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OSCILLATION CRITERIA FOR A CLASS OF NONLINEAR FOURTH ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, sufficient conditions are obtained for oscillation of a class of nonlinear fourth order mixed neutral differential equations of the form

$$\left(\frac{1}{a(t)}\left(\left(y(t)+p(t)y(t-\tau)\right)''\right)^{\alpha}\right)'' = q(t)f(y(t-\sigma_1)) + r(t)g(y(t+\sigma_2)) \qquad \text{(E)}$$

under the assumption

$$\int_0^{\infty} \left(a(t)\right)^{\frac{1}{\alpha}}\,\mathrm{d}t = \infty.$$

where α is a ratio of odd positive integers. (E) is studied for various ranges of $p(t)$.

1. Introduction

Interest in Functional differential/difference equations is growing due to the development in science and technology and the challenges that the new classes of such equations provide in these application areas. Equations involving delay and those involving advance and a combination of both arise in nerve conduction (Life Sciences), organizational behaviour (Social sciences), signal processing pantograph equations (mechanical engineering), to mention a few (see e.g. [2, 3, 6] and [12]). Study of such equations has been an active area of research of many researchers. In this work, an attempt is made to study the oscillatory behaviour of solutions of a class of nonlinear neutral differential equations of the form

$$\left(\frac{1}{a(t)}\left(\left(y(t)+p(t)y(t-\tau)\right)''\right)^{\alpha}\right)'' = q(t)f(y(t-\sigma_1)) + r(t)g(y(t+\sigma_2)), \quad (1.1)$$

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where $a \in C([0,\infty),(0,\infty))$, $p, f, g \in C([0,\infty),\mathcal{R})$, $q, r \in C([0,\infty),[0,\infty))$, f and g are nondecreasing with $uf(u) > 0$, $ug(u) > 0$, for $u \neq 0, \alpha$ is the quotient of odd positive integers, $\tau > 0$, $\sigma_1 > 0$ and $\sigma_2 \geq 0$ under the assumption

(A$_0$) $\int\limits_0^\infty (a(t))^{\frac{1}{\alpha}}\, \mathrm{d}t = \infty.$

In [10] and [11], Parhi and Tripathy have discussed the oscillatory and asymptotic behaviour of solutions of a class of fourth order nonlinear differential equations of the form

$$\big(r(t)\big(y(t) + p(t)y(t-\tau)\big)''\big)'' + q(t)G(y(t-\sigma)) = 0 \tag{1.2}$$

under the assumptions

$$\int\limits_0^\infty \frac{t}{r(t)}\,\mathrm{d}t = \infty, \qquad \int\limits_0^\infty \frac{t}{r(t)}\,\mathrm{d}t < \infty.$$

It is observed that, the associated forced equations

$$\big(r(t)\big(y(t) + p(t)y(t-\tau)\big)''\big)'' + q(t)G(y(t-\sigma)) = f(t)$$

is oscillatory but not the unforced equations (1.2), for various ranges of $p(t)$. This is happened due to the analysis incorporated there. However, the study of (1.2) to investigate the oscillatory results is still under progress. For recent contribution in this area we refer the reader to ([1,4,5,7,9,13]) and the references cited there in.

We may note that the study of oscillatory and asymptotic behaviour of solutions of (1.1) and the following equations

$$\left(\frac{1}{a(t)}\big((y(t) + p(t)y(t-\tau))''\big)^{\alpha}\right)'' = q(t)f(y(t-\sigma_1)) - r(t)g(y(t-\sigma_2)), \tag{1.3}$$

are more or less similar(e.g. refer to [4]). Hence (1.1) and (1.3) are comparable. Interestingly, Eq. (1.2) is a particular case of Eq. (1.3). But unlike the analysis in [10] and [11], the objective of this work is to establish new oscillation criteria for Eq. (1.1) under the assumption (A$_0$).

By a solution of (1.1) we understand a function $y \in C([-\rho,\infty),\mathcal{R})$ such that $(y(t)+p(t)y(t-\tau))$ is twice continuously differentiable, $\frac{1}{a(t)}((y(t)+p(t)y(t-\tau))'')^{\alpha}$ is twice continuously differentiable and (1.1) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \sigma_1\}$, and $\sup\big\{|y(t)| : \ t \geq t_0\big\} > 0$ for every $t_0 \geq 0$. A solution of (1.1) is said to be oscillatory if it has arbitrary large zeros; otherwise, it is called nonoscillatory.

2. Oscillation results

We define the operators:

$$\begin{aligned}
L_0 z(t) &= z(t),\\
L_1 z(t) &= \frac{\mathrm{d}}{\mathrm{d}t} L_0 z(t),\\
L_2 z(t) &= \frac{1}{a(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t} L_1 z(t) \right)^{\alpha},\\
L_3 z(t) &= \frac{\mathrm{d}}{\mathrm{d}t} L_2 z(t),\\
L_4 z(t) &= \frac{\mathrm{d}}{\mathrm{d}t} L_3 z(t),
\end{aligned}$$

where $z(t) = y(t) + p(t)y(t-\tau)$. Then Eq. (1.1) becomes

$$L_4 z(t) = q(t) f(y(t-\sigma_1)) + r(t) g(y(t+\sigma_2)). \tag{2.1}$$

For our use in the sequel, we assume the following conditions:

(A$_1$) $f(uv) \geq f(u)f(v)$, $g(uv) \geq g(u)g(v)$, for $u, v \in \mathcal{R}$ and $u, v > 0$,

(A$_2$) $f(-u) = -f(u)$, $g(-u) = -g(u)$, for $u \in \mathcal{R}$,

(A$_3$) there exist $\lambda > 0$, $\mu > 0$ such that $f(u) + f(v) \geq \lambda f(u+v)$, $g(u) + g(v) \geq \mu g(u+v)$, for $u, v \in \mathcal{R}$ and $u, v > 0$,

(A$_4$) $Q(t) = \min\big\{q(t), q(t-\tau)\big\}$, $R(t) = \min\big\{r(t), r(t-\tau)\big\}$, for $t \geq \tau$,

(A$_5$) $\frac{f\left(u^{\frac{1}{\alpha}}\right)}{u} \geq \beta_1 > 0$, $\frac{g\left(u^{\frac{1}{\alpha}}\right)}{u} \geq \beta_2 > 0$, $u \neq 0$,

(A$_6$) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} Q(\theta) f\big[B(\theta-\sigma_1, t_1)\big]\,\mathrm{d}\theta > \frac{1+f(b)}{\lambda\beta_1}$,

(A$_7$) $\limsup\limits_{s\to\infty} \int\limits_{s}^{s+\sigma_2} R(\theta) g\big[A(\theta+\sigma_2, s+\sigma_2)\big]\,\mathrm{d}\theta > \frac{1+g(b)}{\lambda\beta_2}$,

(A$_8$) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} Q(\theta) f\big[C(s-\sigma_1, \theta-\sigma_1)\big]\,\mathrm{d}\theta > \frac{1+f(b)}{\lambda\beta_1}$,

(A$_9$) $\limsup\limits_{s\to\infty} \int\limits_{s}^{s+\sigma_2} r(\theta) g[A(\theta+\sigma_2, s+\sigma_2)]\,\mathrm{d}\theta > \frac{1}{\beta_2}$,

(A$_{10}$) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} q(\theta) f[B(\theta-\sigma_1, t_1)]\,\mathrm{d}\theta > \frac{1}{\beta_1}$,

(A$_{11}$) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} q(\theta) f[C(s-\sigma_1, \theta-\sigma_1)]\,\mathrm{d}\theta > \frac{1}{\beta_1}$,

(A_{12}) $\limsup\limits_{s\to\infty} \int\limits_{s+\tau-\sigma}^{s} q(\theta) f[D(\theta+\tau-\sigma_1, v+\tau-\sigma_1)]\,\mathrm{d}\theta > \frac{1}{\beta_1 f(\frac{1}{b})}$, $b>0$, $\tau \le \sigma_1$,

(A_{13}) $\frac{f^{\frac{1}{\alpha}}(u)}{u} \ge \beta_3 > 0$, $\frac{g^{\frac{1}{\alpha}}(u)}{u} \ge \beta_4 > 0$, for $u \ne 0$,

(A_{14}) $\limsup\limits_{s\to\infty} \int\limits_{s}^{s+\sigma_2} \int\limits_{s}^{t_4} \Big[a(t_3) \int\limits_{s}^{t_3} \int\limits_{s}^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\Big]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 > \left(\frac{1+g(b)}{\mu}\right)^{\frac{1}{\alpha}} \frac{1}{\beta_4}$,
where $b>0$ and $t_4 > t_3 > t_2$,

(A_{15}) $\limsup\limits_{s\to\infty} \left(\int\limits_{s}^{\infty} Q(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} B[s-\tau, t_1] > \frac{1}{\beta_3}\left(\frac{1+f(b)}{\lambda}\right)^{\frac{1}{\alpha}}$,

(A_{16}) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} \int\limits_{t_4}^{s} \Big[a(t_3) \int\limits_{t_2}^{s} (t-t_3) Q(t)\,\mathrm{d}t\Big]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 > \left(\frac{1+f(b)}{\lambda}\right)^{\frac{1}{\alpha}} \frac{1}{\beta_3}$,

(A_{17}) $\limsup\limits_{s\to\infty} \int\limits_{s}^{s+\sigma_2} \int\limits_{s}^{t_4} [a(t_3) \int\limits_{s}^{t_3} \int\limits_{s}^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 > \frac{1}{\beta_4}$,

(A_{18}) $\limsup\limits_{s\to\infty} B(s-\sigma_1, t_1) \left[\int\limits_{s}^{\infty} q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}} > \frac{1}{\beta_3}$,

(A_{19}) $\limsup\limits_{s\to\infty} \int\limits_{s-\sigma_1}^{s} \int\limits_{t_4}^{s} \Big[a(t_3) \int\limits_{t_3}^{s} (t-t_3) q(t)\,\mathrm{d}t\Big]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 > \frac{1}{\beta_3}$,

(A_{20}) $\limsup\limits_{t\to\infty} D[t-\sigma_1, \theta-\sigma_1] \left(\int\limits_{t}^{s} q(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{\alpha}} > \frac{b}{\beta_3}$, $b>0$,

(A_{21}) $\int\limits_{0}^{\infty} q(t)\,\mathrm{d}t = +\infty$,

where

$$A[s,v] = \int_{v}^{s} (s-t)(t-v)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t,$$

$$B[s,t_1] = \int_{t_1}^{s} (s-t)^{1+\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t,$$

$$C[v,u] = \int_{u}^{v} (t-u)(v-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t,$$

$$D[s,\theta] = \int_{u}^{s} \int_{\theta}^{s} (s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t\,\mathrm{d}\theta = \int_{\theta}^{s} (t-u)(s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t.$$

Theorem 2.1. *Let $0 \le p(t) \le b < \infty$. If* (A_0)–(A_8) *hold, then* (1.1) *is oscillatory.*

Proof. If possible, let $y(t)$ be a non-oscillatory solution of (1.1), for $t \ge t_0$. Without loss of generality, we may assume that $y(t) > 0$, for $t \ge t_0$. Ultimately, it follows from (2.1) that $L_4 z(t) \ge 0$ eventually and hence $L_i z(t)$, $i = 1, 2, 3$ are eventually of one sign. In what follows, we shall consider the following three cases:

$$
\begin{array}{llll}
\text{(a)} & L_i z(t) > 0, \quad i = 1,2,3, & \text{for} & t \ge t_1 > t_0 + \rho,\\
\text{(b)} & L_i z(t) > 0, \quad i = 1,2, & \text{for} & t \ge t_1,\\
 & L_3 z(t) < 0, & \text{for} & t \ge t_1,\\
\text{(c)} & (-1)^i L_i z(t) > 0, \quad i = 1,2,3, & \text{for} & t \ge t_1.
\end{array}
$$

Case (a): Assume that $L_i z(t) > 0$, $i = 1, 2, 3$, for $t \ge t_1$. Then for $u \ge v > t_1$,

$$L_2 z(u) - L_2 z(v) = \int_v^u L_3 z(s)\,\mathrm{d}s \ge (u - v) L_3 z(v)$$

implies that $L_2 z(u) \ge (u - v) L_3 z(v)$, that is,

$$z''(u) \ge (u - v)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(u) L_3^{\frac{1}{\alpha}} z(v). \tag{2.2}$$

Applying Taylor's expansion to $z(t)$, we obtain

$$
\begin{aligned}
z(s) &= z(t_1) + (s - t_1) z'(t_1) + \int_{t_1}^{s} (s - t) z''(t)\,\mathrm{d}t, \qquad s \ge t_1\\
&\ge z(t_1) + (s - t_1) z'(t_1) + \int_{t_1}^{s} (s - t)(t - v)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t) L_3^{\frac{1}{\alpha}} z(v)\,\mathrm{d}t\\
&> \int_{t_1}^{s} (s - t)(t - v)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t) L_3^{\frac{1}{\alpha}} z(v)\,\mathrm{d}t\\
&> L_3^{\frac{1}{\alpha}} z(v) \int_{v}^{s} (s - t)(t - v)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\,\mathrm{d}t\\
&= A[s, v] L_3^{\frac{1}{\alpha}} z(v), \qquad s \ge v \ge t_1.
\end{aligned}
$$

Consequently,

$$z(\theta + \sigma_2) \ge A[\theta + \sigma_2, s + \sigma_2] L_3^{\frac{1}{\alpha}} z(s + \sigma_2), \qquad \theta + \sigma_2 \ge s + \sigma_2 \ge t_1 + \sigma_2. \tag{2.3}$$

From (1.1), it follows that

$$L_4 z(t) \geq r(t) g(y(t+\sigma_2))$$

and hence

$$\begin{aligned} L_4 z(t) + g(b) L_4 z(t-\tau) &\geq r(t) g(y(t+\sigma_2)) + g(b) r(t-\tau) g(y(t-\tau+\sigma_2)) \\ &\geq R(t)[g(y(t+\sigma_2)) + g(b) g(y(t-\tau+\sigma_2))] \\ &\geq R(t)[g(y(t+\sigma_2)) + g(by(t-\tau+\sigma_2))] \\ &\geq \mu R(t) g(z(t+\sigma_2)), \end{aligned}$$

where we have used the fact that $z(t+\sigma_2) \leq y(t+\sigma_2) + by(t+\sigma_2-\tau)$. Upon using (2.3), the last inequality becomes

$$\begin{aligned} L_4 z(\theta) + g(b) L_4 z(\theta-\tau) &\geq \mu R(\theta) g(z(\theta+\sigma_2)) \\ &\geq \mu R(\theta) g[A(\theta+\sigma_2, s+\sigma_2) L_3^{\frac{1}{\alpha}} z(s+\sigma_2)] \\ &\geq \mu R(\theta) g[A(\theta+\sigma_2, s+\sigma_2)] g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big] \end{aligned}$$

due to (A_1), (A_2) and (A_4). Thus

$$\begin{aligned} &\mu g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big] \int_s^{s+\sigma_2} R(\theta) g[A(\theta+\sigma_2, s+\sigma_2)]\,\mathrm{d}\theta \\ &\qquad \leq L_3 z(s+\sigma_2) + g(b) L_3 z(s-\tau+\sigma_2) - L_3 z(s) - g(b) L_3 z(s-\tau) \\ &\qquad \leq L_3 z(s+\sigma_2) + g(b) L_3 z(s-\tau+\sigma_2), \end{aligned}$$

that is,

$$\int_s^{s+\sigma_2} R(\theta) g[A(\theta+\sigma_2, s+\sigma_2)]\,\mathrm{d}\theta \leq \frac{L_3 z(s+\sigma_2) + g(b) L_3 z(s-\tau+\sigma_2)}{\mu g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big]}.$$

Since $L_3 z(t)$ is nondecreasing, then the last inequality yields that

$$\int_s^{s+\sigma_2} R(\theta) g[A(\theta+\sigma_2, s+\sigma_2)]\,\mathrm{d}\theta \leq \frac{1+g(b)}{\mu\beta_2}$$

due to (A_5), a contradiction to our assumption (A_7).

Case (b): Let $L_i z(t) > 0$, $i = 1, 2$ and $L_3 z(t) < 0$, for $t \geq t_1$. Then for $s > u \geq t_1$, we have

$$L_2 z(s) - L_2 z(u) = \int_u^s L_3 z(t)\,\mathrm{d}t,$$

which then yields

$$L_2 z(u) \geq (s-u)(-L_3 z(s)),$$

that is,

$$z''(u) \geq (s-u)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(u)\bigl(-L_3^{\frac{1}{\alpha}} z(s)\bigr).$$

Applying Taylor's expansion to $z(t)$, we obtain

$$\begin{aligned} z(s) &\geq \int_{t_1}^{s} (s-t)z''(t)\,\mathrm{d}t, \qquad s \geq t_1 \\ &\geq \int_{t_1}^{s} (s-t)(s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\bigl(-L_3^{\frac{1}{\alpha}} z(s)\bigr)\,\mathrm{d}t \\ &= \bigl(-L_3^{\frac{1}{\alpha}} z(s)\bigr)B[s,t_1] \end{aligned}$$

and hence

$$z(\theta-\sigma_1) \geq \bigl(-L_3^{\frac{1}{\alpha}} z(\theta-\sigma_1)\bigr)B[\theta-\sigma_1,t_1], \qquad \theta-\sigma_1 \geq t_1. \tag{2.4}$$

Since (2.1) can be reduced to

$$L_4 z(t) \geq q(t)f(y(t-\sigma_1))$$

then following Case (a), we obtain

$$L_4 z(t) + f(b)L_4 z(t-\tau) \geq \lambda Q(t) f(z(t-\sigma_1)) \tag{2.5}$$

due to (A_1), (A_3), and (A_4). Consequently,

$$L_4 z(\theta) + f(b)L_4 z(\theta-\tau) \geq \lambda Q(\theta) f[B(\theta-\sigma_1,t_1)] f\bigl[-L_3^{\frac{1}{\alpha}} z(\theta-\sigma_1)\bigr]$$

due to (2.4). Proceeding as in Case (a), we obtain a contradiction to our assumption (A_6).

Case(c): Using the fact that $L_3 z(t) < 0$, we obtain from the Case (b) that

$$z''(u) \geq (s-u)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(u)\bigl(-L_3^{\frac{1}{\alpha}} z(s)\bigr), \qquad s \geq u \geq t_1.$$

On the other hand,

$$z(u) = z(v) - (v-u)z'(v) + \int_{u}^{v} (t-u)z''(t)\,\mathrm{d}t$$

$$> \int_{u}^{v} (t-u)z''(t)\,\mathrm{d}t, \qquad v \geq t_1$$

implies that

$$z(u) \geq \int\limits_u^v (t-u)(v-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t)\big(-L_3^{\frac{1}{\alpha}} z(v)\big)\,\mathrm{d}t$$
$$= C[v,u]\big(-L_3^{\frac{1}{\alpha}} z(v)\big), \qquad v \geq u \geq t_1.$$

Consequently, for $v-\sigma_1 \geq \theta-\sigma_1 \geq t_1$

$$z(\theta-\sigma_1) \geq C[v-\sigma_1, \theta-\sigma_1]\big(-L_3^{\frac{1}{\alpha}} z(v-\sigma_1)\big).$$

Proceeding as in the Case (b), we obtain

$$L_4 z(\theta) + f(b) L_4 z(\theta-\tau) \geq \lambda Q(\theta) f[C(v-\theta, \theta-\sigma_1)] f\big[-L_3^{\frac{1}{\alpha}} z(v-\sigma_1)\big],$$

that is,

$$\lambda \int\limits_{v-\sigma_1}^{v} Q(\theta) f[C(v-\theta,\theta-\sigma_1)] f\big[-L_3^{\frac{1}{\alpha}} z(v-\sigma_1)\big]\,\mathrm{d}\theta$$
$$\leq \int\limits_{v-\sigma_1}^{v} [L_4 z(\theta) + f(b) L_4 z(\theta-\tau)]\,\mathrm{d}\theta$$
$$< -L_3 z(v-\sigma_1) - f(b) L_3 z(v-\sigma_1-\tau)$$

and hence

$$\int\limits_{v-\sigma_1}^{v} Q(\theta) f[C(v-\theta,\theta-\sigma_1)]\,\mathrm{d}\theta < \frac{-L_3 z(v-\sigma_1) - f(b) L_3 z(v-\sigma_1-\tau)}{\lambda f\big[-L_3^{\frac{1}{\alpha}} z(v-\sigma_1)\big]}$$
$$\leq \frac{1+f(b)}{\lambda \beta_1}$$

due to (A_2) and (A_5), a contradiction to (A_8). This completes the proof of the theorem. □

Theorem 2.2. *Let* $-\infty < -b \leq p(t) \leq 0$, $b > 0$. *If* (A_0), (A_1), (A_2), (A_9)–(A_{12}) *and* (A_{21}) *hold, then every solution of* (1.1) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1) such that $y(t) > 0$, for $t \geq t_0$. Proceeding as in the proof of Theorem 2.1, we have three Cases (a), (b) and (c). Further, from these three cases, it follows that $z(t)$ is a monotonic function on $[t_1, \infty)$, $t_1 > t_0 + \max\{\tau, \sigma_1\}$. Suppose that $z(t) > 0$, for $t \geq t_1$. In what follows, we shall consider the three Cases (a), (b) and (c).

Using the same type of reasoning as in the Case (a) of Theorem 2.1, we get the inequality (2.3). From (2.1), it follows that $L_4 z(t) \geq r(t) g(y(t+\sigma_2))$ and hence

$$L_4 z(t) \geq r(t) g(z(t+\sigma_2)),$$

where $z(t) = y(t) + p(t)y(t-\tau) \leq y(t)$. Consequently,

$$\begin{aligned} L_4 z(\theta) &\geq r(\theta) g(z(\theta+\sigma_2)) \\ &\geq r(\theta) g[A(\theta+\sigma_2, s+\sigma_2)] g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big] \end{aligned}$$

due to (2.3). Therefore,

$$g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big] \int_s^{s+\sigma_2} r(\theta) g[A(\theta+\sigma_2, s+\sigma_2)] \, \mathrm{d}\theta \leq L_3 z(s+\sigma_2) - L_3 z(s),$$

that is,

$$\int_s^{s+\sigma_2} r(\theta) g[A(\theta+\sigma_2, s+\sigma_2)] \, \mathrm{d}\theta \leq \frac{L_3 z(s+\sigma_2)}{g\big[L_3^{\frac{1}{\alpha}} z(s+\sigma_2)\big]} \leq \frac{1}{\beta_2},$$

a contradiction to (A_9). Case (b) and Case (c) can similarly be dealt with.

Next, we suppose that $z(t) < 0$, for $t \geq t_1$. Then $z(t) \geq -by(t-\tau)$, implies that $y(t-\sigma_1) \geq \big(\frac{1}{b}\big) z(t+\tau-\sigma_1))$, for $t \geq t_2 \geq t_1$. Also, in this case we consider the above three cases along with

(d) $L_i z(t) < 0, \quad i = 1, 2, 3, \quad \text{for} \quad t \geq t_1 > t_0 + \rho.$

It is easy to verify that the Cases (a) and (b) imply that $z(t) > 0$, for $t \geq t_2$, a contradiction. Consider the Case (c), for $t \geq t_2$. Consequently,

$$z''(u) \geq (s-u)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(u) \big(-L_3^{\frac{1}{\alpha}} z(s)\big), \qquad s \geq u \geq t_1,$$

that is,

$$z'(s) - z'(u) \geq \big(-L_3^{\frac{1}{\alpha}} z(s)\big) \int_u^s (s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t) \, \mathrm{d}t$$

implies that

$$-z'(u) \geq \big(-L_3^{\frac{1}{\alpha}} z(s)\big) \int_u^s (s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t) \, \mathrm{d}t.$$

Integrating the above inequality from u to s, we get

$$-z(s) + z(u) \geq \big(-L_3^{\frac{1}{\alpha}} z(s)\big) \int_u^s \int_\theta^s (s-t)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(t) \, \mathrm{d}t \, \mathrm{d}\theta,$$

that is,

$$-z(s) \geq \big(-L_3^{\frac{1}{\alpha}} z(s)\big) D[s, \theta].$$

Upon the last inequality, it follows that

$$-z(v+\tau-\sigma_1) \geq (-L_3^{\frac{1}{\alpha}} z(v+\tau-\sigma_1)) D[v+\tau-\sigma_1, \theta+\tau-\sigma_1].$$

In what follows, the Eq. (2.1) reduces to

$$\begin{aligned} L_4(z(v)) &\geq q(v) f(y(v-\sigma_1)) \\ &\geq q(v) f\left(-\frac{1}{b} z(v+\tau-\sigma_1)\right) \\ &\geq q(v) f\left(\frac{1}{b}\right) f(-L_3^{\frac{1}{\alpha}} z(v+\tau-\sigma_1)) f[D(v+\tau-\sigma_1, \theta+\tau-\sigma_1)]. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} &\int\limits_{s+\tau-\sigma_1}^{s} L_4(z(v))\,\mathrm{d}v \\ &\geq f\left(\frac{1}{b}\right) \int\limits_{s+\tau-\sigma_1}^{s} q(v) f\big(-L_3^{\frac{1}{\alpha}} z(v+\tau-\sigma_1)\big) f[D(v+\tau-\sigma_1, \theta+\tau-\sigma_1)]\,\mathrm{d}v \\ &\geq f\left(\frac{1}{b}\right) f(-L_3^{\frac{1}{\alpha}} z(s+\tau-\sigma_1)) \int\limits_{s+\tau-\sigma_1}^{s} q(v) f[D(v+\tau-\sigma_1, \theta+\tau-\sigma_1)]\,\mathrm{d}v \end{aligned}$$

implies that

$$\frac{-L_3 z(s+\tau-\sigma_1}{f(-L_3^{\frac{1}{\alpha}} z(s+\tau-\sigma_1))} \geq f\left(\frac{1}{b}\right) \int\limits_{s+\tau-\sigma_1}^{s} q(v) f[D(v+\tau-\sigma_1, \theta+\tau-\sigma_1)]\,\mathrm{d}v,$$

that is,

$$\limsup_{s\to\infty} \int\limits_{s+\tau-\sigma_1}^{s} q(v) f[D(v+\tau-\sigma_1, \theta+\tau-\sigma_1)]\,\mathrm{d}v \leq \frac{1}{\beta_1 f(\frac{1}{b})},$$

a contradiction to our assumption (A_{12}). In Case (d), we use the fact that $z(t)$ is nonincreasing for $t \geq t_2$. Hence there exist a constant $C > 0$ and $t_3 \geq t_2$ such that $z(t) \leq -C$, for $t \geq t_3$. As a result, (2.7) becomes

$$L_4(z(t)) \geq q(t) f\left(-\frac{1}{b} z(t+\tau-\sigma_1)\right) \geq q(t) f\left(\frac{C}{b}\right),$$

for $t \geq t_3$ and on integration from t_1 to t we get a contradiction due to (A_{21}). Thus the proof of the theorem is complete. $\square$

Theorem 2.3. *Let* $0 \le p(t) \le b < \infty$ *and* $\tau \ge \sigma_1$. *If* (A_0)–(A_4) *and* (A_{13})–(A_{16}) *hold, then* (1.1) *is oscillatory.*

Proof. If possible, let $y(t)$ be a non-oscillatory solution of (1.1) such that $y(t) > 0$, for $t \ge t_0$. From (2.1), it follows that $L_3 z(t)$ is nondecreasing on $[t_1, \infty)$, $t_1 \ge t_0 + \tau$. Thus we have three cases as in Theorem 2.1. Assume that the Case (a) holds. Integrating the inequality

$$L_4 z(t) + g(b) L_4 z(t-\tau) \ge \mu R(t) g(z(t+\sigma_2))$$

from s to t_2, we get

$$\begin{aligned} L_3 z(t_2) + g(b) L_3 z(t_2 - \tau) &\ge \mu \int_s^{t_2} R(t) g(z(t+\sigma_2))\,\mathrm{d}t \\ &\ge \mu g(z(s+\sigma_2)) \int_s^{t_2} R(t)\,\mathrm{d}t, \qquad t_2 \ge s \ge t_1 \end{aligned}$$

due to nondecreasing $z(t)$, for $t \ge t_1$. Moreover, for $t_3 \ge t_2 \ge s \ge t_1$ the last inequality yields

$$\int_s^{t_3} [L_3 z(t_2) + g(b) L_3 z(t_2 - \tau)]\,\mathrm{d}t_2 \ge \mu \int_s^{t_3} g(z(s+\sigma_2)) \int_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2,$$

that is,

$$L_2 z(t_3) + g(b) L_2 z(t_3 - \tau) \ge \mu g(z(s+\sigma_2)) \int_s^{t_3} \int_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2.$$

Consequently,

$$(1 + g(b)) L_2 z(t_3) \ge \mu g(z(s+\sigma_2)) \int_s^{t_3} \int_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2.$$

Hence

$$z''(t_3) \ge \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(s+\sigma_2)) \left[a(t_3) \int_s^{t_3} \int_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}}.$$

Integrating the last inequality from s to t_4 for $t_4 \ge t_3 \ge t_2 \ge s \ge t_1$, we get

$$z'(t_4) \ge \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(s+\sigma_2)) \int_s^{t_4} \left[a(t_3) \int_s^{t_3} \int_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3$$

and hence further integration from s to $s+\sigma_2$, we obtain

$$z(s+\sigma_2) \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(s+\sigma_2)) \int\limits_s^{s+\sigma_2} \int\limits_s^{t_4} \left[a(t_3) \int\limits_s^{t_3} \int\limits_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4,$$

that is,

$$\frac{z(s+\sigma_2)}{g^{\frac{1}{\alpha}}(z(s+\sigma_2))} \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} \int\limits_s^{s+\sigma_2} \int\limits_s^{t_4} \left[a(t_3) \int\limits_s^{t_3} \int\limits_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4.$$

Applying (A_{13}), it follows that

$$\limsup_{s\to\infty} \int\limits_s^{s+\sigma_2} \int\limits_s^{t_4} \left[a(t_3) \int\limits_s^{t_3} \int\limits_s^{t_2} R(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 \leq \left(\frac{1+g(b)}{\mu}\right)^{\frac{1}{\alpha}} \frac{1}{\beta_4},$$

a contradiction to our assumption (A_{14}).

Let the Case (b) hold. From (2.1) and (2.5), it follows that

$$\int\limits_s^{\infty} [L_4 z(t) + f(b) L_4 z(t-\tau)]\,\mathrm{d}t \geq \lambda \int\limits_s^{\infty} Q(t) f(z(t-\sigma_1))\,\mathrm{d}t$$

$$\geq \lambda f(z(s-\sigma_1)) \int\limits_s^{\infty} Q(t)\,\mathrm{d}t,$$

that is,

$$-L_3 z(s) - f(b) L_3 z(s-\tau) \geq \lambda f(z(s-\sigma_1)) \int\limits_s^{\infty} Q(t)\,\mathrm{d}t,$$

for $s \geq t_1$. Upon using $\sigma_1 \leq \tau$, the last inequality becomes

$$-(1+f(b)) L_3 z(s-\tau) \geq \lambda f(z(s-\tau)) \int\limits_s^{\infty} Q(t)\,\mathrm{d}t$$

and hence (2.4) yields that

$$z(s-\tau) \geq B[s-\tau, t_1] \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(s-\tau)) \left(\int\limits_s^{\infty} Q(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}}.$$

Consequently,

$$\frac{z(s-\tau)}{f^{\frac{1}{\alpha}}(z(s-\tau))} \geq B[s-\tau, t_1] \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} \left(\int\limits_s^{\infty} Q(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}},$$

that is,

$$\limsup_{s\to\infty} B[s-\tau, t_1]\left(\int_s^\infty Q(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \le \left(\frac{1+f(b)}{\lambda}\right)^{\frac{1}{\alpha}} \frac{1}{\beta_3},$$

a contradiction to our assumption (A_{15}).

Finally, we consider the Case (c). Proceeding as in the Case (b), we obtain

$$(1+f(b))(-L_3 z(t_2-\tau)) \ge \lambda f(z(s-\sigma_1)) \int_{t_2}^{s} Q(t)\,\mathrm{d}t.$$

Thus for $s \ge t_3 \ge t_2 \ge t_1$,

$$(1+f(b)) \int_{t_3}^{s} (-L_3 z(t_2-\tau))\,\mathrm{d}t_2 \ge \lambda f(z(s-\sigma_1)) \int_{t_3}^{s}\int_{t_2}^{s} Q(t)\,\mathrm{d}t\,\mathrm{d}t_2$$

implies that

$$(1+f(b)) L_2 z(t_3-\tau) \ge \lambda f(z(s-\sigma_1)) \int_{t_3}^{s} (t-t_3) Q(t)\,\mathrm{d}t.$$

Hence

$$(1+f(b))^{\frac{1}{\alpha}} z''(t_3-\tau) \ge \lambda^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(s-\sigma_1)) \left[a(t_3)\int_{t_3}^{s} (t-t_3) Q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}}.$$

Integrating the last inequality from t_4 to s, we get

$$z'(t_4-\tau) \ge \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(s-\sigma_1)) \int_{t_4}^{s} \left[a(t_3)\int_{t_3}^{s} (t-t_3) Q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}} \mathrm{d}t_3$$

Using the fact that $z'(t)$ is nondecreasing and further integration of the above inequality, we obtain

$$\int_{s-\sigma_1}^{s} z'(t_4)\,\mathrm{d}t_4$$

$$\ge \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(s-\sigma_1)) \int_{s-\sigma_1}^{s}\int_{t_4}^{s} \left[a(t_3)\int_{t_3}^{s} (t-t_3) Q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4,$$

that is,

$$z(s) \ge \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(s)) \int_{s-\sigma_1}^{s}\int_{t_4}^{s} \left[a(t_3)\int_{t_3}^{s} (t-t_3) Q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4.$$

Therefore,

$$\limsup_{s\to\infty}\int\limits_{s-\sigma_1}^{s}\int\limits_{t_4}^{s}\left[a(t_3)\int\limits_{t_3}^{s}(t-t_3)Q(t)\,\mathrm{d}t\right]^{\frac{1}{\alpha}}\mathrm{d}t_3\,\mathrm{d}t_4 \le \left(\frac{1+f(b)}{\lambda}\right)^{\frac{1}{\alpha}}\frac{1}{\beta_3},$$

a contradiction to (A_{16}). Hence the proof of the theorem is complete. □

Theorem 2.4. *Let* $-\infty < -b \le p(t) \le 0$, $b > 0$. *If* (A_0), (A_2), (A_{13}) *and* (A_{17})–(A_{21}) *hold, then* (1.1) *is oscillatory.*

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (1.1) such that $y(t) > 0$, for $t \ge t_0$. Then proceeding as in the proof of Theorem 2.2, we obtain

$$L_4 z(t) \ge r(t) g(z(t+\sigma_2)).$$

Thus

$$\int\limits_{s}^{t_2} L_4 z(t)\,\mathrm{d}t \ge \int\limits_{s}^{t_2} r(t) g(z(t+\sigma_2))\,\mathrm{d}t$$

$$\ge g(z(s+\sigma_2))\int\limits_{s}^{t_2} r(t)\,\mathrm{d}t$$

due to nondecreasing $z(t)$ with $z(t) > 0$ and $L_3 z(t) > 0$, for $t \ge t_1$. Consequently, the last inequality yields

$$L_3 z(t_2) \ge g(z(s+\sigma_2))\int\limits_{s}^{t_2} r(t)\,\mathrm{d}t$$

which on further integration from s to t_3, we get

$$L_2 z(t_3) \ge g(z(s+\sigma_2))\int\limits_{s}^{t_3}\int\limits_{s}^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2,$$

that is,

$$z''(t_3) \ge g^{\frac{1}{\alpha}}(z(s+\sigma_2))\left[a(t_3)\int\limits_{s}^{t_3}\int\limits_{s}^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}},$$

where $t_3 \ge t_2 \ge s \ge t_1$. Consequently,

$$\int\limits_{s}^{t_4} z''(t_3)\,\mathrm{d}t_3 \ge g^{\frac{1}{\alpha}}(z(s+\sigma_2))\int\limits_{s}^{t_4}\left[a(t_3)\int\limits_{s}^{t_3}\int\limits_{s}^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}}\mathrm{d}t_3$$

implies that

$$z'(t_4) \geq g^{\frac{1}{\alpha}}(z(s+\sigma_2)) \int_s^{t_4} \left[a(t_3) \int_s^{t_3} \int_s^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3. \tag{2.7}$$

Integrating (2.7) from s to $s+\sigma_2$, we obtain

$$z(s+\sigma_2) \geq g^{\frac{1}{\alpha}}(z(s+\sigma_2)) \int_s^{s+\sigma_2} \int_s^{t_4} \left[a(t_3) \int_s^{t_3} \int_s^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4$$

and hence

$$\limsup_{s\to\infty} \int_s^{s+\sigma_2} \int_s^{t_4} \left[a(t_3) \int_s^{t_3} \int_s^{t_2} r(t)\,\mathrm{d}t\,\mathrm{d}t_2\right]^{\frac{1}{\alpha}} \mathrm{d}t_3\,\mathrm{d}t_4 \leq \frac{1}{\beta_4},$$

a contradiction to (A_{17}).

Consider the Case (b). From Eq. (2.1), it follows that

$$\begin{aligned} -L_3 z(t) &\geq \int_t^{\infty} q(s) f(y(s-\sigma_1))\,\mathrm{d}s \\ &\geq \int_t^{\infty} q(s) f(z(s-\sigma_1))\,\mathrm{d}s \\ &\geq f(z(t-\sigma_1)) \int_t^{\infty} q(s)\,\mathrm{d}s, \end{aligned}$$

where $z(t) \leq y(t)$, for $t \geq t_1$. Thus the inequality (2.4) yields

$$\begin{aligned} z(t-\sigma_1) &\geq B[t-\sigma_1, t_1]\big(-L_3^{\frac{1}{\alpha}} z(t-\sigma_1)\big) \\ &\geq B[t-\sigma_1, t_1] f^{\frac{1}{\alpha}}(z(t-\sigma_1)) \left(\int_t^{\infty} q(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}, \end{aligned}$$

for $t-\sigma_1 \geq t_1$. Consequently,

$$\limsup_{t\to\infty} B[t-\sigma_1, t_1] \left(\int_t^{\infty} q(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \leq \frac{1}{\beta_3}$$

due to (A_{13}), a contradiction to our assumption (A_{18}).

Suppose that the Case (c) holds. For $t \geq s \geq t_1$ and we have from Eq. (2.1) that

$$
\begin{aligned}
-L_3 z(s) &\geq \int_s^t q(\theta) f(y(\theta - \sigma_1))\, \mathrm{d}\theta \\
&\geq \int_s^t q(\theta) f(z(\theta - \sigma_1))\, \mathrm{d}\theta \\
&\geq f(z(\theta - \sigma_1)) \int_s^t q(\theta)\, \mathrm{d}\theta.
\end{aligned}
$$

The rest of the proof follows from the Case (c) of Theorem 2.3 to obtain a contradiction.

Next, we assume that $z(t) < 0$, for $t \geq t_1$. Then $z(t) \geq -by(t-\tau)$ implies that $y(t) \geq \left(-\frac{1}{b}\right) z(t+\tau-\sigma_1)$, for $t \geq t_2 > t_1$. Also for this case, we consider the Cases (a), (b), (c) and (d). It is easy to verify that the Cases (a) and (b) are not possible and the Case (d) is same as in Theorem 2.2. Let the Case (c) hold. From (2.1), it follows that

$$
\begin{aligned}
L_4 z(t) &\geq q(t) f(y(t-\sigma_1)) \\
&\geq q(t) f\left(-\frac{1}{b} z(t+\tau-\sigma_1)\right) \\
&\geq q(t) f\left(-\frac{1}{b} z(t-\sigma_1)\right)
\end{aligned}
$$

and hence for $s \geq t_2$,

$$
\begin{aligned}
\int_t^s L_4 z(\theta)\, \mathrm{d}\theta &\geq \int_t^s q(\theta) f\left(-\frac{1}{b} z(\theta-\sigma_1)\right) \mathrm{d}\theta \\
&\geq f\left(-\frac{1}{b} z(t-\sigma_1)\right) \int_t^s q(\theta)\, \mathrm{d}\theta,
\end{aligned}
$$

that is, for $s \geq t \geq t_2$

$$
-L_3 z(t) \geq f\left(-\frac{1}{b} z(t-\sigma_1)\right) \int_t^s q(\theta)\, \mathrm{d}\theta.
$$

On the other hand, (2.6) yields that

$$\begin{aligned}-z(t-\sigma_1) &\geq D[t-\sigma_1,\theta-\sigma_1]\big(-L_3^{\frac{1}{\alpha}} z(t-\sigma_1)\big)\\ &\geq D[t-\sigma_1,\theta-\sigma_1]\big(-L_3^{\frac{1}{\alpha}} z(t)\big),\end{aligned}$$

that is,

$$-z(t-\sigma_1) \geq D[t-\sigma_1,\theta-\sigma_1] f^{\frac{1}{\alpha}}\left(-\frac{1}{b} z(t-\sigma_1)\right)\left(\int_t^s q(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{\alpha}}.$$

Hence

$$\frac{-z(t-\sigma_1)}{f^{\frac{1}{\alpha}}\left(-\frac{1}{b} z(t-\sigma_1)\right)} \geq D[t-\sigma_1,\theta-\sigma_1]\left(\int_t^s q(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{\alpha}}$$

implies that

$$\limsup_{t\to\infty} D[t-\sigma_1,\theta-\sigma_1]\left(\int_t^s q(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{\alpha}} \leq \frac{b}{\beta_3},$$

a contradiction to our assumption (A_{20}). This completes the proof of the theorem. $\square$

Example 2.5. Consider

$$\begin{aligned}&\left(t\left(y(t)+\left(1+\frac{1}{t}\right)y(t-\pi)\right)''\right)''\\ &= \left(\frac{2}{t}-\frac{12}{t^3}\right) y\left(t-\frac{\pi}{2}\right)+\left(1-\frac{6}{t^2}+\frac{12}{t^4}\right) y(t+\pi),\end{aligned} \tag{2.8}$$

for $t \geq 3$, where $1 \leq 1+\frac{1}{t} = p(t) \leq 2$, $a(t) = \frac{1}{t}$, $\alpha = 1$, $\lambda = 1$ and $f(u) = g(u) = u$. Clearly, all the conditions of Theorem 2.1 are satisfied. Hence (2.8) is oscillatory. In particular, $y(t) = \sin t$ is such an oscillatory solution of (2.8).

Theorem 2.6. *Let $-\infty < -b \leq p(t) \leq 0$, $b > 0$ and $\tau \leq \sigma_1$. Assume that* (A_1), (A_2) *and* (A_{21}) *hold. For $\beta(m) \leq m$, if*

$$x'(t) - r(t) g[A(t+\sigma_2, t+\beta(\sigma_2))] g\big(x^{\frac{1}{\alpha}}(t+\beta(\sigma_2))\big) = 0 \tag{2.9}$$

$$u'(t) + q(t) f[B(t-\sigma_1, t_1)] f\big(u^{\frac{1}{\alpha}}(t-\sigma_1)\big) = 0 \tag{2.10}$$

$$v'(t) + q(t) f[C(t-\beta(\sigma_1), t-\sigma_1)] f\big(v^{\frac{1}{\alpha}}(t-\beta(\sigma_1))\big) = 0 \tag{2.11}$$

$$w'(t) + f\left(\frac{1}{b}\right) q(t) f[D(t+\tau-\sigma_1, t+\beta(\tau-\sigma_1))] f\big(w^{\frac{1}{\alpha}}(t+\tau-\sigma_1)\big) = 0 \tag{2.12}$$

are oscillatory, then every solution of (1.1) *oscillates.*

P r o o f. If possible, let $y(t)$ be a non-oscillatory solution of of (1.1) such that $y(t) > 0$, for $t \geq t_0$. Proceeding as in the proof of Theorem 2.2, we have three Cases (a), (b) and (c) when $z(t) > 0$.

For the Case (a), $L_i z(t) > 0, i = 1, 2, 3$, for $t \geq t_1$. Then $z(s) \geq A[s, v] L_3^{\frac{1}{\alpha}} z(v)$, $s \geq v \geq t_1$. Thus

$$z(t+\sigma_2) \geq A[t+\sigma_2, t+\beta(\sigma_2)] L_3^{\frac{1}{\alpha}} z(t+\beta(\sigma_2)),$$

where $\beta(\sigma_2) \leq \sigma_2$. Consequently, Eq. (2.1) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} L_3 z(t) &\geq r(t) g(y(t+\sigma_2)) \\ &\geq r(t) g(z(t+\sigma_2)) \\ &\geq r(t) g\big(A(t+\sigma_2, t+\beta(\sigma_2))\big) g\big(L_3^{\frac{1}{\alpha}} z(t+\beta(\sigma_2))\big)\end{aligned}$$

for $t \geq t_1$ due to (A_1). Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} L_3 z(t) - r(t) g\big(A(t+\sigma_2, t+\beta(\sigma_2))\big) g\big(L_3^{\frac{1}{\alpha}} z(t+\beta(\sigma_2))\big) \geq 0$$

has a positive solution $L_3 z(t)$, for $t \geq t_1$. By [4: Corollary 2.4.2], (2.9) has an eventually positive solution, a contradiction to our hypothesis. Case (b) and Case (c) are similar.

Next, we suppose that $z(t) < 0$, for $t \geq t_1$ and for this case, Case (c) and (d) are the desired cases. Proceeding as in the proof of Theorem 2.2, we have

$$L_4 z(t) \geq q(t) f\left(\frac{1}{b}\right) f\big(-L_3^{\frac{1}{\alpha}} z(t+\tau-\sigma_1)\big) f\big[D\big(t+\tau-\sigma_1, t+\beta(\tau-\sigma_1)\big)\big]$$

for $t \geq t_1$, that is,

$$L_4 z(t) + q(t) f\left(\frac{1}{b}\right) f\big(L_3^{\frac{1}{\alpha}} z(t+\tau-\sigma_1)\big) f\big[D\big(t+\tau-\sigma_1, t+\beta(\tau-\sigma_1)\big)\big] \geq 0$$

has a negative solution $L_3 z(t)$, for $t \geq t_1$. By [2: Corollary 2.4.1], (2.12) has a negative solution, a contradiction to our hypothesis. The Case (d) follows from the proof of Theorem 2.2. This completes the proof of the theorem. $\square$

Theorem 2.7. *Let* $0 \leq p(t) \leq b < 1$. *Assume that* (A_1) *and* (A_2) *hold. For* $\beta(m) \leq m$, *if*

$$x'(t) - g(1-b) r(t) g[A(t+\sigma_2, t+\beta(\sigma_2))] g\big(x^{\frac{1}{\alpha}}(t+\beta(\sigma_2))\big) = 0$$

and

$$u'(t) + g(1-b) q(t) f[B(t-\sigma_1, t_1)] f\big(u^{\frac{1}{\alpha}}(t-\sigma_1)\big) = 0$$

are oscillatory, then every unbounded solution of (1.1) *oscillates.*

P r o o f. Suppose on the contrary that $y(t)$ is an unbounded non-oscillatory solution of (1.1) such that $y(t) > 0$, for $t \ge t_0$. Using the same type of reasoning as in the proof of Theorem 2.1, we consider the Cases (a), (b) and (c). For each of the Cases (a) and (b), $z(t)$ is nondecreasing on $[t_1, \infty)$. Hence for $t \ge t_1$,

$$\begin{aligned}(1 - p(t))z(t) &\le z(t) - p(t)z(t)\\ &\le z(t) - p(t)z(t-\tau)\\ &= y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t).\end{aligned}$$

Following to the proof of Theorem 2.1 and 2.6, it is easy to verify the Case (a) and (b). For the Case (c), $z(t)$ happens to be bounded due to $(-1)^i L_i z(t) > 0$, $i = 1, 2, 3$. Hence this case does't arise. Thus the theorem is proved. □

Remark 2.8. With a suitable transformation (see e.g. [10,11] and [14]), we can establish the existence of positive bounded solutions of (1.1).

Remark 2.9. Due to the method employed here, it is ensure to consider the assumption (A_0). However, we can not apply the present method if

$$(\mathrm{A}_{22})\quad \int_0^\infty a(t)^{\frac{1}{\alpha}}\,\mathrm{d}t < \infty.$$

It seems that some more conditions or a different method along with (A_{22}) is necessary to show that Eq. (1.1) is oscillatory.

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