VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0100-3
Math. Slovaca **63** (2013), No. 2, 321–330

ON (STRONG) α-FAVORABILITY OF THE VIETORIS HYPERSPACE

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(*Communicated by Ľubica Holá*)

ABSTRACT. For a normal space X, α (i.e. the nonempty player) having a winning strategy (resp. winning tactic) in the strong Choquet game $Ch(X)$ played on X is equivalent to α having a winning strategy (resp. winning tactic) in the strong Choquet game played on the hyperspace $CL(X)$ of nonempty closed subsets endowed with the Vietoris topology τ_V. It is shown that for a non-normal X where α has a winning strategy (resp. winning tactic) in $Ch(X)$, α may or may not have a winning strategy (resp. winning tactic) in the strong Choquet game played on the Vietoris hyperspace. If X is quasi-regular, then having a winning strategy (resp. winning tactic) for α in the Banach-Mazur game $BM(X)$ played on X is sufficient for α having a winning strategy (resp. winning tactic) in $BM(CL(X), \tau_V)$, but not necessary, not even for a separable metric X. In the absence of quasi-regularity of a space X where α has a winning strategy in $BM(X)$, α may or may not have a winning strategy in the Banach-Mazur game played on the Vietoris hyperspace.

1. Introduction

Various completeness properties of the *Vietoris* topology τ_V on the hyperspace $CL(X)$ of nonempty closed subsets of a T_1 space X are well-established. For example, the strongest completeness properties of compactness, resp. complete metrizability of $CL(X)$ are characterized through compactness, resp. metrizable compactness of X [Mi]. On the other end, for the weakest completeness property of Baireness, it is known that Baireness of X is necessary, and in case, say, of 2nd countable X, also sufficient for Baireness of $(CL(X), \tau_V)$ (see [Mc],

2010 Mathematics Subject Classification: Primary 54B20; Secondary 54E52.
Keywords: Vietoris topology, Banach-Mazur game, strong Choquet game, (strongly) (weakly) α-favorable space, Baire space, Tychonoff plank, Bernstein set.

and more recently [CT]). Motivated by the fact that these, and other completeness properties can be viewed through various topological games, in particular the so-called Banach-Mazur and strong Choquet game, respectively, we study the effect of these two games on the hyperspace from the nonempty player's (i.e. α's) point of view (see section 3 for definitions, and basic results about these games). It is worth mentioning, that similar questions have been recently investigated for other hypertopologies, e.g. the so-called *Wijsman topology* in [CJ], [Zs], [PZ]

2. Preliminaries

Given a T_1 topological space X, denote by $CL(X)$ the set of all nonempty closed subsets of X. For any $S \subseteq X$ put

$$S^- = \{A \in CL(X) : \ A \cap S \neq \emptyset\} \qquad \text{and} \qquad S^+ = \{A \in CL(X) : \ A \subseteq S\}.$$

The *Vietoris topology* [Mi] τ_V on $CL(X)$ has subbase elements of the form U^- and U^+, where $\emptyset \neq U \subseteq X$ is open; thus, a base $\mathcal{B}_V$ for $(CL(X), \tau_V)$ consists of the sets

$$\langle U_0, \dots, U_n \rangle = \bigcap_{i \le n} U_i^- \cap \Big(\bigcup_{i \le n} U_i \Big)^+,$$

where $U_0, \dots, U_n$ are nonempty open subsets of X, $n < \omega$; we will also use the notation $\langle \mathcal{U} \rangle$ where $\mathcal{U} = \{U_0, \dots, U_n\}$. If X is Hausdorff and we require the U_i's to be pairwise disjoint, we get a π-base $\mathcal{P}_V$ for τ_V.

The next result is well-known (cf. [Mi: Lemma 2.3.1]):

Lemma 2.1. *The following are equivalent:*

(i) $\langle U_0, \dots, U_m \rangle \subseteq \langle V_0, \dots, V_n \rangle$;

(ii) $\bigcup_{i \le m} U_i \subseteq \bigcup_{j \le n} V_j$ *and* $\forall j \le n \ \exists i \le m : \quad U_i \subseteq V_j$.

3. Topological games

Let X be a topological space, and $\mathcal{P}$ a fixed π-base for X. The *Banach-Mazur game $BM(X)$* is played as follows: players β, and α alternate in choosing elements of $\mathcal{P}$, with β choosing first, so that

$$B_0 \supseteq A_0 \supseteq B_1 \supseteq A_1 \supseteq \dots \supseteq B_n \supseteq A_n \supseteq \dots$$

Then $B_0, A_0, \dots, B_n, A_n, \dots$ is a *play* (or *run*) in $BM(X)$, and α wins this play if

$$\bigcap_{n\in\omega} A_n \Big(= \bigcap_{n\in\omega} B_n\Big) \neq \emptyset,$$

otherwise, β wins. A *strategy* in $BM(X)$ is a function $\sigma\colon \mathcal{P}^{<\omega} \to \mathcal{P}$ such that

$$\sigma(W_0, \dots, W_n) \subseteq W_n$$

for all $n \in \omega$, and $(W_0, \dots, W_n) \in \mathcal{P}^{n+1}$. A *tactic* in $BM(X)$ is a function $\tau\colon \mathcal{P} \to \mathcal{P}$ such that $\tau(W) \subseteq W$ for all $W \in \mathcal{P}$. A *winning strategy* (resp. *winning tactic*) for α is a strategy (tactic) σ such that α wins every play of $BM(X)$ *compatible with* σ, i.e. such that $\sigma(B_0, \dots, B_n) = A_n$ (resp. $\sigma(B_n) = A_n$) for all $n \in \omega$. A *winning strategy* (resp. *winning tactic*) for β is defined analogously. The space X is called *(weakly) α-favorable*, if α has a winning tactic (resp. winning strategy) in $BM(X)$ (these are distinct properties in general [De1, De2]). The space X is called *β-favorable*, if β has a winning strategy in $BM(X)$ (this is equivalent to β having a winning tactic in $BM(X)$ [GT]). It is known that a metrizable X is (weakly) α-favorable iff X contains a dense completely metrizable subspace (see [Wh, GT] for generalizations of this characterization); on the other side, a topological space X is not β-favorable iff X is a *Baire space* (i.e. each sequence of dense open subsets of X intersects in a dense subset of X [HMC, Ke]).

Let $\mathcal{B}$ be a base for X. Denote

$$\mathcal{E} = \mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \big\{(x, U) \in X \times \mathcal{B} :\ x \in U\big\}.$$

The *strong Choquet game* $Ch(X)$ is played similarly to the Banach-Mazur game, but in addition to the open B, β also chooses a point $x \in B$. More precisely, players β and α alternate in choosing $(x_n, B_n) \in \mathcal{E}$ and $A_n \in \mathcal{B}$, respectively, with β choosing first, so that for each $n \in \omega$, $x_n \in A_n \subseteq B_n$, and $B_{n+1} \subseteq A_n$. The play

$$(x_0, B_0), A_0, \dots, (x_n, B_n), A_n, \dots$$

is won by α, if $\bigcap_{n\in\omega} A_n (= \bigcap_{n\in\omega} B_n) \neq \emptyset$; otherwise, β wins.

A *strategy* in $Ch(X)$ for α (resp. β) is a function $\sigma\colon \mathcal{E}^{<\omega} \to \mathcal{B}$ (resp. $\sigma\colon \mathcal{B}^{<\omega} \to \mathcal{E}$) such that $x_n \in \sigma((x_0, B_0), \dots, (x_n, B_n)) \subseteq B_n$ for all $((x_0, B_0), \dots, (x_n, B_n)) \in \mathcal{E}^{<\omega}$ (resp. $\sigma(\emptyset) = (x_0, B_0)$, $B_n \subseteq A_{n-1}$, where $\sigma(A_0, \dots, A_{n-1}) = (x_n, B_n)$, for all $(A_0, \dots, A_{n-1}) \in \mathcal{B}^n$, $n \geq 1$). A strategy σ for α (resp. β) is a *winning strategy*, if α (resp. β) wins every run of $Ch(X)$ compatible with σ, i.e. such that $\sigma((x_0, B_0), \dots, (x_n, B_n)) = A_n$ for all $n \in \omega$ (resp. $\sigma(\emptyset) = (x_0, B_0)$ and $\sigma(A_0, \dots, A_{n-1}) = (x_n, B_n)$ for all $n \geq 1$). A *tactic* in $Ch(X)$ for α is a function $t\colon \mathcal{E} \to \mathcal{B}$ such that $x \in t(x, B) \subseteq B$ for all $(x, B) \in \mathcal{E}$; t is a *winning tactic*, if α wins every run of $Ch(X)$ compatible with t, i.e. such that $t(x_n, B_n) = A_n$ for all $n \in \omega$. We could analogously define a winning tactic for β, however,it is known that having a winning tactic or strategy for β in $Ch(X)$ are equivalent [GT]. This is not the case for α though [De1, De2], so the following definitions make sense: X is *strongly α-favorable* [Te] (resp. *strongly Choquet* — see [Ke]), provided α has a winning tactic (resp. winning strategy) in $Ch(X)$. A metrizable space is strongly α-favorable (strongly Choquet) iff it is completely metrizable [Ke]. Also note that Čech-complete (hence locally compact) spaces are strongly α-favorable [Po].

4. Strong α-favorability of the Vietoris topology

We will say that X is *ω-normal*, provided for any open $U \subseteq X$ and a nonempty closed $A \subseteq U$ there is an open V with $A \subseteq V \subseteq U$ such that for any countable $C \subseteq V$ we have $\overline{C} \subseteq U$. Using ω-normality instead of normality, we can generalize [Zs: Theorem 5.1] about strong α-favorability of the Vietoris hyperspace as follows:

Theorem 4.1.

(i) *If X is an ω-normal strongly Choquet (strongly α-favorable) space, then $(CL(X), \tau_V)$ is strongly Choquet (strongly α-favorable).*

(ii) *If $(CL(X), \tau_V)$ is strongly Choquet (strongly α-favorable), then X is strongly Choquet (strongly α-favorable).*

Proof.

(i) We will provide the proof only for strong Choquetness, since strong α-favorability is analogous (cf. [Zs: Theorem 5.1]). Let σ be a winning strategy for α in $Ch(X)$. We will inductively define a winning strategy σ_V for α in $Ch(CL(X), \mathcal{B}_V)$ as follows: if $(F_i, \mathbf{V}_i) \in \mathcal{E}(CL(X), \mathcal{B}_V)$ for all $i \le k$, where $\mathbf{V}_i = \langle V_{i,0}, \dots, V_{i,n_i} \rangle$, and $\mathbf{V}_0 \supseteq \dots \supseteq \mathbf{V}_k$, then by Theorem 2.1, we can assume that (n_i) is strictly increasing and for all $i \le k$ and $n_{i-1} < j \le n_i$ (put $n_{-1} = -1$), $V_{i,j} \supseteq \dots \supseteq V_{k,j}$. Also, $F_i \in \mathbf{V}_i$, so for each $i \le k$ and $j \le n_i$ we can find $x_{i,j} \in F_i \cap V_{i,j}$, and since $F_k \subseteq \bigcup_{j \le n_k} V_{k,j}$ and X is ω-normal, we can find an open W_k with $F_k \subseteq W_k \subseteq \bigcup_{j \le n_k} V_{k,j}$ such that for any countable $C \subseteq W_k$, $\overline{C} \subseteq \bigcup_{j \le n_k} V_{k,j}$. Finally, define

$$\sigma_V((F_0, \mathbf{V}_0), \dots, (F_k, \mathbf{V}_k)) = W_k^+ \cap \bigcap_{i \le k} \bigcap_{n_{i-1} < j \le n_i} \sigma((x_{i,j}, V_{i,j}), \dots, (x_{k,j}, V_{k,j}))^-.$$

If $(F_0, \mathbf{V}_0), \sigma_V(F_0, \mathbf{V}_0), \dots, (F_k, \mathbf{V}_k), \sigma_V((F_0, \mathbf{V}_0), \dots, (F_k, \mathbf{V}_k)), \dots$ is a run of the strong Choquet game in $(CL(X), \tau_V)$, then for each $i \in \omega$ and $n_{i-1} < j \le n_i$, there exists some $a_j \in \bigcap_{k \ge i} V_{k,j}$, and $A = \overline{\{a_i : i < \omega\}} \subseteq \bigcup_{j \le n_k} V_{k,j}$; thus, $A \in \bigcap_k \mathbf{V}_k$.

(ii) Let σ_V be a winning strategy for α in $Ch(CL(X), \mathcal{B}_V)$. Given $k < \omega$ and $(x_i, V_i) \in \mathcal{E}(X)$ for all $i \leq k$, let

$$\sigma_V\big((\{x_0\}, V_0^+), \ldots, (\{x_k\}, V_k^+)\big) = \langle W_{k,0}, \ldots, W_{k,n_k}\rangle.$$

Then a winning strategy σ for α in $Ch(X)$ can be defined via

$$\sigma\big((x_0, V_0), \ldots, (x_k, V_k)\big) = \bigcap_{i \leq n_k} W_{k,i}.$$

Indeed, if $(x_0, V_0), U_0, \ldots, (x_k, V_k), U_k, \ldots$ is a run of $Ch(X)$ compatible with σ, then $x_{k+1} \in V_{k+1} \subseteq U_k = \bigcap_{i \leq n_k} W_{k,i}$, and clearly $\{x_{k+1}\} \in V_{k+1}^+ \subseteq \langle W_{k,0}, \ldots$ $\ldots, W_{k,n_k}\rangle$; thus,

$$(\{x_0\}, V_0^+), \sigma_V(\{x_0\}, V_0^+), \ldots, (\{x_k\}, V_k^+), \sigma_V((\{x_0\}, V_0^+), \ldots, (\{x_k\}, V_k^+)), \ldots$$

is a run of $Ch(CL(X), \mathcal{B}_V)$ compatible with σ_V, so there is some $F \in \bigcap_{k<\omega} V_k^+$, whence, $\bigcap_{k<\omega} V_k \neq \emptyset$. The proof for strong α-favorability is similar. □

The next example will show that the previous theorem is truly a generalization of [Zs: Theorem 5.1], since there are non-normal, ω-normal, strongly Choquet (strongly α-favorable) spaces:

Example 4.2. There exists a Hausdorff, ω-normal, non-quasi-regular, strongly Choquet space.

P r o o f. Let $X = \mathbb{R}$ be endowed with the topology having

$$\mathcal{B} = \{I \setminus C : \ I \subseteq \mathbb{R} \text{ bounded open interval}, \ C \subset \mathbb{R} \text{ countable}\}$$

as its base. Then X is clearly T_2, but not quasi-regular, since if $I \setminus C \in \mathcal{B}$ is such that $I \setminus C \subseteq \mathbb{R} \setminus \mathbb{Q}$, then $\overline{I \setminus C} = \overline{I} \nsubseteq \mathbb{R} \setminus \mathbb{Q}$. Also, by [HZ: Example 2.7], X is strongly Choquet, and, since the countable subsets of X are closed, X is ω-normal. □

It is known that the above example is not strongly α-favorable (not even α-favorable — cf. [De1]), however, we have the following:

Example 4.3. There exists a Tychonoff, ω-normal, non-normal, strongly α-favorable space.

P r o o f. Consider $X_0 = \omega_2 + 1 \setminus \{x < \omega_2 : \ \mathrm{cf}(x) = \omega\}$ and $X_1 = \omega_2 \setminus \{x < \omega_2 : \mathrm{cf}(x) = \omega\}$, both with the order topology, and define $X = X_0 \times X_1$. It is not hard to adjust the proof of non-normality of the Tychonoff square to our case.

Notice that X_0 is strongly α-favorable, since putting $t(x, U) = U$ for all $(x, U) \in \mathcal{E}(X_0)$ where $U = X_0 \cap (a, x]$, we get a winning tactic for α in $Ch(X_0)$. Analogously, X_1 is strongly α-favorable, and so is X, since strong α-favorability is productive. As countable subsets of X do not cluster in X, they are closed, and hence, X is ω-normal. $\square$

In the next example we will show that in Theorem 4.1(i) ω-normality is essential:

Example 4.4. There exists a non-ω-normal locally compact space X such that $Ch(CL(X), \tau_V)$ is β-favorable.

Proof. If $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ is the Tychonoff plank, then X is locally compact; but X is not ω-normal: indeed, let $U = (\omega_1 + 1) \times \omega$ and $A = \{\omega_1\} \times \omega$. Then for any X-open subset V of U containing A there is a $\lambda < \omega_1$ with $[\lambda, \omega_1] \times \omega \subseteq V$, and for the countable set $C = \{\lambda\} \times \omega \subseteq V$ we have $(\lambda, \omega) \in \overline{C} \setminus U$.

Inductively define a winning strategy σ_V for β in $Ch(CL(X), \tau_V)$ as follows: let β's initial choice be $\sigma_V(\emptyset) = (F_0, \mathbf{V}_0)$, where

$$F_0 = \big\{(\delta_0, 0)\big\} \cup \big\{(\omega_1, \lambda) : \ 1 \le \lambda < \omega\big\} \quad \text{and} \quad \mathbf{V}_0 = ((\omega_1 + 1) \times \omega)^+ \cap \{(\delta_0, 0)\}^-$$

for a fixed successor $\delta_0 < \omega_1$. Assume that σ_V has been defined for initial plays of length $k - 1$ so that $\sigma_V(\mathbf{U}_0, \dots, \mathbf{U}_{k-1}) = (F_k, \mathbf{V}_k)$, where

$$F_k = \big\{(\delta_i, i) : \ i \le k\big\} \cup \big\{(\omega_1, \lambda) : \ k + 1 \le \lambda < \omega\big\}$$

and

$$\mathbf{V}_k = \mathbf{U}_{k-1} \cap \bigcap_{i \le k} \{(\delta_i, i)\}^-$$

for some successors $\delta_0, \dots, \delta_k < \omega_1$. Consider $\mathbf{U}_k = \langle \mathcal{U}_k \rangle$ such that $F_k \in \mathbf{U}_k \subseteq \mathbf{V}_k$. Then $(\omega_1, k{+}1) \in F_k \subseteq \bigcup \mathcal{U}_k$, so there is some $\gamma < \omega_1$ with $(\gamma, \omega_1] \times \{k{+}1\} \subseteq \bigcup \mathcal{U}_k$ and we can choose a successor $\delta_{k+1} \in (\gamma, \omega_1]$. Then $(\delta_{k+1}, k + 1)$ is an isolated point in X and we can define $F_{k+1} = \big\{(\delta_i, i) : \ i \le k + 1\big\} \cup \big\{(\omega_1, \lambda) : k + 2 \le \lambda < \omega\big\}$, $\mathbf{V}_{k+1} = \mathbf{U}_k \cap \{(\delta_{k+1}, k + 1)\}^-$ and put $\sigma(\mathbf{U}_0, \dots, \mathbf{U}_k) = (F_{k+1}, \mathbf{V}_{k+1})$.

Now, if $(F_0, \mathbf{V}_0), \dots, \mathbf{U}_k, (F_{k+1}, \mathbf{V}_{k+1}), \dots$ is a play of the strong Choquet game on $(CL(X), \tau_V)$ compatible with σ_V, assume there exists $A \in \bigcap_{k<\omega} \mathbf{V}_k$. Then $A \supseteq \big\{(\delta_k, k) : \ k < \omega\big\}$, which is not closed in X (since $\delta = \sup\{\delta_k : \ k < \omega\} < \omega_1$, and $(\delta, \omega) \in \omega_1 \times \{\omega\} \setminus A$ is a limit point of A), and $\overline{A} \notin \mathbf{V}_0$. This implies that $\bigcap_{k<\omega} \mathbf{V}_k$ is empty; hence, β wins in $Ch(CL(X), \tau_V)$. $\square$

Problem 4.5. *Is ω-normality of X necessary for strong Choquetness (strong α-favorability) of the Vietoris hyperspace $(CL(X), \tau_V)$?*

5. α-favorability of the Vietoris topology

We will say that X is *ω-quasi-regular*, provided for any nonempty open $U \subseteq X$ there is a nonempty open $V \subseteq U$ such that $\overline{C} \subseteq U$ for any countable $C \subseteq V$. We now turn to investigating (weak) α-favorability of the hyperspace:

Theorem 5.1. *If X is an ω-quasi-regular, (weakly) α-favorable space, then $(CL(X), \tau_V)$ is (weakly) α-favorable.*

Proof. We prove the theorem only for α-favorability, weak α-favorability is analogous. Let t be a winning tactic for α in $BM(X)$. For each nonempty open $U \subseteq X$ fix a nonempty open $\widetilde{U}$ such that $\overline{C} \subseteq U$ for any countable $C \subseteq \widetilde{U}$. Whenever $\langle U_0, \dots, U_n \rangle \in \mathcal{B}_V$, define the tactic t_V for α in $BM(CL(X), \tau_V)$ via $t_V(\langle U_0, \dots, U_n \rangle) = \langle t(\widetilde{U}_0), \dots, t(\widetilde{U}_n) \rangle$. Then t_V is a winning tactic for α, since if $\mathbf{V}_0, t_V(\mathbf{V}_0), \dots, \mathbf{V}_k, t_V(\mathbf{V}_k), \dots$ is a run of $BM(CL(X), \tau_V)$ compatible with t_V, where $\mathbf{V}_k = \langle V_{k,0}, \dots, V_{k,n_k} \rangle$, by Theorem 2.1 we can assume the (n_k) is an increasing sequence of positive integers, and given a $k < \omega$, $V_{k+1,i} \subseteq t(\widetilde{V}_{k,i})$ for all $i \leq n_k$ and $\bigcup_{i \leq n_{k+1}} V_{k+1,i} \subseteq \bigcup_{i \leq n_k} V_{k,i}$; moreover, $\overline{C} \subseteq \bigcup_{i \leq n_k} V_{k,i}$ for each countable $C \subseteq \bigcup_{i \leq n_{k+1}} V_{k+1,i}$. Since t is a winning tactic for α, for every $k < \omega$ and $n_{k-1} < i \leq n_k$ (where $n_{-1} = -1$), there exists some $a_i \in \bigcap_{k \geq i} V_{k,i}$. Also, $A = \overline{\{a_i : \ i \leq \omega\}} \subseteq \bigcup_{i \leq n_k} V_{k,i}$ for any $k < \omega$, so $A \in \bigcap_k \mathbf{V}_k$. □

Note that the above theorem has been established for quasi-regular base spaces in [Zs: Theorem 4.3]; our result is more general however, since the space from Example 4.2 is non-quasi-regular, but clearly ω-quasi-regular and weakly α-favorable.

We will now give an example to show that in the absence of ω-quasi-regularity, (weak) α-favorability of X does not always guarantee (weak) α-favorability of $(CL(X), \tau_V)$:

Example 5.2. There exists a non-ω-quasi-regular, strongly α-favorable space X such that $BM(X)$ is β-favorable.

Proof. The space X is the closed unit interval $I = [0, 1]$ with an extra point ∞. The topology on I is as usual, and the open base at ∞ consists of the union of $\{\infty\}$ and the co-finite subsets of I. Then X is not ω-quasi-regular, because the X-closure of any infinite subset of I contains ∞; moreover, it is strongly α-favorable, since the following tactic is winning for α in $Ch(X)$: given $(x, V) \in \mathcal{E}(X)$ with $x \neq \infty$, let α choose an open $U \subseteq I$ containing x such that $\overline{U}^I \subseteq V$, and, if $x = \infty$, let α choose U.

Define a winning tactic t_V for β in $BM(CL(X))$ as follows: put $t_V(\emptyset) = I^+$, and for $\langle V_0, \dots, V_n\rangle \in \mathcal{P}_V$, put $t_V(\langle V_0, \dots, V_n\rangle) = \langle U_{0,0}, U_{0,1}, \dots, U_{n,0}, U_{n,1}\rangle$, where $U_{i,0}, U_{i,1}$ are disjoint open intervals such that $\overline{U_{i,0} \cup U_{i,1}}^I \subseteq V_i$ for each $i < n$. Now assume that there is a run of $BM(CL(X))$ compatible with t_V which is not won by β, i.e. there is some X-closed set A in the final intersection of this run. Then A must have cardinality continuum; thus, $\infty \in A$, which cannot be, since $A \subset I$. □

Remark 5.3. Note that the previous example, in a sense, is stronger than Example 4.4, since if $BM(CL(X))$ is β-favorable, so is $Ch(CL(X))$; however, the Tychonoff plank in Example 4.4 is $T_{3\frac{1}{2}}$, while the previous example is only T_1.

Recall that $X \subset \mathbb{R}$ is a *Bernstein set*, provided both X and $\mathbb{R} \setminus X$ intersect each dense-in-itself G_δ-subset of $\mathbb{R}$ [HMC]. It is well-known, that a Bernstein set is neither α-favorable, nor β-favorable [HMC].

Remark 5.4. The following argument gives a direct proof of this undecidedness of $BM(X)$ for the Bernstein set X: assume first, that α has a winning strategy σ on X. For each nonempty bounded open interval U choose two disjoint open intervals V_U^0 and V_U^1 such that $\overline{V_U^j} \subset U$ for all $j = 0, 1$. Each $\gamma \in 2^\omega$ defines a strategy σ_γ for β on X as follows: let $\sigma_\gamma(\emptyset) = X \cap (0, 1)$ and

$$\sigma_\gamma(U_0, \dots, U_n) = X \cap V_{U_n'}^{\gamma(n)},$$

where for an X-open U, U' denotes the $\mathbb{R}$-open set for which $U' \cap X = U$. Consider all σ-compatible runs, where β follows one of these 2^ω-many strategies σ_γ. Since σ is a winning strategy for α, α wins all these runs; thus, X contains a closed dense-in-itself subset, a contradiction.

If we interchange α with β in the above argument, then the σ-compatible runs as viewed from $\mathbb{R}$ still have a nonempty intersection for each α-strategy σ_γ, but they are empty in X, since now σ is a winning strategy for β in $BM(X)$. This implies that $\mathbb{R} \setminus X$ contains a closed dense-in-itself subset, a contradiction. □

In our last result, we will show that, unlike strong α-favorability or strong Choquetness (see Theorem 4.1(ii)), (weak) α-favorability of X is not necessary for (weak) α-favorability of $(CL(X), \tau_V)$:

Example 5.5. There exists a metrizable non-weakly α-favorable space X with an α-favorable hyperspace $(CL(X), \tau_V)$.

P r o o f. Let $X \subset \mathbb{R}$ be a Bernstein set. In what follows, $\overline{A}$ will stand for the closure of $A \subseteq \mathbb{R}$ in $\mathbb{R}$. To define a winning tactic t_V for α on $(CL(X), \tau_V)$, first let $\mathcal{B}'$ be the collection of bounded open intervals in $\mathbb{R}$, and $\mathcal{B} = \{V' \cap X : V' \in \mathcal{B}'\}$. Then for any $V = V' \cap X \in \mathcal{B}$, $V' = \operatorname{int}(\overline{V})$ and for $U, V \in \mathcal{B}$, $U \subseteq V$ iff $U' \subseteq V'$.

If $\{V_0, \dots, V_n\}$ is a pairwise disjoint collection of elements of $\mathcal{B}$, then $\langle V_0, \dots \dots, V_n\rangle \in \mathcal{P}_V$. For each $i \le n$, define disjoint $U'_{i,0}, U'_{i,1} \in \mathcal{B}'$ such that

$$\overline{U'_{i,0} \cup U'_{i,1}} \subseteq V'_i. \qquad (*)$$

Put $t_V(\langle V_0, \dots, V_n\rangle) = \langle U_{0,0}, U_{0,1}, \dots, U_{n,0}, U_{n,1}\rangle$.

Let $\mathbf{V}_0, t_V(\mathbf{V}_0), \dots, \mathbf{V}_k, t_V(\mathbf{V}_k), \dots$ be a run of $BM(CL(X), \tau_V)$ compatible with t_V. If $\mathbf{V}_k = \langle V_0^k, \dots, V_{n_k}^k\rangle$, then $t_V(\mathbf{V}_k) = \langle U_{0,0}^k, U_{0,1}^k, \dots, U_{n_k,0}^k, U_{n_k,1}^k\rangle$ for some strictly increasing sequence (n_k) of positive integers $(n_{k+1} \ge 2n_k)$.

Then $F' = \bigcap_{k<\omega} \overline{\bigcup_{i \le n_k} (V_i^k)'}$ is a nonempty compact subset of $\mathbb{R}$; moreover, in view of $(*)$, F' is dense-in-itself, and consequently $F = F' \cap X \in CL(X)$. Furthermore by $(*)$, $\overline{(V_i^k)'}$ is dense-in-itself for each k, i, so $F \cap V_i^k = F' \cap (V_i^k)' \cap X \ne \emptyset$, and clearly, $F' \subset \bigcup_{i \le n_k} V_i^k$; thus, $F' \in \mathbf{V}_k$ for each k, and α wins the run. □

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Received 29. 9. 2010
Accepted 7. 3. 2011

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