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A CONSTRUCTION OF MIXED POISSON PROCESSES VIA DISINTEGRATIONS

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ABSTRACT. A new construction of mixed Poisson processes with prescribed distributions for their claim interarrival times is given. As a consequence, some concrete examples of constructing such processes useful for applications are presented and the corresponding disintegrating and claim measures are computed.

Introduction

In [5: Theorem 4.10], some characterizations of *mixed Poisson processes*, defined on a probability space (Ω, Σ, P) (P-MPPs for short), via *disintegrations* were obtained. In particular, it was proven there, that a claim number process $\{N_t\}_{t\in\mathbb{R}_+}$ is a P-MPP with parameter a random variable Θ on (Ω, Σ) if and only if it is a Poisson process with parameter θ under the disintegrating measure P_θ for P_Θ-almost all (P_Θ-a.a. for short) $\theta \in \mathbb{R}$, equivalently if the sequence $\{W_n\}_{n\in\mathbb{N}}$ of its claim interarrival times is conditionally independent and for each $n \in \mathbb{N}$ the equality $P_{W_n|\Theta} = \mathbf{Exp}(\Theta)$ holds true almost surely (see Section 1 for the notations).

Based on the last result, the existence of MPPs with prescribed distributions for their claim interarrival processes is proven, see Theorem 3.1.

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The proof of the above theorem also relies on a result concerning *disintegrations and product regular conditional probabilities*, see Proposition 2.5, which may be of independent interest, since it asserts that the existence of a product regular conditional probability (product r.c.p. for short) is equivalent to that of a disintegration consistent with an inverse-measure-preserving map (see Section 2 for the definitions).

As an application of Theorem 3.1, some concrete examples of constructions of MPPs are presented. More precisely, given Borel probability measures $Q_n(\theta)$ and ν on $\Upsilon := (0,\infty)$ for $n \in \mathbb{N}$ and $\theta \in \Upsilon$, we construct a probability space (Ω, Σ, P), a disintegration $\{P_\theta\}_{\theta\in\Upsilon}$ of P over ν, an inverse-measure-preserving random variable Θ from Ω onto Υ, such that $\{P_\theta\}_{\theta\in\Upsilon}$ is consistent with Θ, and a P-MPP $\{N_t\}_{t\in\mathbb{R}_+}$ having a claim interarrival process $\{W_n\}_{n\in\mathbb{N}}$ distributed according to $Q_n(\theta)$. Moreover, we compute the corresponding claim measures for each case.

The standard construction of a MPP (cf. e.g. [4: pp. 61–63]) is given by totally different methods than ours, since in that case Kolmogorov's forward differential equations are involved and the underlying probability space (Ω, Σ, P) with $\Omega = \mathbb{N}_0^{\mathbb{R}_+}$ is much more enlarged than that of Theorem 3.1.

Since each MPP is a Markov process (cf. e.g. [8: Theorem 4.2.3]), the construction of MPPs is guaranteed by a refinement of Kolmogorov's Existence Theorem for stochastic processes (cf. e.g. [3: Theorem 455A]). But in such a construction the underlying probability space is a triple (Ω, Σ, Q) again with enlarged $\Omega = \mathbb{N}_0^{\mathbb{R}_+}$, while the computation of Q becomes quite complicated, since Q is represented as a mixture of projective limit (hence non-direct product) probability measures.

1. Preliminaries

By $\mathbb{N}$ is denoted the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $m \in \mathbb{N}$, then $\mathbb{N}_m := \{1, \dots, m\}$. The symbol $\mathbb{R}$ stands for the set of all real numbers, while $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

Given a probability space (Ω, Σ, P), a set $N \in \Sigma$ with $P(N) = 0$ is called a *P-null set* (or a null set for simplicity). The family of all P-null sets is denoted by Σ_0. For arbitrary sets $A, B \in \Sigma$ we write $A = B$ P-almost surely (P-a.s. for short), if $A \triangle B$, the symmetric difference of A and B, is a P-null set. For measurable functions $X, Y\colon \Omega \longrightarrow \mathbb{R}$ we write $X = Y$ P-a.s., if $\{X \neq Y\} \in \Sigma_0$.

If $A \subseteq \Omega$, then $A^c := \Omega \setminus A$, while by χ_A is denoted the indicator (or characteristic) function of the set A. A family $\mathcal{D}$ of subsets of Ω is a *Dynkin class* if it satisfies the following conditions: $\Omega \in \mathcal{D}$, $B \setminus A \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$ and $A \subseteq B$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$ whenever $\{A_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{D}$ (cf. e.g. [2: 136]). For any Hausdorff topology $\mathcal{T}$ on Ω by $\mathfrak{B}(\Omega)$ is denoted the *Borel σ-algebra* on Ω, i.e. the σ-algebra generated by $\mathcal{T}$. In particular, by $\mathfrak{B} := \mathfrak{B}(\mathbb{R})$ is denoted the Borel σ-algebra of subsets of $\mathbb{R}$.

The family of all real-valued P-integrable functions on Ω, that is of all real-valued measurable functions f on Ω with $\int |f| \,\mathrm{d}P < \infty$, is denoted by $\mathcal{L}^1(P)$. Functions that are P-a.s. equal are not identified. By β is denoted the restriction of the Lebesgue measure λ to $\mathfrak{B}$, while any restriction of β to $\mathfrak{B}(A)$, where A is any Borel subset of $\mathbb{R}$ will denoted again by β. The σ-algebra generated by a family $\mathcal{G}$ of subsets of Ω is denoted by $\sigma(\mathcal{G})$, $\sigma(f) := \{f^{-1}(B) : \ B \in \mathfrak{B}\}$ stands for the σ-algebra generated by any real-valued function f on Ω, and $\sigma(\{f_i\}_{i\in I}) := \sigma\big(\bigcup_{i\in I} \sigma(f_i)\big)$ denotes the σ-algebra generated by a family $\{f_i\}_{i\in I}$ of real-valued functions on Ω.

For any given random variable X on (Ω, Σ, P), setting $T_X := \{B \subseteq \mathbb{R} : X^{-1}(B) \in \Sigma\}$, we clearly get that $\mathfrak{B} \subseteq T_X$. Denote by $P_X : T_X \longrightarrow \mathbb{R}$ the image measure of P under X and again by P_X its restriction to $\mathfrak{B}$. The notation $P_X = \mathbf{K}(\theta)$ will state that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in \widetilde{\Theta}$ and $\widetilde{\Theta}$ is the parametric space.

In particular, we denote by $\mathbf{P}(\theta)$, $\mathbf{Exp}(\theta)$, $\mathbf{Ga}(r,\gamma,\alpha)$, $\mathbf{Be}(r,\alpha,\gamma)$, $\mathbf{U}(c_1,c_2)$ and $\mathbf{Del}(r,\gamma,\alpha)$, where r, c_1, c_2 and θ, γ, α are real and positive real parameters, respectively, the law of Poisson, exponential, shifted (or generalized) gamma, beta, uniform and Delaporte distribution, respectively (cf. e.g. [8] and [4]).

If $X \in \mathcal{L}^1(P)$ and T is a σ-subalgebra of Σ, then each function $Y \in \mathcal{L}^1(P \mid T)$ satisfying for each $A \in T$ the equality $\int_A X \,\mathrm{d}P = \int_A Y \,\mathrm{d}P$ is said to be a *version of the conditional expectation* of X with respect to (or given) T and it will be denoted by $\mathbb{E}_P[X \mid T]$. For $X := \chi_B \in \mathcal{L}^1(P)$ with $B \in \Sigma$ we set $P(B \mid T) := \mathbb{E}_P[\chi_B \mid T]$.

Given a partially ordered set I, any non-decreasing family $\{\Sigma_i\}_{i\in I}$ of σ-subalgebras of Σ is said to be a *filtration* for (Ω, Σ). For any family $\{Z_i\}_{i\in I}$ of random variables on (Ω, Σ), the filtration $\{\mathcal{Z}_i\}_{i\in I}$ with $\mathcal{Z}_i := \sigma\big(\bigcup_{j\le i} \sigma(Z_j)\big)$ for each $i \in I$, is called *the canonical filtration* for $\{Z_i\}_{i\in I}$.

We denote by $(\Omega \times \Upsilon, \Sigma \otimes T, P \otimes Q)$ the product probability space of the probability spaces (Ω, Σ, P) and (Υ, T, Q), and by p_Ω and p_Υ the canonical projections of $\Omega \times \Upsilon$ onto Ω and Υ, respectively. If $C \subseteq \Omega \times \Upsilon$ and $(\omega, y) \in \Omega \times \Upsilon$ is fixed, then we shall be using the ordinary notation C_ω and C^y for the ω- and y-section of C, respectively.

2. A basic result on disintegrations

Throughout this section (Ω, Σ, P) and (Υ, T, Q) are arbitrary but fixed probability spaces. We first give some basic results on *disintegrations of P over Q* as well as on *regular conditional probabilities* and *product regular conditional probabilities*, required for the purposes of this paper.

The following definition is a special instance of that in [3: 452E], proper for our investigation.

Definition 2.1. A *disintegration of P over Q* is a family $\{P_y\}_{y \in \Upsilon}$ of probability measures $P_y \colon \Sigma \longrightarrow \mathbb{R}$ such that

(d1) for each $D \in \Sigma$ the function $P_{\cdot}(D) \colon \Upsilon \longrightarrow \mathbb{R}$ is T-measurable,

(d2) the integral $\int P_y(D) Q(\mathrm{d}y)$ is defined in $\mathbb{R}_+$ and equal to $P(D)$ for each $D \in \Sigma$.

If $f \colon \Omega \longrightarrow \Upsilon$ is an inverse-measure-preserving map (i.e. $P(f^{-1}(B)) = Q(B)$ for each $B \in T$), a disintegration $\{P_y\}_{y \in \Upsilon}$ of P over Q is called *consistent* with f if, for each $B \in T$, the equality $P_y(f^{-1}(B)) = 1$ holds for Q-almost every $y \in B$.

For detailed information about a more general definition and its related existence results for disintegrations, see Pachl [6] or Fremlin [3: 452, 453].

Definition 2.2. Let $\mathcal{F}$ be a σ-subalgebra of Σ. A function $P_{\mathcal{F}} \colon \Sigma \times \Omega \longrightarrow \mathbb{R}$ is said to be a *regular conditional probability* (r.c.p. for short) *of Σ given $\mathcal{F}$* if

(cp1) for each $A \in \Sigma$ the equality $P_{\mathcal{F}}(A, \cdot) = \mathbb{E}_P[\chi_A \mid \mathcal{F}](\cdot)$ holds true $P \mid \mathcal{F}$-a.s., where $P_{\mathcal{F}}(A, \cdot)$ is $\mathcal{F}$-measurable,

(cp2) for all $\omega \in \Omega$ the set function $P_{\mathcal{F}}(\cdot, \omega) \colon \Sigma \longrightarrow \mathbb{R}$ is a probability.

It is known that for a perfect probability space (Ω, Σ, P) (cf. e.g. [7] for the definition and properties of perfect measures), if Σ is countably generated and if $\mathcal{F}$ is a σ-subalgebra of Σ, a r.c.p. $P_{\mathcal{F}}$ of Σ given $\mathcal{F}$ always exists, such that for every $\omega \in \Omega$ the measure $P_{\mathcal{F}}(\cdot, \omega)$ is perfect (cf. [7: Theorem 4.2.1]).

At this point, let us recall that for any two random variables X and Z on Ω a *conditional distribution of* X *given* Z is a function $P_{X|Z}\colon \mathfrak{B}\times\Omega \longrightarrow [0,1]$ such that the following holds true:

(cd1) For each $\omega\in\Omega$ the set-function $P_{X|Z}(\cdot,\omega)$ is a probability measure on $\mathfrak{B}$,

(cd2) for each $B\in\mathfrak{B}$ we have

$$P_{X|Z}(B,\cdot) = P\big(X^{-1}(B) \mid \sigma(Z)\big)(\cdot) \quad P \mid \sigma(Z)\text{-a.s.},$$

where $P_{X|Z}(B,\cdot)$ is $\sigma(Z)$-measurable.

Definition 2.3. Assume that M is a probability on the σ-algebra $\Sigma\otimes T$ such that P and Q are the marginals of M. Assume also that for each $y\in\Upsilon$ there exists a probability P_y on Σ, satisfying the following properties:

(D1) For every $A\in\Sigma$ the map $y\longmapsto P_y(A)$ is T-measurable;

(D2) $M(A\times B) = \int_B P_y(A)Q(\mathrm{d}y)$ for each $A\times B\in\Sigma\times T$.

Then, $\{P_y\}_{y\in\Upsilon}$ is said to be a *product r.c.p. on* Σ *for* M *with respect to* Q (cf. e.g. [9: Definition 1.1]).

It is well known that, if Σ is countably generated and P is perfect, then a product r.c.p. always exists (cf. [1: Theorem 6]). For a more general definition of a product r.c.p. and the related existence result see Pachl [6].

It should be pointed out that most of the probability spaces appearing in applications are countably compact; hence perfect (see [7]).

Lemma 2.4. *Let M be a probability measure on $\Sigma\otimes T$ such that $\{\widetilde{P}_y\}_{y\in\Upsilon}$ is a product r.c.p. on Σ for M with respect to Q. Put $P_y := \widetilde{P}_y\otimes\delta_y$ for $y\in\Upsilon$, where δ_y is the Dirac probability measure on T defined by $\delta_y(B) := \chi_B(y)$ for each $B\in T$. Then $\{P_y\}_{y\in\Upsilon}$ is a disintegration of M over Q consistent with the canonical projection p_Υ from $\Omega\times\Upsilon$ onto Υ.*

Proof. Clearly P_y is a probability measure on $\Sigma\otimes T$ for $y\in\Upsilon$.

(a) The family $\{P_y\}_{y\in\Upsilon}$ satisfies property (d1).

In fact, it can be easily shown that the family

$$\mathcal{D}_1 := \big\{E\in\Sigma\otimes T :\ P_.(E) \text{ is } T\text{-measurable}\big\}$$

is a Dynkin class. Furthermore, for each $A\times B\in\Sigma\times T$ and $y\in\Upsilon$ we have $P_y(A\times B) = \widetilde{P}_y(A)\chi_B(y)$, implying that $P_.(A\times B)$ is T-measurable; hence $A\times B\in\mathcal{D}_1$. But since $\Sigma\times T$ is closed under finite intersections, taking into account the fact that $\mathcal{D}_1$ is a Dynkin class, we may apply the Monotone Class Theorem (cf. e.g. [2: 136B]) to get $\Sigma\otimes T\subseteq\mathcal{D}_1$; hence $\Sigma\otimes T = \mathcal{D}_1$.

(b) The family $\{P_y\}_{y\in\Upsilon}$ satisfies property (d2).
In fact, it can be easily seen that the family

$$\mathcal{D}_2 := \big\{E \in \Sigma\otimes T:\ M(E) = \int P_y(E)Q(\mathrm{d}y)\big\}$$

is a Dynkin class. For each $A\times B \in \Sigma\times T$ we have

$$M(A\times B) = \int \widetilde{P}_y(A)\delta_y(B)Q(\mathrm{d}y) = \int P_y(A\times B)Q(\mathrm{d}y);$$

hence $A\times B\in\mathcal{D}_2$. Again a monotone class argument as in (a) yields $\Sigma\otimes T = \mathcal{D}_2$.
(c) The disintegration $\{P_y\}_{y\in\Upsilon}$ of M over Q is consistent with p_Υ.
In fact, for each $B\in T$ we get

$$\begin{aligned} M(\Omega\times B) = Q(B) &= \int_B \widetilde{P}_y(\Omega)Q(\mathrm{d}y) = \int_B \chi_B(y)\widetilde{P}_y(\Omega)Q(\mathrm{d}y)\\ &= \int_B P_y(\Omega\times B)Q(\mathrm{d}y) = \int_B P_y\big(p_\Upsilon^{-1}(B)\big)Q(\mathrm{d}y),\end{aligned}$$

implying that $\int_B Q(\mathrm{d}y) = \int_B P_y\big(p_\Upsilon^{-1}(B)\big)Q(\mathrm{d}y)$ or equivalently $P_y\big(p_\Upsilon^{-1}(B)\big) = 1$ for Q-a.a. $y\in B$; hence (c) holds true. This completes the whole proof. □

Proposition 2.5. *Let M be a probability measure on $\Sigma\otimes T$ such that P and Q are the marginals of M. Then the following assertions are equivalent:*

(i) *There exists a product r.c.p. on Σ for M with respect to Q;*

(ii) *there exists a disintegration of M over Q consistent with p_Υ.*

Proof. The implication (i) $\Longrightarrow$ (ii) is immediate by Lemma 2.4.

To show the inverse implication, assume that a disintegration $\{\widetilde{P}_y\}_{y\in\Upsilon}$ of M over Q consistent with p_Υ exists. For each $y\in\Upsilon$ define the set-function $P_y\colon\Sigma\longrightarrow\mathbb{R}$ by means of

$$P_y(A) := \widetilde{P}_y(A\times\Upsilon) \qquad \text{for all} \quad A\in\Sigma.$$

Clearly $\{P_y\}_{y\in\Upsilon}$ is a family of probability measures on Σ satisfying property (D1).

To show (D2), fix on $A\times B\in\Sigma\times T$. Since $\{\widetilde{P}_y\}_{y\in\Upsilon}$ is consistent with p_Υ, we get $\widetilde{P}_y(\Omega\times B) = 1$ for Q-a.a. $y\in B$, and since also $\widetilde{P}_y(\Omega\times B^c) = 1$ for Q-a.a. $y\in B^c$, we get $\widetilde{P}_y(\Omega\times B) = 0$ for Q-a.a. $y\in B^c$, implying

$$\widetilde{P}_y(A\times B) = 0 \qquad \text{for} \quad Q\text{-a.a.} \quad y\in B^c. \tag{1}$$

Again the consistency of $\{\widetilde{P}_y\}_{y\in\Upsilon}$ with p_Υ yields for Q-a.a. $y \in B$ that

$$\widetilde{P}_y(A \times B) = \widetilde{P}_y\big((A \times \Upsilon) \cap (\Omega \times B)\big) = \widetilde{P}_y(A \times \Upsilon) = P_y(A),$$

i.e.

$$\widetilde{P}_y(A \times B) = P_y(A) \qquad \text{for} \quad Q\text{-a.a.} \quad y \in B. \tag{2}$$

Applying now conditions (1) and (2) we obtain

$$\begin{aligned} M(A \times B) &= \int \widetilde{P}_y(A \times B) Q(\mathrm{d}y) \\ &= \int_B \widetilde{P}_y(A \times B) Q(\mathrm{d}y) + \int_{B^c} \widetilde{P}_y(A \times B) Q(\mathrm{d}y) \\ &= \int_B P_y(A) Q(\mathrm{d}y); \end{aligned}$$

hence property (D2) follows. As a consequence, we get that assertion (i) holds true. □

3. The construction

Let (Ω, Σ, P) be a probability space and Θ a random variable on it. A family $\{\Sigma_i\}_{i\in I}$ of σ-subalgebras of Σ is *(P-) conditionally (stochastically) independent* relative to the σ-subalgebra T of Σ, if for each $n \in \mathbb{N}$ with $n \geq 2$ we have

$$P(E_1 \cap \cdots \cap E_n \mid T) = \prod_{j=1}^{n} P(E_j \mid T) \quad P \mid T\text{-a.s.}$$

whenever $i_1, \ldots, i_n$ are distinct members of I and $E_j \in \Sigma_{i_j}$ for every $j \leq n$. A family $\{X_i\}_{i\in I}$ of random variables on Ω is *(P-) conditionally (stochastically) independent* relative to a random variable Y on Ω, if the family $\{\sigma(X_i)\}_{i\in I}$ of σ-algebras is conditionally independent relative to the σ-algebra $\sigma(Y)$.

A stochastic process (or merely a process for short) $\{X_t\}_{t\in\mathbb{R}_+}$ on (Ω, Σ) has *conditionally independent increments* (relative to Θ), if for each $m \in \mathbb{N}$ and for each $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$, such that $0 = t_0 < t_1 < \cdots < t_m$ the increments $X_{t_j} - X_{t_{j-1}}$ $(j \in \mathbb{N}_m)$ are conditionally independent (relative to Θ). The process $\{X_t\}_{t\in\mathbb{R}_+}$ has *conditionally stationary increments* (relative to Θ) if for each $m \in \mathbb{N}$, $h \in \mathbb{R}_+$ and for each $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ such that $0 = t_0 < t_1 < \cdots < t_m$ condition

$$P_{X_{t_j+h} - X_{t_{j-1}+h} \mid \Theta} = P_{X_{t_j} - X_{t_{j-1}} \mid \Theta} \quad P \mid \sigma(\Theta)\text{-a.s.}$$

holds true.

Since the term *conditionally* is always used relative to the random variable Θ, throughout what follows, we simply write “conditionally” in the place of “conditionally relative to Θ”.

For the definition of a claim number process $\{N_t\}_{t\in\mathbb{R}_+}$ with exceptional null set Ω_N we refer to [8: Chapt. 2, Sect. 2.1, p. 17]. *In what follows, we may assume without loss of generality, that $\Omega_N = \emptyset$.*

A claim number process $\{N_t\}_{t\in\mathbb{R}_+}$ is said to be a *mixed Poisson process* on (Ω, Σ, P) (or a P-MPP for short) with parameter Θ such that $P_\Theta\big((0,\infty)\big) = 1$, if it has conditionally stationary independent increments, such that

$$P_{N_t|\Theta} = \mathbf{P}(t\Theta) \quad P \mid \sigma(\Theta)\text{-a.s.}$$

holds true for each $t \in (0,\infty)$. In particular, if the distribution of Θ is degenerate at $\theta_0 > 0$ (i.e. $P_\Theta(\{\theta_0\}) = 1$), then $\{N_t\}_{t\in\mathbb{R}_+}$ is a *P-Poisson process* with parameter θ_0 in the sense of [8: Chapt. 2, Sect. 2.3, p. 23].

Recall the following notations concerning product probability spaces. Let I be an arbitrary non-empty index set. If $\{(\Omega_i, \Sigma_i, P_i)\}_{i\in I}$ is a family of probability spaces, then for each $\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, P_J)$ the product probability space $\bigotimes_{i\in J}(\Omega_i, \Sigma_i, P_i) := \big(\prod_{i\in J}\Omega_i, \bigotimes_{i\in J}\Sigma_i, \bigotimes_{i\in J}P_i\big)$. If (Ω, Σ, P) is a probability space, we write P_I for the product measure on Ω^I and Σ_I for its domain.

Recall also some notions from the topological measure theory. Assume that $\mathcal{T}$ is a Hausdorff topology on Ω and that P is a probability measure on $\mathfrak{B}(\Omega)$. Then P is called *inner regular with respect to a family $\mathcal{F} \subseteq \mathfrak{B}(\Omega)$* if

$$P(A) = \sup\big\{P(F): \ F \in \mathcal{F}, \ F \subseteq A\big\} \qquad \text{for all} \quad A \in \mathfrak{B}(\Omega),$$

while P is said to be *outer regular with respect to the family $\mathcal{T}$* if

$$P(A) = \inf\big\{P(G): \ G \in \mathcal{T}, \ G \supseteq A\big\} \qquad \text{for all} \quad A \in \mathfrak{B}(\Omega).$$

A probability measure P on $\mathfrak{B}(\Omega)$ is called *τ-additive* if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets then

$$P\Big(\bigcup \mathcal{G}\Big) = \sup\big\{P(G): \ G \in \mathcal{G}\big\}.$$

Throughout what follows, we write $\Upsilon := (0,\infty)$, $\widetilde{\Omega} := \Upsilon^{\mathbb{N}}$, $\Omega := \widetilde{\Omega} \times \Upsilon$, $\widetilde{\Sigma} := \mathfrak{B}(\widetilde{\Omega})$ and $\Sigma := \mathfrak{B}(\Omega)$ for simplicity.

Denote by $\widetilde{\mathcal{C}}$ the family of all measurable cylinders $\widetilde{B} \in \mathfrak{B}(\widetilde{\Omega})$, i.e. of all sets $\widetilde{B} \subseteq \widetilde{\Omega}$ expressible as $\prod_{n\in\mathbb{N}} \widetilde{B}_n$, where $\widetilde{B}_n \in \mathfrak{B}(\Upsilon)$ for every $n \in \mathbb{N}$, and $\widetilde{L} := \{n \in \mathbb{N} : \ \widetilde{B}_n \neq \Upsilon\}$ is finite. Set $\widetilde{C}_n := \widetilde{B}_n$ for each $n \in \widetilde{L}$. Then

$\widetilde{B} = \prod_{k \in \widetilde{L}} \widetilde{C}_k \times \Upsilon^{\mathbb{N} \setminus \widetilde{L}}$. In the same way, $\mathcal{C}$ stands for the family of all measurable cylinders $B \in \mathfrak{B}(\Omega)$.

Theorem 3.1. *Let ν be an arbitrary probability measure on $\mathfrak{B}(\Upsilon)$, and let $Q_n(\theta)$ be probability measures on $\mathfrak{B}(\Upsilon)$ such that $Q_n(\theta) = \mathbf{Exp}(\theta)$ for all $n \in \mathbb{N}$ and for any fixed $\theta \in \Upsilon$. Then there exist a random variable Θ from Ω onto Υ, a family of probability measures $\{P_\theta\}_{\theta \in \Upsilon}$ on Σ, a unique probability measure P on Σ such that $P_\Theta = \nu$ and $\{P_\theta\}_{\theta \in \Upsilon}$ is a disintegration of P over ν consistent with Θ, and a P-MPP $\{N_t\}_{t \in \mathbb{R}_+}$ with parameter Θ, the claim interarrival process $\{W_n\}_{n \in \mathbb{N}}$ of which satisfies condition*

$$(P_\theta)_{W_n} = Q_n(\theta) \qquad \textit{for all} \quad n \in \mathbb{N},$$

if $\theta \in \Upsilon$ is fixed.

Proof. Fix on arbitrary $\theta \in \Upsilon$. If $Q_n(\theta) = \mathbf{Exp}(\theta)$ for each $n \in \mathbb{N}$, it follows that there exist a unique probability measure $\widetilde{P}_\theta := \bigotimes_{n \in \mathbb{N}} Q_n(\theta)$ on $\widetilde{\Sigma}$, and a sequence $\{\widetilde{W}_n\}_{n \in \mathbb{N}}$ of $\widetilde{P}_\theta$-independent random variables on $(\widetilde{\Omega}, \widetilde{\Sigma})$ such that

$$\widetilde{W}_n(\omega) = \omega_n = p_n(\omega) \qquad \text{for each} \quad \omega \in \widetilde{\Omega} \quad \text{and} \quad n \in \mathbb{N},$$

where $p_n \colon \Upsilon^{\mathbb{N}} \longrightarrow \Upsilon$ is the canonical projection, satisfying

$$(\widetilde{P}_\theta)_{\widetilde{W}_n} = Q_n(\theta) \qquad \text{for all} \quad n \in \mathbb{N}. \tag{3}$$

(a) The function $\theta \longmapsto \widetilde{P}_\theta(E)$ for fixed $E \in \widetilde{\Sigma}$ is $\mathfrak{B}(\Upsilon)$-measurable.
In fact, the measurability of $Q_n(\cdot)(B)$ for fixed $B \in \mathfrak{B}(\Upsilon)$ is elementary; for each $n \in \mathbb{N}$ we have $Q_n(\theta)(B) = \int h(x,\theta)\beta(dx)$, where $h(x,\theta) := \chi_B(x)\theta e^{-\theta x}$ for every $x, \theta \in \Upsilon$, hence $\theta \longmapsto Q_n(\theta)(B)$ is actually a continuous function of θ, because the integrand is continuous as a function of two variables and is dominated by an integrable function as θ varies over any bounded interval. Consequently, for any fixed $E \in \widetilde{\mathcal{C}}$ the function $\widetilde{P}_{.}(E)$ is $\mathfrak{B}(\Upsilon)$-measurable as a product of such functions.

Denote also by $\widetilde{\mathcal{D}}$ the set of all $E \in \widetilde{\Sigma}$ such that $\widetilde{P}_{.}(E)$ is a $\mathfrak{B}(\Upsilon)$-measurable function. It then can be easily seen that $\widetilde{\mathcal{D}}$ is a Dynkin class. Since $\widetilde{\mathcal{C}}$ is closed under finite intersections, we may apply the Monotone Class Theorem (cf. e.g. [2: 136B]) to get $\widetilde{\Sigma} \subseteq \widetilde{\mathcal{D}}$; hence $\widetilde{\Sigma} = \widetilde{\mathcal{D}}$.
(b) According to (a), we may now define the set-function $\widetilde{P} \colon \widetilde{\Sigma} \longrightarrow \mathbb{R}$ by means of

$$\widetilde{P}(E) := \int \widetilde{P}_\theta(E)\nu(\mathrm{d}\theta) \qquad \text{for all} \quad E \in \widetilde{\Sigma}.$$

Then $\widetilde{P}$ is a probability measure on $\widetilde{\Sigma}$ and $\{\widetilde{P}_\theta\}_{\theta\in\Upsilon}$ is a disintegration of $\widetilde{P}$ over ν.

(c) Put $P(E) := \int \widetilde{P}_\theta(E^\theta)\nu(\mathrm{d}\theta)$ for each $E \in \Sigma$. It is easy to see, that P is a probability measure on Σ such that $\{\widetilde{P}_\theta\}_{\theta\in\Upsilon}$ is a product r.c.p. on $\widetilde{\Sigma}$ for P with respect to ν.

For each $\theta \in \Upsilon$ put $P_\theta := \widetilde{P}_\theta \otimes \delta_\theta$. Obviously, each P_θ is a probability measure on Σ. So, we may apply Proposition 2.5, to get that $\{P_\theta\}_{\theta\in\Upsilon}$ is a disintegration of P over ν consistent with the canonical projection p_Υ from $\widetilde{\Omega}\times\Upsilon$ onto Υ. Clearly, putting $\Theta := p_\Upsilon$ we get $P_\Theta = \nu$.

(d) For each $n \in \mathbb{N}$, let $W_n := \widetilde{W}_n \circ p_{\widetilde{\Omega}}$. Then each W_n is the canonical projection from Ω onto Υ. As a consequence, the above fact together with condition (3) yields

$$(P_\theta)_{W_n} = P_\theta \circ p_{\widetilde{\Omega}}^{-1} \circ \widetilde{W}_n^{-1} = \widetilde{P}_\theta \circ \widetilde{W}_n^{-1} = (\widetilde{P}_\theta)_{\widetilde{W}_n} = Q_n(\theta) \qquad \text{for all} \quad n \in \mathbb{N} \tag{4}$$

and for any fixed $\theta \in \Upsilon$. But the latter together with the fact that the sequence $\{\widetilde{W}_n\}_{n\in\mathbb{N}}$ consists of $\widetilde{P}_\theta$-independent random variables implies, by applying a standard computation, that $\{W_n\}_{n\in\mathbb{N}}$ is P_θ-independent for all $\theta \in \Upsilon$.

(e) Setting $T_n := \sum_{k=1}^{n} W_k$ for all $n \in \mathbb{N}_0$ and $N_t := \sum_{n=1}^{\infty} \chi_{\{T_n \le t\}}$ for all $t \in \mathbb{R}_+$, we get a claim arrival process $\{T_n\}_{n\in\mathbb{N}_0}$ (cf. e.g. [8: Chapt. 1, Sect. 1.1, p. 6], for the definition) and a claim number process $\{N_t\}_{t\in\mathbb{R}_+}$ with empty exceptional null sets (cf. e.g. [8: Theorem 2.1.1]).

Since the process $\{W_n\}_{n\in\mathbb{N}}$ is P_θ-independent for all $\theta \in \Upsilon$ and $\{P_\theta\}_{\theta\in\Upsilon}$ is a disintegration of P over ν consistent with Θ, we may apply [5: Lemma 4.1] to get that the process $\{W_n\}_{n\in\mathbb{N}}$ is P-conditionally independent. Furthermore, condition (4) together with [5: Lemma 4.3] implies that for each $n \in \mathbb{N}$ the equality $P_{W_n|\Theta} = \mathbf{Exp}(\Theta)$ holds $P \mid \sigma(\Theta)$-a.s. true. Consequently, we may apply [5: Proposition 4.5] to obtain that $\{N_t\}_{t\in\mathbb{R}_+}$ is a P-MPP with parameter Θ. □

Remark 3.2. Denote by $\mathbf{F} := \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ the canonical filtration of the claim number process $\{N_t\}_{t\in\mathbb{R}_+}$. If $\mathcal{A}_t := \sigma\big(\mathcal{F}_t \cup \sigma(\Theta)\big)$ for each $t \in \mathbb{R}_+$, then $\mathbf{A} := \{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ is a filtration for (Ω, Σ). Moreover, set $\mathcal{F}_\infty := \sigma\big(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t\big)$ and $\mathcal{A}_\infty := \sigma\big(\mathcal{F}_\infty \cup \sigma(\Theta)\big)$. The latter yields $\mathcal{F}_\infty = \sigma(\{N_t\}_{t\in\mathbb{R}_+}) = \sigma(\{T_n\}_{n\in\mathbb{N}_0}) = \mathfrak{B}(\Omega)$, where for the second and the third equality cf. e.g. [8: Lemmas 2.1.3, 1.1.1], respectively; hence $\mathcal{A}_\infty \subseteq \mathfrak{B}(\Omega) = \mathcal{F}_\infty$ since $\sigma(\Theta) \subseteq \mathfrak{B}(\Omega)$. Consequently, $\mathcal{F}_\infty = \mathfrak{B}(\Omega) = \Sigma = \mathcal{A}_\infty$.

The above shown equality $\Sigma = \mathcal{A}_\infty$ is an assumption under which [5: Theorem 4.10] has been proven. But this equality is used in the proof of that theorem, exclusively for showing the equivalences of assertions involving claim measures (cf. e.g. [8: Chapt. 1, Sect. 1.1, p. 8] for the definition) and martingales. The equivalence (i) $\Longleftrightarrow$ (ii) of [5: Theorem 4.10], which is exactly [5: Proposition 4.5] used in the proof of Theorem 3.1, holds true without this assumption.

In the following examples, assuming the situation of Theorem 3.1, the probability measure P, the disintegration $\{P_\theta\}_{\theta\in\varUpsilon}$ of P over ν as well as the P_θ-claim measures μ_θ, for $\theta \in \varUpsilon$, and the P-claim measure μ are computed, for some important cases of MPPs.

For simplicity, write $\mathcal{E}$ for the euclidean topology on $\varUpsilon$ and $\mathcal{E}^{\mathbb{N}}$ for the product topology $\prod\limits_{n\in\mathbb{N}} \mathcal{E}_n$ with $\mathcal{E}_n = \mathcal{E}$ for all $n \in \mathbb{N}$. It is well known, that $(\widetilde{\varOmega}, \mathcal{E}^{\mathbb{N}})$ is a Polish space (cf. e.g. [3: Definitions 4A2A, Proposition 4A2Q]).

Example 3.3. Denote by $\widetilde{\mu}_\theta$ and $\widetilde{\mu}$ the $\widetilde{P}_\theta$- and $\widetilde{P}$-claim measures, where $\theta \in \varUpsilon$. Assume that $P_\varTheta = \mathbf{Ga}(\gamma, \alpha)$, i.e. that $\{N_t\}_{t\in\mathbb{R}_+}$ is a *Pólya-Lundberg process* with parameters γ and α (cf. e.g. [8: Chapt. 4, Sect. 4.3, p. 93] for the definition).
(a) Let us fix an arbitrary $\widetilde{B} = \prod\limits_{k\in\widetilde{L}} \widetilde{C}_k \times \varUpsilon^{\mathbb{N}\setminus\widetilde{L}} \in \widetilde{\mathcal{C}}$. Then we get

$$\widetilde{P}_\theta(\widetilde{B}) = \Big(\bigotimes_{n\in\mathbb{N}} Q_n(\theta)\Big)(\widetilde{B}) = \prod_{k\in\widetilde{L}} Q_k(\theta)(\widetilde{C}_k) = \prod_{k\in\widetilde{L}} \int_{\widetilde{C}_k} \theta \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k); \tag{5}$$

hence

$$\widetilde{P}(\widetilde{B}) = \int \Big[\prod_{k\in\widetilde{L}} \int_{\widetilde{C}_k} \theta \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k)\Big] P_\varTheta(\mathrm{d}\theta).$$

Then from (5) we get, for each $E \in \mathfrak{B}(\varUpsilon)$, that

$$P_\theta(\widetilde{B} \times E) = (\widetilde{P}_\theta \otimes \delta_\theta)(\widetilde{B} \times E) = \widetilde{P}_\theta(\widetilde{B})\delta_\theta(E) = \chi_E(\theta) \prod_{k\in\widetilde{L}} \int_{\widetilde{C}_k} \theta \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k);$$

hence

$$P(\widetilde{B} \times E) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int_E \Big[\prod_{k\in\widetilde{L}} \int_{\widetilde{C}_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k)\Big] \theta^\alpha \mathrm{e}^{-\gamma\theta} \beta(\mathrm{d}\theta).$$

As a consequence we get

$$P_\theta(B) = \prod_{k\in L} \int_{C_k} \theta \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k); \tag{6}$$

hence

$$P(B) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int_0^\infty \Big[\prod_{k\in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \theta^\alpha \mathrm{e}^{-\gamma\theta} \,\mathrm{d}\theta$$

for each $B = \prod_{k\in L} C_k \times \Upsilon^{\mathbb{N}\setminus L} \in \mathcal{C}$.

(b) Let $\mathcal{U} \subseteq \mathcal{C}$ be the standard basis for the product topology $\mathcal{E}^{\mathbb{N}} \times \mathcal{E}$, consisting of sets expressible as $\prod_{n\in\mathbb{N}} G_n$, where $G_n \in \mathcal{E}$ for every $n \in \mathbb{N}$, and the set $L := \{n \in \mathbb{N} : \ G_n \neq \Upsilon\}$ is finite. Write $\mathcal{U}_f$ for the set of all finite unions of members of $\mathcal{U}$, and $\mathcal{H}$ for the set of all non-empty upwards-directed families in $\mathcal{U}_f$.

If $G \in \mathcal{E}^{\mathbb{N}} \times \mathcal{E}$, then

$$\mathcal{V}_G := \{V \in \mathcal{U}_f : \ V \subseteq G\}$$

belongs in $\mathcal{H}$ and $\bigcup \mathcal{V}_G = G$. Since $(\Omega, \mathcal{E}^{\mathbb{N}} \times \mathcal{E})$ is a Polish space, it follows that the measures P and P_θ (for $\theta \in \Upsilon$) are inner regular with respect to the closed sets (cf. e.g. [3: Theorem 412E]). Moreover, they are τ-additive (cf. e.g. [3: Proposition 414O]). As a consequence we obtain

$$P(G) = \sup_{V\in\mathcal{V}_G} P(V) \qquad \text{and} \qquad P_\theta(G) = \sup_{V\in\mathcal{V}_G} P_\theta(V) \quad \text{for all } \theta \in \Upsilon. \tag{7}$$

But since P and all P_θ are inner regular with respect to the closed sets, they are outer regular with respect to the open sets, which together with (7) yields

$$P(E) = \inf_{G\in\mathcal{E}^{\mathbb{N}}\times\mathcal{E}, G\supseteq E} \ \sup_{V\in\mathcal{V}_G} P(V) \qquad \text{for all} \quad E \in \mathfrak{B}(\Omega)$$

and

$$P_\theta(E) = \inf_{G\in\mathcal{E}^{\mathbb{N}}\times\mathcal{E}, G\supseteq E} \ \sup_{V\in\mathcal{V}_G} P_\theta(V) \qquad \text{for all} \quad E \in \mathfrak{B}(\Omega) \quad \text{and all} \quad \theta \in \Upsilon.$$

(c) We focus now our interest on computing claim measures. First let $\widetilde{\mathcal{F}}_t$, $\widetilde{\mathcal{A}}_t$, $\widetilde{\mathbf{F}} := \{\widetilde{\mathcal{F}}_t\}_{t\in\mathbb{R}_+}$, $\widetilde{\mathbf{A}} := \{\widetilde{\mathcal{A}}_t\}_{t\in\mathbb{R}_+}$, $\widetilde{\mathcal{F}}_\infty$ and $\widetilde{\mathcal{A}}_\infty$ be defined in the same way as in Remark 3.2 but for the claim number process $\{\widetilde{N}_t\}_{t\in\mathbb{R}_+}$ given by $\widetilde{N}_t := \sum_{n=1}^\infty \chi_{\{\widetilde{T}_n\le t\}}$ for all $t \in \mathbb{R}_+$, where $\widetilde{T}_n := \sum_{k=1}^n \widetilde{W}_k$ for all $n \in \mathbb{N}_0$. Then we get $\widetilde{\Sigma} = \widetilde{\mathcal{A}}_\infty$. As a consequence, by [5: Theorem 4.10], we have $\widetilde{\mu}_\theta \mid \sigma(\widetilde{\mathcal{Q}}) = (\theta\widetilde{P}_\theta \otimes \lambda) \mid \sigma(\widetilde{\mathcal{Q}})$ for P_Θ-a.a. $\theta \in \Upsilon$, where

$$\widetilde{\mathcal{Q}} := \big\{\widetilde{A} \times (s,t] : \ s,t \in \mathbb{R}_+ \text{ with } s \le t, \ \ \widetilde{A} \in \widetilde{\mathcal{A}}_s\big\} \subseteq \widetilde{\mathcal{A}}_\infty \otimes \mathfrak{B};$$

hence for every $\widetilde{A} \times (s,t] \in \widetilde{\mathcal{Q}}$ we get

$$\widetilde{\mu}_\theta(\widetilde{A} \times (s,t]) = \theta(t-s)\widetilde{P}_\theta(\widetilde{A}) \qquad \text{for} \quad P_\Theta\text{-a.a. } \theta \in \Upsilon \tag{8}$$

and by [5: Lemma 4.9], we infer

$$\widetilde{\mu}(\widetilde{A} \times (s,t]) = (t-s) \int \theta \widetilde{P}_\theta(\widetilde{A}) P_\Theta(\mathrm{d}\theta). \tag{9}$$

In particular, if $\widetilde{B} \times (s,t]$ is an arbitrary set in $\widetilde{\mathcal{Q}}$ with $\widetilde{B} = \prod_{k \in \widetilde{L}} \widetilde{C}_k \times \Upsilon^{\mathbb{N} \setminus \widetilde{L}} \in \widetilde{\mathcal{A}}_s \cap \widetilde{\mathcal{C}}$, applying (5) and (8) as well as (5) and (9), we get

$$\widetilde{\mu}_\theta(\widetilde{B} \times (s,t]) = \theta^2 (t-s) \prod_{k \in \widetilde{L}} \int_{\widetilde{C}_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \qquad \text{for} \quad P_\Theta\text{-a.a.} \quad \theta \in \Upsilon$$

and

$$\widetilde{\mu}(\widetilde{B} \times (s,t]) = (t-s) \int \theta^2 \left[\prod_{k \in \widetilde{L}} \int_{\widetilde{C}_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \right] P_\Theta(\mathrm{d}\theta),$$

respectively.

Put $\mathcal{Q} := \{ A \times (s,t] : \ s,t \in \mathbb{R}_+ \ \text{with} \ s \le t, \ A \in \mathcal{A}_s \} \subseteq \mathcal{A}_\infty \otimes \mathfrak{B}$.

In the same way, since by Remark 3.2 we have $\Sigma = \mathcal{A}_\infty$, we get

$$\mu_\theta(A \times (s,t]) = \theta(t-s) P_\theta(A) \qquad \text{for} \quad P_\Theta\text{-a.a.} \ \theta \in \Upsilon \tag{10}$$

and

$$\mu(A \times (s,t]) = (t-s) \int \theta P_\theta(A) P_\Theta(\mathrm{d}\theta) \tag{11}$$

for every $A \times (s,t] \in \mathcal{Q}$. In particular, applying (6) and (10) as well as (6) and (11), we get

$$\mu_\theta(B \times (s,t]) = \theta^2 (t-s) \prod_{k \in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \qquad \text{for} \quad P_\Theta\text{-a.a.} \ \theta \in \Upsilon$$

and

$$\mu(B \times (s,t]) = \frac{\gamma^\alpha (t-s)}{\Gamma(\alpha)} \int_0^\infty \left[\prod_{k \in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \right] \theta^{\alpha+1} \mathrm{e}^{-\gamma\theta} \, \mathrm{d}\theta,$$

respectively, for every $B \times (s,t] \in \mathcal{Q}$ with $B = \prod_{k \in L} C_k \times \Upsilon^{\mathbb{N} \setminus L} \in \mathcal{A}_s \cap \mathcal{C}$.

Next we give examples for some other distributions of Θ being of particular interest for applications. We merely provide the formulas for the probability measure P and the claim measure μ in each case, without repeating the analytic computations by which they were extracted, since the latter are similar with those of Example 3.3. Note that the disintegrating probability measures P_θ as well as the claim measures μ_θ, for $\theta \in \Upsilon$, remain the same as in Example 3.3.

Example 3.4. For $P_\Theta = \mathbf{Ga}(r,\gamma,\alpha)$, that is for the case of $\{N_t\}_{t\in\mathbb{R}_+}$ being a *Delaporte process* with $P_{N_t} = \mathbf{Del}(rt,\gamma,\alpha/(\alpha+t))$ for each $t \in \mathbb{R}_+$ (cf. e.g. [4: p. 34]), we obtain

$$P(\widetilde{B}\times E) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int\limits_{E\cap[r,\infty)} \Big[\prod_{k\in\widetilde{L}} \int\limits_{\widetilde{C}_k} \mathrm{e}^{-\theta\omega_k}\beta(\mathrm{d}\omega_k)\Big]\theta(\theta-r)^{\alpha-1}\mathrm{e}^{-\gamma(\theta-r)}\beta(\mathrm{d}\theta)$$

for all $\widetilde{B}\times E \in \widetilde{\mathcal{C}}\times\mathfrak{B}(\varUpsilon)$ as in Example 3.3, implying

$$P(B) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int\limits_r^\infty \Big[\prod_{k\in L} \int\limits_{C_k} \mathrm{e}^{-\theta\omega_k}\beta(\mathrm{d}\omega_k)\Big]\theta(\theta-r)^{\alpha-1}\mathrm{e}^{-\gamma(\theta-r)}\,\mathrm{d}\theta$$

for all $B = \prod\limits_{k\in L} C_k \times \varUpsilon^{\mathbb{N}\setminus L} \in \mathcal{C}$. Moreover, by (6) and (11) we get

$$\mu(B\times(s,t]) = \frac{\gamma^\alpha(t-s)}{\Gamma(\alpha)} \int\limits_r^\infty \Big[\prod_{k\in L} \int\limits_{C_k} \mathrm{e}^{-\theta\omega_k}\beta(\mathrm{d}\omega_k)\Big]\theta^2(\theta-r)^{\alpha-1}\mathrm{e}^{-\gamma(\theta-r)}\,\mathrm{d}\theta$$

for all $B\times(s,t] \in \mathcal{Q}$ as in Example 3.3.

Example 3.5. For $P_\Theta = p\delta_{\theta_1} + (1-p)\delta_{\theta_2}$ with $\theta_1,\theta_2 \in \varUpsilon$ and $p \in (0,1)$, that is for the case of $\{N_t\}_{t\in\mathbb{R}_+}$ being a *double Poisson process* (cf. e.g. [4: p. 77]), we get

$$P(\widetilde{B}\times E) = p\chi_E(\theta_1)\prod_{k\in\widetilde{L}}\int\limits_{\widetilde{C}_k}\theta_1\mathrm{e}^{-\theta_1\omega_k}\beta(\mathrm{d}\omega_k) + (1-p)\chi_E(\theta_2)\prod_{k\in\widetilde{L}}\int\limits_{\widetilde{C}_k}\theta_2\mathrm{e}^{-\theta_2\omega_k}\beta(\mathrm{d}\omega_k)$$

for all $\widetilde{B}\times E \in \widetilde{\mathcal{C}}\times\mathfrak{B}(\varUpsilon)$ as in Example 3.3, implying

$$P(B) = p\prod_{k\in L}\int\limits_{C_k}\theta_1\mathrm{e}^{-\theta_1\omega_k}\beta(\mathrm{d}\omega_k) + (1-p)\prod_{k\in L}\int\limits_{C_k}\theta_2\mathrm{e}^{-\theta_2\omega_k}\beta(\mathrm{d}\omega_k)$$

for all $B = \prod\limits_{k\in L} C_k \times \varUpsilon^{\mathbb{N}\setminus L} \in \mathcal{C}$. Moreover, by (6) and (11) we get

$$\mu(B\times(s,t]) = (t-s)\Big[p\theta_1^2\prod_{k\in L}\int\limits_{C_k}\mathrm{e}^{-\theta_1\omega_k}\beta(\mathrm{d}\omega_k) + (1-p)\theta_2^2\prod_{k\in L}\int\limits_{C_k}\mathrm{e}^{-\theta_2\omega_k}\beta(\mathrm{d}\omega_k)\Big]$$

for all $B\times(s,t] \in \mathcal{Q}$ as in Example 3.3.

Example 3.6. For $P_\Theta = \mathbf{U}(0,\alpha)$ with $\alpha > 0$, that is for the case of $\{N_t\}_{t\in\mathbb{R}_+}$ being a *uniform-Poisson process* (cf. e.g. [4: p. 76]), we obtain

$$P(\widetilde{B}\times E) = \frac{1}{\alpha}\int\limits_{E\cap[0,\alpha)}\Big[\prod_{k\in\widetilde{L}}\int\limits_{\widetilde{C}_k}\theta\mathrm{e}^{-\theta\omega_k}\beta(\mathrm{d}\omega_k)\Big]\beta(\mathrm{d}\theta)$$

for all $\widetilde{B} \times E \in \widetilde{\mathcal{C}} \times \mathfrak{B}(\Upsilon)$ as in Example 3.3, implying

$$P(B) = \frac{1}{\alpha} \int_0^\alpha \Big[\prod_{k \in L} \int_{C_k} \theta \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \, \mathrm{d}\theta$$

for all $B = \prod_{k \in L} C_k \times \Upsilon^{\mathbb{N} \setminus L} \in \mathcal{C}$. Moreover, by (6) and (11) we get

$$\mu(B \times (s,t]) = \frac{t-s}{\alpha} \int_0^\alpha \theta^2 \Big[\prod_{k \in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \, \mathrm{d}\theta$$

for all $B \times (s,t] \in \mathcal{Q}$ as in Example 3.3.

Example 3.7. For $P_\Theta = \mathbf{Be}(r, \alpha, \gamma)$, that is for the case of *beta-Poisson distributed claim numbers* (cf. e.g. [4: p. 42]), the probability density function of Θ is given by $g_\Theta(\theta) = \big[1/B(\alpha,\gamma)\big] \theta^{\alpha-1} (r-\theta)^{\gamma-1} r^{1-(a+\gamma)}$ for all $\theta \in (0,r)$; hence we get

$$P(\widetilde{B} \times E) = \frac{r^{1-(a+\gamma)}}{B(\alpha,\gamma)} \int_{E \cap (0,r)} \Big[\prod_{k \in \widetilde{L}} \int_{\widetilde{C}_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \theta^\alpha (r-\theta)^{\gamma-1} \beta(\mathrm{d}\theta)$$

for all $\widetilde{B} \times E \in \widetilde{\mathcal{C}} \times \mathfrak{B}(\Upsilon)$ as in Example 3.3, implying

$$P(B) = \frac{r^{1-(a+\gamma)}}{B(\alpha,\gamma)} \int_0^r \Big[\prod_{k \in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \theta^\alpha (r-\theta)^{\gamma-1} \, \mathrm{d}\theta$$

for all $B = \prod_{k \in L} C_k \times \Upsilon^{\mathbb{N} \setminus L} \in \mathcal{C}$. Moreover, by (6) and (11) we get

$$\mu(B \times (s,t]) = \frac{r^{1-(a+\gamma)}(t-s)}{B(\alpha,\gamma)} \int_0^r \Big[\prod_{k \in L} \int_{C_k} \mathrm{e}^{-\theta\omega_k} \beta(\mathrm{d}\omega_k) \Big] \theta^{\alpha+1} (r-\theta)^{\gamma-1} \, \mathrm{d}\theta$$

for all $B \times (s,t] \in \mathcal{Q}$ as in Example 3.3.

At this point it is worth noticing that in the above examples the structure distributions P_Θ, that is the corresponding MPPs, under consideration find a wide range of applications in insurance mathematics and risk theory. The beta distribution in the last example is useful for certain biological ones as well. For more on applications of Examples 3.4, 3.6 and 3.7, cf. e.g. [4: pp. 34, 46, 43], respectively.

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