\mathcal{P}}$) then $x \in \mathcal{C}_0^*$ ($y \in \mathcal{C}_1^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_0} > 0_{\mathcal{P}}$ and $y_{a_2} > 0_{\mathcal{P}}$ then $(x_{v_0} \vee x_{a_0} \vee x_{v_1}) \vee (y_{v_2} \vee y_{a_2} \vee y_{v_3}) = w^{\perp}$, so $x \vee y = w^{\perp} \oplus (x_w \vee y_w)$.

Finally we can say that $x \vee y$ exists in $\mathcal{P}$ whenever $x \in \mathcal{A}_i^*$ and $y \in \mathcal{A}_j^*$ for every $i, j \in \{0, 1, 2, 3\}$. □

**Theorem 24.** *Let $\mathcal{P} = \mathcal{A}_0^* \cup \mathcal{A}_1^* \cup \mathcal{A}_2^* \cup \mathcal{A}_3^* \cup \mathcal{B}_0^* \cup \mathcal{B}_1^* \cup \mathcal{B}_2^* \cup \mathcal{B}_3^*$ be a pasting of an admissible system of eight MV-algebras. Let the blocks $\mathcal{A}_i^*$, $i = 0, 1, 2, 3$, form a 4-loop with the sets $V_i$ ($i = 0, 1, 2, 3$) of nodal vertices and the set $W$ of central nodal vertices. Let the blocks $\mathcal{B}_i^*$, $i = 0, 1, 2, 3$, form an astroid with the sets $U_i$ of nodal vertices and the set $W_0$ of central nodal vertices, such that $V_i \subset U_i$ and $W \subset W_0$ for every $i = 0, 1, 2, 3$. Then $\mathcal{P}$ is a D-lattice.*

Proof. Let $V_i$, $A_i$, $W$, $v_i$, $a_i$, $w$ be defined as in the proof of Theorem 16, $U_i = \{u_{i1}, u_{i2}, \dots, u_{im_i}\}$, $B_i = U_i \setminus V_i = \{b_{i1}, b_{i2}, \dots, b_{i\beta_i}\}$ for $i = 0, 1, 2, 3$, and $W_0 = \{w_{01}, \dots, w_{0q}\}$, $\overline{W} = W_0 \setminus W = \{\overline{w}_1, \overline{w}_2, \dots, \overline{w}_p\}$. Let us put

$$u_i = \bigoplus_{j=1}^{m_i} \tau(u_{ij}) u_{ij}, \quad b_i = \bigoplus_{s=1}^{\beta_i} \tau(b_{is}) b_{is}, \quad w_0 = \bigoplus_{r=1}^{q} \tau(w_{0r}) w_{0r}, \quad \overline{w} = \bigoplus_{t=1}^{p} \tau(\overline{w}_t) \overline{w}_t.$$

Since $B_i \neq \emptyset$, it follows that $b_i > 0_{\mathcal{P}}$, $u_i = v_i \oplus b_i$, and moreover, $w_0 = w \oplus \overline{w}$. The equalities

$$v_i \oplus a_i \oplus v_{i+1} \oplus w = 1_{\mathcal{P}} = u_i \oplus u_{i+1} \oplus w_0 = (v_i \oplus b_i) \oplus (v_{i+1} \oplus b_{i+1}) \oplus (w \oplus \overline{w})$$

imply that $a_i = (v_i \oplus v_{i+1} \oplus w)^\perp = b_i \oplus b_{i+1} \oplus \overline{w}$ and $u_i = u_{i+2}$ for $i = 0, 1, 2, 3$ (mod 4).

Let $x \in \mathcal{A}_i^*$, $x = x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w$, where $x_{v_i} \leq v_i$, $x_{a_i} \leq a_i$, $x_{v_{i+1}} \leq v_{i+1}$, $x_w \leq w$ for some $i \in \{0, 1, 2, 3\}$ (mod 4). Likewise as in the proof of Theorem 18 it can be proved that $x = x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} \vee x_w$.

Let $y \in \mathcal{A}_{i+1}^*$, $y = y_{v_{i+1}} \oplus y_{a_{i+1}} \oplus y_{v_{i+2}} \oplus y_w$, $y_{v_{i+1}} \leq v_{i+1}$, $y_{a_{i+1}} \leq a_{i+1}$, $y_{v_{i+2}} \leq v_{i+2}$, $y_w \leq w$. Then $y = y_{v_{i+1}} \vee y_{a_{i+1}} \vee y_{v_{i+2}} \vee y_w$. If $x_{v_i} \vee x_{a_i} = 0_\mathcal{P}$ ($y_{a_{i+1}} \vee y_{v_{i+2}} = 0_\mathcal{P}$) then $x \in \mathcal{A}_{i+1}^*$ ($y \in \mathcal{A}_i^*$), so the supremum of $x$ and $y$ exists in $\mathcal{P}$. If $x_{v_i} \vee x_{a_i} > 0_\mathcal{P}$ and $y_{a_{i+1}} \vee y_{v_{i+2}} > 0_\mathcal{P}$ then $(x_{v_i} \vee x_{a_i}) \vee (y_{a_{i+1}} \vee y_{v_{i+2}}) = (v_{i+1} \oplus w)^\perp$.

Let us denote $(x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)$ by $c$. Then $x \leq (v_{i+1} \oplus w)^\perp \oplus c$ and also $y \leq (v_{i+1} \oplus w)^\perp \oplus c$ and in the same manner as in the proof of Theorem 18 it can be proved that $x \vee y = (v_{i+1} \oplus w)^\perp \oplus c$.

Let $y \in \mathcal{B}_i^*$, $y = y_{v_i} \oplus y_{b_i} \oplus y_{v_{i+1}} \oplus y_{b_{i+1}} \oplus y_w \oplus y_{\overline{w}}$, $y_{v_i} \leq v_i$, $y_{b_i} \leq b_i$, $y_{v_{i+1}} \leq v_{i+1}$, $y_{b_{i+1}} \leq b_{i+1}$, $y_w \leq w$, $y_{\overline{w}} \leq \overline{w}$. Then $y = y_{v_i} \vee y_{b_i} \vee y_{v_{i+1}} \vee y_{b_{i+1}} \vee y_w \vee y_{\overline{w}}$. If $x_{a_i} = 0_\mathcal{P}$ ($y_{b_i} \vee y_{b_{i+1}} \vee y_{\overline{w}} = 0_\mathcal{P}$) then $x \in \mathcal{B}_i^*$ ($y \in \mathcal{A}_i^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_\mathcal{P}$ and $y_{b_i} \vee y_{b_{i+1}} \vee y_{\overline{w}} > 0_\mathcal{P}$, then $x_{a_i} \vee (y_{b_i} \vee y_{b_{i+1}} \vee y_{\overline{w}}) = (v_i \oplus v_{i+1} \oplus w)^\perp$ and by putting $d = (x_{v_i} \vee y_{v_i}) \vee (x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)$ we get $x \vee y = (v_i \oplus v_{i+1} \oplus w)^\perp \oplus d$.

Let $y \in \mathcal{B}_{i+1}^*$, $y = y_{v_{i+1}} \oplus y_{b_{i+1}} \oplus y_{v_{i+2}} \oplus y_{b_{i+2}} \oplus y_w \oplus y_{\overline{w}}$, $y_{v_{i+1}} \leq v_{i+1}$, $y_{b_{i+1}} \leq b_{i+1}$, $y_{v_{i+2}} \leq v_{i+2}$, $y_{b_{i+2}} \leq b_{i+2}$, $y_w \leq w$, $y_{\overline{w}} \leq \overline{w}$. Then $y = y_{v_{i+1}} \vee y_{b_{i+1}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_w \vee y_{\overline{w}}$. If $x_{a_i} = 0_\mathcal{P}$ ($y_{b_{i+1}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}} = 0_\mathcal{P}$) then $x \in \mathcal{B}_i^*$ ($y \in \mathcal{A}_i^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_\mathcal{P}$ and $y_{b_{i+1}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}} > 0_\mathcal{P}$ then $x_{a_i} \vee (y_{b_{i+1}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}}) = (v_{i+1} \oplus w)^\perp$, therefore $x \vee y = (v_{i+1} \oplus w)^\perp \oplus \big((x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)\big)$.

Let $y \in \mathcal{B}_{i+2}^*$, $y = y_{v_{i+2}} \oplus y_{b_{i+2}} \oplus y_{v_{i+3}} \oplus y_{b_{i+3}} \oplus y_w \oplus y_{\overline{w}}$, such that $y_{v_{i+2}} \leq v_{i+2}$, $y_{b_{i+2}} \leq b_{i+2}$, $y_{v_{i+3}} \leq v_{i+3}$, $y_{b_{i+3}} \leq b_{i+3}$, $y_w \leq w$, $y_{\overline{w}} \leq \overline{w}$. As in previous cases we have $y = y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{v_{i+3}} \vee y_{b_{i+3}} \vee y_w \vee y_{\overline{w}}$. Obviously $x_{a_i} = 0_\mathcal{P}$ gives $x \in \mathcal{B}_i^*$ and on the other hand $y_{v_{i+3}} \vee y_{b_{i+3}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}} = 0_\mathcal{P}$ gives $y \in \mathcal{A}_i^*$, so $x \vee y \in \mathcal{P}$. If $x_{a_i} > 0_\mathcal{P}$ and $y_{v_{i+3}} \vee y_{b_{i+3}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}} > 0_\mathcal{P}$ then $x_{a_i} \vee (y_{v_{i+3}} \vee y_{b_{i+3}} \vee y_{v_{i+2}} \vee y_{b_{i+2}} \vee y_{\overline{w}}) = w^\perp$ and thus $x \vee y = w^\perp \oplus (x_w \vee y_w)$.

Finally let $y \in \mathcal{A}_{i+2}^*$, $y = y_{v_{i+3}} \oplus y_{a_{i+2}} \oplus y_{v_{i+2}} \oplus y_w$, where $y_{v_{i+3}} \leq v_{i+3}$, $y_{a_{i+2}} \leq a_{i+2}$, $y_{v_{i+2}} \leq v_{i+2}$, $y_w \leq w$. Clearly $y = y_{v_{i+3}} \vee y_{a_{i+2}} \vee y_{v_{i+2}} \vee y_w$. If $x_{a_i} = 0_\mathcal{P}$ ($y_{a_{i+2}} = 0_\mathcal{P}$) then $x \in \mathcal{B}_i^*$ ($y \in \mathcal{B}_{i+2}^*$), so as above, the supremum of $x$ and $y$ exists in $\mathcal{P}$. If $x_{a_i} > 0_\mathcal{P}$ and $y_{a_{i+2}} > 0_\mathcal{P}$ then $x_{a_i} \vee y_{a_{i+2}} = w^\perp$ and $x \vee y = w^\perp \oplus (x_w \vee y_w)$. □

**Definition 25.** Let $\mathcal{P}$ be a pasting of an admissible system of MV-algebras containing a 4-loop $\mathcal{A}_i^*$, $i = 0, 1, 2, 3$, with the sets $V_i$ (i=0,1,2,3) of nodal vertices and the set $W$ of central nodal vertices.

(i) We say that the 4-loop $\mathcal{A}_i^*$ $(i = 0,1,2,3)$ is *bound* in the pasting $\mathcal{P}$ if at least one of the following conditions is satisfied.

(1) There are blocks $\mathcal{C}_0^*, \mathcal{C}_1^*$ in $\mathcal{P}$ (not necessarily different) such that $V_0 \cup V_1 \cup V_2 \cup V_3 \cup W \subset \mathrm{At}(\mathcal{C}_0^*) \cup \mathrm{At}(\mathcal{C}_1^*)$ and $V_i \cup V_{i+2} \subset \mathrm{At}(\mathcal{C}_0^*) \cap \mathrm{At}(\mathcal{C}_1^*)$ for some $i \in \{0,1,2,3\}$ (mod 4).

(2) There is an astroid $\mathcal{B}_i^*$, $i = 0,1,2,3$, with the sets $U_i$ (i=0,1,2,3) of nodal vertices and the set $W_0$ of central nodal vertices, such that $V_i \subset U_i$ and $W \subset W_0$ for every $i = 0,1,2,3$.

(ii) The 4-loop $\mathcal{A}_i^*$ $(i = 0,1,2,3)$ is *unbound* in the pasting $\mathcal{P}$ if it is not bound.

Comments on Definition 25:

(1) A 4-loop is bound in a pasting of an admissible system of MV-algebras $\mathcal{P}$ if there is a block in $\mathcal{P}$ containing all its nodal vertices. In this case $\mathcal{C}_0^* = \mathcal{C}_1^*$.

(2) Every astroid in a pasting of an admissible system of MV-algebras is a bound 4-loop.

**THEOREM 26.** *A pasting of an admissible system of MV-algebras containing an unbound* 4*-loop is not a lattice-ordered D-poset.*

Proof. The proof may be made in the same manner as the proof of Theorem 16. □

The results of the previous theorems (Theorem 14, Theorem 16, Theorem 20 and Theorem 26) can be summarized into the following corollary.

**COROLLARY 27.** *Every pasting of an admissible system of MV-algebras containing an unbound* 3*-loop or an unbound* 4*-loop is not a lattice-ordered D-poset.*

Finally we prove that a pasting of an admissible system MV-algebras forming an $n$-loop is a lattice-ordered D-poset for every $n > 4$.

**THEOREM 28.** *Let* $\mathcal{P} = \bigcup_{i=0}^{n-1} \mathcal{A}_i^*$ *be a pasting of an admissible system of MV-algebras, such that the blocks* $\mathcal{A}_i^*$ $(i = 0,1,\dots,n-1)$ *form an* $n$*-loop, where* $n > 4$. *Then* $\mathcal{P}$ *is a lattice-ordered D-poset.*

Proof. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{ik_i}\}$ $(i = 0,1,\dots,n-1)$ be the sets of nodal vertices and $W = \{w_1, w_2, \dots, w_k\}$ be the set of central nodal vertices of the $n$-loop. Denote

$$A_i = \mathrm{At}(\mathcal{A}_i^*) \setminus (V_i \cup V_{i+1} \cup W) = \{a_{it} : t \in T_i\}, \qquad a_i = \bigoplus_{t \in T_i} \tau(a_{it}) a_{it},$$

$$v_i = \bigoplus_{j=1}^{k_i} \tau(v_{ij}) v_{ij}, \quad w = \bigoplus_{s=1}^{k} \tau(w_s) w_s, \qquad i = 0,1,\dots,n-1 \pmod{n},$$

where $T_i$ are countable index sets. From Lemma 13 it follows that $a_i \vee a_j$ exists in $\mathcal{P}$ for every $i, j \in \{0, 1, \ldots, n-1\}$. It is not difficult to see that

$$v_i \vee v_{i+1} = v_i \oplus v_{i+1}, \quad v_i \vee v_{i+2} = (v_{i+1} \oplus w)^{\perp}, \quad v_i \vee v_{i+m} = w^{\perp},$$

where $m = 3, 4, \ldots, n-3$ and $i = 0, 1, \ldots, n-1 \pmod{n}$.

Let $x \in \mathcal{A}_i^*$, $y \in \mathcal{A}_{i+1}^*$, $i \in \{0, 1, \ldots, n-1\} \pmod{n}$, $x = x_{v_i} \oplus x_{a_i} \oplus x_{v_{i+1}} \oplus x_w$, $y = y_{v_{i+1}} \oplus y_{a_{i+1}} \oplus y_{v_{i+2}} \oplus y_w$, such that $x_{v_i} \leq v_i$, $x_{a_i} \leq a_i$, $x_{v_{i+1}} \leq v_{i+1}$, $x_w \leq w$, $y_{v_{i+1}} \leq v_{i+1}$, $y_{a_{i+1}} \leq a_{i+1}$, $y_{v_{i+2}} \leq v_{i+2}$, $y_w \leq w$. Surely $x = x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} \vee x_w$ and $y = y_{v_{i+1}} \vee y_{a_{i+1}} \vee y_{v_{i+2}} \vee y_w$. If $x_{v_i} \vee x_{a_i} = 0_{\mathcal{P}}$ ($y_{a_{i+1}} \vee y_{v_{i+2}} = 0_{\mathcal{P}}$) then $x \in \mathcal{A}_{i+1}^*$ ($y \in \mathcal{A}_i^*$), so $x \vee y$ exists in $\mathcal{P}$.

If $x_{v_i} \vee x_{a_i} > 0_{\mathcal{P}}$ ($y_{a_{i+1}} \vee y_{v_{i+2}} > 0_{\mathcal{P}}$) then $(x_{v_i} \vee x_{a_i}) \vee (y_{a_{i+1}} \vee y_{v_{i+2}}) = (v_{i+1} \oplus w)^{\perp}$ and thus

$$x \vee y = (v_{i+1} \oplus w)^{\perp} \oplus \big((x_{v_{i+1}} \vee y_{v_{i+1}}) \vee (x_w \vee y_w)\big).$$

Let $x \in \mathcal{A}_i^*$, $y \in \mathcal{A}_{i+r}^*$, $r = 2, 3, \ldots, n-2$, $y = y_{v_{i+r}} \oplus y_{a_{i+r}} \oplus y_{v_{i+r+1}} \oplus y_w$, $y_{v_{i+r}} \leq v_{i+r}$, $y_{a_{i+r}} \leq a_{i+r}$, $y_{v_{i+r+1}} \leq v_{i+r+1}$, $y_w \leq w$, where all indices are assumed modulo $n$. If $x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} = 0_{\mathcal{P}}$ ($y_{v_{i+r}} \vee y_{a_{i+r}} \vee y_{v_{i+r+1}} = 0_{\mathcal{P}}$) then $x = x_w \in \mathcal{A}_{i+r}^*$ ($y = y_w \in \mathcal{A}_i^*$), so $x \vee y$ exists in $\mathcal{P}$. If $x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}} > 0_{\mathcal{P}}$ and $y_{v_{i+r}} \vee y_{a_{i+r}} \vee y_{v_{i+r+1}} > 0_{\mathcal{P}}$ then $(x_{v_i} \vee x_{a_i} \vee x_{v_{i+1}}) \vee (y_{v_{i+r}} \vee y_{a_{i+r}} \vee y_{v_{i+r+1}}) = w^{\perp}$, therefore $x \vee y = w^{\perp} \oplus (x_w \vee y_w)$. □

## REFERENCES


[1] CHANG, C. C.: *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[2] CHOVANEC, F.—KÔPKA, F.: *Boolean D-posets*, Tatra Mt. Math. Publ. **10** (1997), 183–197.

[3] CHOVANEC, F.—KÔPKA, F.: *D-posets*. In: Handbook of Quantum Logic and Quantum Structures: Quantum Structures (K. Engesser, D. M. Gabbay, D. Lehmanm, eds.), Elsevier, Amsterdam, 2007, pp. 367–428.

[4] CHOVANEC, F.—JUREČKOVÁ, M.: *MV-algebra pastings*, Internat. J. Theoret. Phys. **42** (2003), 1913–1926.

[5] DICHTL, M.: *Astroids and pastings*, Algebra Universalis **18** (1984), 380–385.

[6] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Acad. Publishers/Ister Sciences, Dordrecht/Bratislava, 2000.

[7] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[8] FOULIS, D. J.—GEECHIE, R. J.—RÜTTIMANN, G. T.: *Filters and supports in orthoalgebras*, Internat. J. Theoret. Phys. **31** (1992), 789–807.

[9] GREECHIE, R. J.: *Orthomodular lattices admitting no states*, J. Combin. Theory Ser. A **10** (1971), 119–132.

[10] JENČA, G.: *Blocs of homogeneous effect algebras*, Bull. Austral. Math. Soc. **64** (2001), 81–98.

[11] JENČA, G.: *The block structure of complete lattice-ordered effect algebras*, J. Aust. Math. Soc. **83** (2007), 181–216.
[12] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.
[13] NAVARA, M.: *State spaces of orthomodular structures*, Rend. Istit. Mat. Univ. Trieste **31** (2000), 143–201.
[14] NAVARA, M.: *Constructions of quantum structures*. In: Handbook of Quantum Logic and Quantum Structures: Quantum Structures (K. Engesser, D. M. Gabbay, D. Lehmanm, eds.), Elsevier, Amsterdam, 2007, pp. 335–366.
[15] NAVARA, M.—ROGALEWICZ, V.: *The pasting constructions for orthomodular posets*, Math. Nachr. **154** (1991), 157–168.
[16] PTÁK, P.—PULMANNOVÁ, S.: *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publ., Dordrecht-Boston-London, 1991.
[17] RIEČANOVÁ, Z.: *Generalization of blocks for D-lattices and lattice-ordered effect algebras*, Internat. J. Theoret. Phys. **39** (2000), 231–237.
[18] RIEČANOVÁ, Z.: *Pasting of MV-effect algebras*, Internat. J. Theoret. Phys. **43** (2004), 1875–1883.
[19] XIE, Y.—LI, Y.—YANG, A.: *The pasting constructions of lattice ordered effect algebras*, Inform. Sci. **180** (2010), 2476–2486.

Received 9. 2. 2010
Accepted 17. 4. 2012

*Department of Informatics*
*Armed Forces Academy*
*of General M. R. Štefánik*
*Demänová 393*
*Liptovský Mikuláš*
*SLOVAK REPUBLIC*

*Mathematical Institute*
*Slovak Academy of Sciences*
*Štefánikova 49*
*Bratislava*
*SLOVAK REPUBLIC*

*E-mail*: ferdinand.chovanec@aos.sk