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COMPARISON RESULTS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

Li Xiao

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ABSTRACT. Comparison principles play an important role in the qualitative and quantitative study of differential equations. In this paper, we investigate a first order functional differential equations with impulses and establish new comparison results.

1. Introduction

Impulsive differential equations are a basic tool to study evolution processes that are subject to abrupt in their state. For instance, many biological, physical and engineering applications exhibit impulsive effects [1,2]. Therefore it is of the utmost importance to develop a general theory for differential equations with impulses.

It is well-known that comparison principles play an important role in the qualitative theory of differential equation, see [5–12], for example, they are essential for developing the monotone iterative method, a powerful theoretical method, which permits us to construct a sequence of approximate solutions converging to a solution of certain differential equations problems [8–11]. So the study of comparison results has attracted the interest of many researchers, see [3, 4] and relevant references therein.

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In this paper, we consider a periodic boundary value problem for a first-order impulsive differential equation of the form

$$\begin{aligned} v'(t) + mv(t) + [\varphi v](t) &= \sigma(t), \quad \text{a.e. } t \in J_0 = J \backslash \{t_1, \dots, t_p\}, \\ \Delta v(t) &= I_k(v(t)), \qquad t = t_k, \quad k = 1, 2, \dots, p, \\ v(0) &= v(T) + \lambda \end{aligned} \tag{1.1}$$

where $J = [0, T]$, $m, \lambda \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, p$, $\sigma \in L^1(J)$, $\varphi \colon L^1(J) \to L^1(J)$, $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, $v(t_k^+)$ and $v(t_k^-)$ denote the right and left limits of $v(t)$ at $t = t_k$, respectively.

Let $v \colon J \to \mathbb{R}$. For $j = 0, 1, \dots, p$, define the functions $v_j \colon (t_j, t_{j+1}) \to \mathbb{R}$, $v_j(t) = v(t)$. Consider the following Banach spaces [13, 14]:

$$E = \bigl\{ v \colon J \to \mathbb{R} : \ v_j \in L^1(t_j, t_{j+1}), \ j = 0, 1, \dots, p \bigr\}$$

and

$$\begin{aligned} E^1 = \bigl\{ v \colon J \to \mathbb{R} : \ v_j \in W^{1,1}(t_j, t_{j+1}), \ j = 0, 1, \dots, p, \ v(0^+) = v(0), \\ v(t_j^-) = v(t_j) \text{ for } j = 1, \dots, p+1, \\ \text{and } v(t_j^+) \text{ exists for } j = 1, \dots, p \bigr\} \end{aligned}$$

with the norms

$$\|v\|_E = \sum_{j=0}^{p} \|v_j\|_{L^1(t_j, t_{j+1})}, \qquad \|v\|_{E^1} = \sum_{j=0}^{p} \|v_j\|_{W^{1,1}(t_j, t_{j+1})}.$$

By a solution of (1.1), we mean a function $v \in E^1$ satisfying (1.1).

In the recent paper [3], the author considered the corresponding form of (1.1) without impulses

$$\begin{aligned} v'(t) + mv(t) + [p(v)](t) = \sigma(t), \quad \text{a.e. } t \in I = [0, T], \\ v(0) = v(T) + \lambda, \end{aligned} \tag{1.2}$$

and presented several comparison results for (1.2). As $p \equiv 0$, the following theorem is one of the main results of [3].

Theorem 1.1. *Consider problem* (1.2) *with $p \equiv 0$, $m > 0$, $\lambda \in \mathbb{R}$, and $\sigma \in L^1(I)$. If*

$$\int_0^T \mathrm{e}^{-m(T-s)} \sigma(s) \, \mathrm{d}s + \lambda \le 0, \tag{1.3}$$

then $v \le 0$ on I.

We note that Theorem 1.1 is not valid as $\sigma(t) > 0$, the following example is an illustration.

Example 1.1. Let $T = 2$, $m = 1$, $\sigma(t) = \mathrm{e}^t - \frac{1}{2}$, $\lambda = \frac{1}{2}(1 - \mathrm{e}^T)$. We consider

$$\begin{aligned} v'(t) + v(t) &= \mathrm{e}^t - \frac{1}{2}, \qquad t \in [0,2], \\ v(0) &= v(2) + \lambda. \end{aligned} \tag{1.4}$$

By inspection, we see that $v(t) = \frac{1}{2}(\mathrm{e}^t - 1)$ is a solution of (1.4), and

$$\int_0^2 \mathrm{e}^{-m(2-s)}\sigma(s)\,\mathrm{d}s + \lambda = 0.$$

That is, the inequality (1.3) is valid, but $v(t) > 0$ in $(0,2]$.

If (1.2) is subject to impulsive perturbations, e.g. (1.1), the sign of the solution of (1.2) may or may not continue to persist under impulsive perturbations. Here we have a question: what is the sufficient condition for the persistence of sign of solutions of (1.2) under impulsive perturbation? Solving the above problem is the main goal of this paper. As a corollary of our results, the above Theorem 1.1 will be improved.

2. Linear problem

In this section, we consider the linear problem of (1.1)

$$\begin{aligned} v'(t) + mv(t) &= \sigma(t), \quad \text{a.e. } t \in J_0, \\ \Delta v(t) &= c_k v(t), \qquad t = t_k, \\ v(0) &= v(T) + \lambda. \end{aligned} \tag{2.1}$$

We know that BVP (2.1) is not always solvable ([7]), but we have the following solvability theorem ([4: Proposition 2]).

Lemma 2.1. *If* $\prod_{k=1}^{p} (1 + c_k) \neq \mathrm{e}^{mT}$, *then problem* (2.1) *admits a unique solution* $v \in E^1$, *and the unique solution is given by*

$$v(t) = \mathrm{e}^{-mt} \prod_{0<t_k<t} (1 + c_k) \left[\lambda + \int_0^T \prod_{s<t_k<T} (1 + c_k)\mathrm{e}^{-m(T-s)}\sigma(s)\,\mathrm{d}s\right] \cdot \left(1 - \mathrm{e}^{-mT} \prod_{k=1}^{p}(1 + c_k)\right)^{-1} + \int_0^t \prod_{s<t_k<t} (1 + c_k)\mathrm{e}^{-m(t-s)}\sigma(s)\,\mathrm{d}s, \qquad t \in J.$$

Considering the sign of λ, we can get the following theorem:

Theorem 2.1. *Suppose $m > 0$, $\lambda \le 0$, $\sigma \le 0$ a.e. on J, $c_k \ge -1$, and $\prod_{k=1}^{p}(1+c_k) < \mathrm{e}^{mT}$. If $v(t)$ is a solution of problem (2.1), then $v(t) \le 0$ on $t \in J$.*

P r o o f. Since $c_k \ge -1$, $\lambda \le 0$, $\sigma \le 0$ a.e. on J and $\mathrm{e}^{-mT} \prod_{k=1}^{p}(1+c_k) < 1$, by Lemma 2.1, we obtain

$$v(t) \le 0.$$

The proof is complete. □

If we don't need the sign of m and λ, we have:

Theorem 2.2. *Suppose $m \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\sigma \le 0$ a.e. on J, $c_k \ge -1$, and $v(0) \le 0$. If $v(t)$ is any solution of problem (2.1), then $v(t) \le 0$ on J.*

P r o o f. For $t \in [0, t_1]$, we have

$$v(t)\mathrm{e}^{mt} = v(0) + \int_0^t \sigma(s)\mathrm{e}^{ms}\,\mathrm{d}s \le 0,$$

thus

$$v(t) \le 0.$$

In particular, $v(t_1) \le 0$, and so

$$v(t_1^+) = (1+c_1)v(t_1) \le 0.$$

For $t \in (t_1, t_2]$, we have

$$v(t)\mathrm{e}^{mt} = v(t_1^+)\mathrm{e}^{mt_1} + \int_{t_1}^t \sigma(s)\mathrm{e}^{ms}\,\mathrm{d}s \le 0,$$

hence

$$v(t) \le 0.$$

By a simple induction, we can prove, in general, that

$$v(t) \le 0 \qquad \text{for} \quad t \in [0, T],$$

which completes the proof. □

Remark 2.1. In this theorem we don't consider the sign of m, so it is a generalization of [3: Proposition 1] even if there are no impulses in (2.1).

Theorem 2.3. *Let $m > 0$, $\lambda \in \mathbb{R}$, $\sigma(t) \le 0$ a.e. on J, $c_k \ge -1$ and*

$$\int_0^T \prod_{s<t_k<T} (1+c_k) \mathrm{e}^{-m(T-s)} \sigma(s) \, \mathrm{d}s + \lambda \le 0, \tag{2.2}$$

$$\prod_{k=1}^{p} (1+c_k) < \mathrm{e}^{mT}. \tag{2.3}$$

If $v(t)$ is any solution of problem (2.1), then $v(t) \le 0$ on J.

P r o o f. For a.e. $t \in J$, by Lemma 2.1, we have

$$\begin{aligned} v(t) = \mathrm{e}^{-mt} \prod_{0<t_k<t} (1+c_k) \Big[\lambda + \int_0^T \prod_{s<t_k<T} (1+c_k) \mathrm{e}^{-m(T-s)} \sigma(s) \, \mathrm{d}s\Big] \\ \cdot \Big(1 - \mathrm{e}^{-mT} \prod_{k=1}^{p} (1+c_k)\Big)^{-1} + \int_0^t \prod_{s<t_k<t} (1+c_k) \mathrm{e}^{-m(t-s)} \sigma(s) \, \mathrm{d}s. \end{aligned}$$

(2.3), (2.4) and the above inequality imply

$$v(t) \le 0 \qquad \text{on} \quad J.$$

The proof is complete. □

Remark 2.2. This theorem improves Theorem 1.1 when $c_k = 0$, for all k.

Define $\sigma^+(t) = \max\{\sigma(t), 0\}$, $\sigma^-(t) = -\min\{\sigma(t), 0\}$, then $\sigma^+(t) \ge 0$, $\sigma^-(t) \ge 0$ and $\sigma(t) = \sigma^+(t) - \sigma^-(t)$.

Theorem 2.4. *Suppose $m > 0$, $\lambda \le 0$, $c_k \ge -1$, $\prod_{k=1}^{p} (1+c_k) < \mathrm{e}^{mT}$ and*

$$\mathrm{e}^{mt} \int_0^t \prod_{s<t_k<t} (1+c_k) \sigma^+(s) \, \mathrm{d}s \le \int_0^t \prod_{s<t_k<t} (1+c_k) \sigma^-(s) \, \mathrm{d}s.$$

Then $v(t) \le 0$ on J.

P r o o f. In view of Lemma 1, we get

$$\begin{aligned} v(t) = \mathrm{e}^{-mt} \prod_{0<t_k<t} (1+c_k) \Big[\lambda + \int_0^T \prod_{s<t_k<T} (1+c_k) \mathrm{e}^{-m(T-s)} \sigma(s) \, \mathrm{d}s\Big] \\ \times \Big(1 - \mathrm{e}^{-mT} \prod_{k=1}^{p} (1+c_k)\Big)^{-1} + \int_0^t \prod_{s<t_k<t} (1+c_k) \mathrm{e}^{-m(t-s)} \sigma(s) \, \mathrm{d}s, \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-mt} \prod_{0<t_k<t} (1+c_k) \\
&\quad \times \Bigg[\lambda + \int_0^T \prod_{s<t_k<T} (1+c_k)\mathrm{e}^{-m(T-s)}\sigma^+(s)\,\mathrm{d}s \\
&\quad - \int_0^T \prod_{s<t_k<T} (1+c_k)\mathrm{e}^{-m(T-s)}\sigma^-(s)\,\mathrm{d}s\Bigg] \\
&\quad \times \Big(1 - \mathrm{e}^{-mT}\prod_{k=1}^{p}(1+c_k)\Big)^{-1} + \int_0^t \prod_{s<t_k<t} (1+c_k)\mathrm{e}^{-m(t-s)}\sigma^+(s)\,\mathrm{d}s \\
&\quad - \int_0^t \prod_{s<t_k<t} (1+c_k)\mathrm{e}^{-m(t-s)}\sigma^-(s)\,\mathrm{d}s \\
&\le \mathrm{e}^{-mt} \prod_{0<t_k<t} (1+c_k) \\
&\quad \times \Bigg[\int_0^T \prod_{s<t_k<T} (1+c_k)\sigma^+(s)\,\mathrm{d}s - \mathrm{e}^{-mT}\int_0^T \prod_{s<t_k<T} (1+c_k)\sigma^-(s)\,\mathrm{d}s\Bigg] \\
&\quad \times \Big(1 - \mathrm{e}^{-mT}\prod_{k=1}^{p}(1+c_k)\Big)^{-1} + \int_0^t \prod_{s<t_k<t} (1+c_k)\sigma^+(s)\,\mathrm{d}s \\
&\quad - \mathrm{e}^{-mt}\int_0^t \prod_{s<t_k<t} (1+c_k)\sigma^-(s)\,\mathrm{d}s \le 0.
\end{aligned}
$$

This proof is complete. □

3. Nonlinear problem

In this section, we consider the equations

$$
\begin{aligned}
v'(t) + mv(t) + [\varphi v](t) &= \sigma(t), \quad \text{a.e. } t \in J_0, \\
\Delta v(t_k) &= I_k(v(t_k)), \qquad k = 1, 2, \ldots, p, \\
v(0) &= v(T) + \lambda,
\end{aligned}
\tag{3.1}
$$

where $\varphi\colon L^1(J) \to L^1(J)$, $I_k \in C(\mathbb{R}, \mathbb{R})$.

THEOREM 3.1. *Let $\lambda \le 0$, $\sigma \le 0$ a.e. on J, $I_k(v) \le c_k v$, $c_k \ge -1$, $\prod\limits_{k=1}^{p} (1+c_k)\mathrm{e}^{-(m-n)T} < 1$, where n is a constant, and assume*

$$-\varphi(v) \le nv \quad a.e.\ on\ J \qquad for\ every \quad v \in L^1(t_j, t_{j+1}).$$

If $v(t)$ is a solution of (3.1)*, then $v(t) \le 0$ on J.*

Proof. For a.e. $t \in J_0$, we have

$$v'(t) + (m-n)v(t) = \sigma(t) - [\varphi v](t) - nv(t) = \sigma_1(t) \le 0,$$

and

$$v(t_k^+) \le (1+c_k)v(t_k).$$

By [1: Corollary 2.4], we obtain

$$\begin{aligned} v(t) &\le v(0) \prod_{0<t_k<t} (1+c_k)\mathrm{e}^{-(m-n)t} + \int_0^t \prod_{s<t_k<t} (1+c_k)\mathrm{e}^{-(m-n)(t-s)}\sigma_1(s)\,\mathrm{d}s \\ &\le v(0) \prod_{0<t_k<t} (1+c_k)\mathrm{e}^{-(m-n)t}. \end{aligned}$$

Clearly, it is sufficient to show that $v(0) \le 0$. If it is not true, then $v(0) > 0$. Therefore we arrive at

$$v(0) = v(T) + \lambda \le v(T) \le v(0) \prod_{k=1}^{p} (1+c_k)\mathrm{e}^{-(m-n)T} < v(0),$$

a contradiction and so $v(0) \le 0$. Therefore $v(t) \le 0$. The proof is complete. □

To deal with perturbations of a more general form, we introduce some new conditions. We consider perturbation φ verifying for every $v_i \in C(t_i, t_{i+1})$, $\varphi(v_i) \in L^\infty(J)$, $i = 0, 1, \dots, p$. For $v \in L^\infty(J)$, we define, as usual, the essential infimum of v on J as the least upper bound of constants β such that $v(t) \ge \beta$ a.e. on J, and it is denoted by $\operatorname*{ess\,inf}\limits_{t\in J} v(t)$.

THEOREM 3.2. *Let $m > 0$, $\lambda \le 0$, $\sigma \le 0$ a.e. on J, $I_k(v) \le c_k v$, $-1 < c_k \le 0$ and for every $\tau \in J$,*

$$v_i \in C(t_i, t_{i+1}), \qquad \operatorname*{ess\,inf}_{t\in[0,\tau]} [\varphi v](t) \ge n \inf_{t\in[0,\tau]} v(t), \quad n > 0. \tag{3.2}$$

Suppose

$$\frac{\prod\limits_{k=1}^{p} (1+c_k)}{\int\limits_0^T \prod\limits_{s<t_k<T} (1+c_k)\,\mathrm{d}s} \ge n\mathrm{e}^{mT} \tag{3.3}$$

holds. If v is a solution to (3.1)*, then $v(t) \le 0$ in J.*

P r o o f. Suppose the conclusion is not valid, then there exists $s \in [0,T]$ such that $\sup\limits_{t\in J} v(t) = v(s) > 0$. Thus we have the following two possible cases.

Case 1. If $v(0) < 0$. Thus there exists $s_1 \in [0,s]$ such that $v(s_1) = \inf\limits_{t\in[0,s]} v(t) = -h, h > 0$. We assume $s_1 \neq t_k^+$, if $s_1 = t_k^+$ for some k, we can prove it in a similar way. For a.e. $t \in [s_1, s] \cap J_0$, we have

$$\begin{aligned} v'(t) + mv(t) &\le -[\varphi v](t) \le \operatorname*{ess\,sup}_{[s_1,s]}\{-[\varphi v](t)\} \le \operatorname*{ess\,sup}_{[0,s]}\{-[\varphi v](t)\} \\ &= -\operatorname*{ess\,inf}_{[0,s]}\,[\varphi v](t) \le -n \inf_{[0,s]} v(t) = nh, \end{aligned}$$

and so

$$v'(t) \le -mv(t) + nh.$$

Since $v(t_k^+) \le (1+c_k)v(t_k)$, thus

$$\begin{aligned} 0 < v(s) &\le v(s_1) \prod_{s_1<t_k<s} (1+c_k)\mathrm{e}^{-m(s-s_1)} + \int_{s_1}^{s} \prod_{u<t_k<s} (1+c_k)\mathrm{e}^{-m(s-u)} nh\,\mathrm{d}u \\ &\le -h \prod_{s_1<t_k<s} (1+c_k)\mathrm{e}^{-mT} + \int_{s_1}^{s} \prod_{u<t_k<s} (1+c_k) nh\,\mathrm{d}u, \end{aligned}$$

we obtain

$$\frac{\prod\limits_{k=1}^{p}(1+c_k)}{\int\limits_0^T \prod\limits_{u<t_k<T}(1+c_k)\,\mathrm{d}u} \le \frac{\prod\limits_{s_1<t_k<s}(1+c_k)}{\int\limits_{s_1}^{s} \prod\limits_{u<t_k<s}(1+c_k)\,\mathrm{d}u} < n\mathrm{e}^{mT},$$

which contradicts (3.3).

Case 2. $v(0) \ge 0$. It is clear that $v(T) \ge 0$.

(i). If $v(t) \ge 0$ for all $t \in [0,T]$, then

$$v'(t) \le -mv(t) - [\varphi v](t) + \sigma(t) \le 0, \quad \text{a.e. } t \in J_0,$$

and

$$v(t_k^+) \le (1+c_k)v(t_k) \le v(t_k), \qquad k = 1, \ldots, p.$$

It follows that $v(t)$ is nonincreasing in J. Thus $v(0) \ge v(t) \ge v(T)$, but $v(0) \le v(T)$, those show that $v(t) = c$ for $t \in [0,T]$. For $c = \text{const} > 0$, we can deduce

$$0 = v'(t) = -mc - nc + \sigma(t) < 0,$$

this is a contradiction.

(ii). There exists $s_2 \in (0,T)$ such that $v(s_2) = \inf_{[0,T]} v(t) = -h_1 < 0$. In a similar way as the proof of case 1, for a.e. $t \in [s_2, T]$, we obtain

$$v'(t) \leq -mv(t) + nh_1,$$
$$v(t_k^+) \leq (1 + c_k)v(t_k).$$

Therefore

$$v(T) \leq v(s_2) \prod_{s_2<t_k<T} (1 + c_k)\mathrm{e}^{-m(T-s_2)} + \int_{s_2}^{T} \prod_{u<t_k<T} (1 + c_k)\mathrm{e}^{-m(T-u)} nh_1 \,\mathrm{d}u,$$

that is

$$0 < -h_1 \prod_{s_2<t_k<T} (1 + c_k)\mathrm{e}^{-mT} + \int_{s_2}^{T} \prod_{u<t_k<T} (1 + c_k)nh_1 \,\mathrm{d}u.$$

Thus

$$\prod_{k=1}^{p}(1+c_k) \leq \prod_{s_2<t_k<T}(1+c_k) < \int_{s_2}^{T} \prod_{u<t_k<T}(1+c_k)n\mathrm{e}^{mT} \,\mathrm{d}u \leq n\mathrm{e}^{mT} \int_{0}^{T} \prod_{u<t_k<T}(1+c_k) \,\mathrm{d}u,$$

so

$$\frac{\prod\limits_{k=1}^{p} (1 + c_k)}{\int\limits_{0}^{T} \prod\limits_{u<t_k<T} (1 + c_k) \,\mathrm{d}u} < n\mathrm{e}^{mT},$$

which is a contradiction with (3.3). Therefore $v(t) \leq 0$ on J. The proof is complete. □

Now, consider problem (3.1) with the perturbation given by

$$[\varphi v](t) = nv(\theta(t)), \tag{3.4}$$

where $n \geq 0$, and $\theta\colon J \to J$ is continuous. If $I_k(v) \equiv 0$, then the problem to consider is

$$\begin{aligned} v'(t) + mv(t) + nv(\theta(t)) &= \sigma(t), \quad \text{a.e. } t \in J, \\ v(0) &= v(T) + \lambda. \end{aligned} \tag{3.5}$$

It is evident that if θ satisfies

$$\theta(t) \leq t, \qquad \text{for a.e.} \quad t \in J, \tag{3.6}$$

then the functional given by (3.4) satisfies (3.2).

In [8], the authors proved the following maximum principle for problem (3.5) when $\sigma \in C(J)$, but the proof is identical for $\sigma \in L^1(J)$. It is an immediate result of Theorem 3.2.

Corollary 3.1. ([8: Theorem 3.1]) *Consider problem* (3.5) *with* $\theta\colon J \to J$ *continuous and satisfying* (3.6). *Let* $v \in W^{1,1}(J)$ *be a solution of* (3.5) *with* $\sigma \le 0$ *a.e. on* J, *and* $m > 0$, $n \ge 0$, $\lambda \le 0$. *If*

$$nT\mathrm{e}^{mT} \le 1,$$

then $v \le 0$ *on* J.

4. Examples

We conclude this section with two examples showing the applications of the above theorems.

Example 4.1. Consider the following boundary value problem

$$\begin{aligned} v'(t) + \frac{1}{2}v(t) + [\varphi(v)](t) &= -t^2, \qquad t \in \Big[0, \frac{1}{2}\Big], \ \ t \ne \frac{1}{4}, \\ \Delta v(t_1) &= -\frac{1}{2}v(t_1), \qquad t_1 = \frac{1}{4}, \\ v(0) &= v(T) + \lambda, \end{aligned} \tag{4.1}$$

where $\lambda \le 0$, $[\varphi(v)](t) = \mathrm{e}^{v(t)}$, $m = \frac{1}{2}$, $c_1 = -\frac{1}{2}$, $T = \frac{1}{2}$. It is clear that $\mathrm{e}^v > v$, so we take $n = 1$. We have

$$\frac{\prod\limits_{k=1}^{p}(1+c_k)}{\int\limits_0^T \prod\limits_{s<t_k<T}(1+c_k)\,\mathrm{d}s} = \frac{4}{3} > \mathrm{e}^{\frac{1}{4}} = n\mathrm{e}^{mT},$$

which shows that (3.3) holds. By Theorem 3.2, if $v(t)$ is a solution of (4.1), then $v(t) \le 0$.

Example 4.2. Consider the following equations

$$\begin{aligned} v'(t) + \frac{1}{12}v(t) &= t - \frac{3}{4}, \qquad t \in [0,1], \ \ t \ne \frac{3}{4}, \\ \Delta v(t_1) &= -\frac{3}{4}v(t_1), \qquad t_1 = \frac{3}{4}, \\ v(0) &= v(T) + \lambda, \end{aligned} \tag{4.2}$$

where $\lambda \le 0$. Taking

$$\sigma^+(t) = \begin{cases} 0, & t \in \big[0, \frac{3}{4}\big] \\ t - \frac{3}{4}, & t \in \big[\frac{3}{4}, 1\big], \end{cases} \qquad \sigma^-(t) = \begin{cases} \frac{3}{4} - t, & t \in \big[0, \frac{3}{4}\big] \\ 0, & t \in \big[\frac{3}{4}, 1\big]. \end{cases}$$

Evidently, for $0 \le t \le \frac{3}{4}$,

$$\mathrm{e}^{mt}\int\limits_0^t \prod_{s<t_k<t}(1+c_k)\sigma^+(s)\,\mathrm{d}s \le \int\limits_0^t \prod_{s<t_k<t}(1+c_k)\sigma^-(s)\,\mathrm{d}s.$$

For $\frac{3}{4} < t \le 1$, we have

$$\begin{aligned} \mathrm{e}^{mt} \int_0^t \prod_{s<t_k<t} (1+c_k)\sigma^+(s)\,\mathrm{d}s &= \mathrm{e}^{\frac{1}{12}t} \int_{\frac{3}{4}}^t \Big(s-\frac{3}{4}\Big)\,\mathrm{d}s \\ &= \frac{1}{2}\mathrm{e}^{\frac{1}{12}t}\Big(t-\frac{3}{4}\Big)^2 \le \frac{1}{32}\mathrm{e}^{\frac{1}{12}} < \frac{9}{128}, \end{aligned}$$

and

$$\int_0^t \prod_{s<t_k<t} (1+c_k)\sigma^-(s)\,\mathrm{d}s = \int_0^{\frac{3}{4}} \frac{1}{4}\Big(\frac{3}{4}-s\Big)\,\mathrm{d}s = \frac{9}{128}.$$

Thus, by Theorem 2.4, if $v(t)$ is a solution of (4.2), then $v(t) \le 0$.

Remark 4.1. We note that

$$\begin{aligned} \int_0^T \prod_{s<t_k<T} (1+c_k)\mathrm{e}^{ms}\sigma(s)\,\mathrm{d}s &= \frac{1}{4}\int_0^{\frac{3}{4}} \mathrm{e}^{\frac{1}{12}s}\Big(s-\frac{3}{4}\Big)\,\mathrm{d}s + \int_{\frac{3}{4}}^1 \mathrm{e}^{\frac{1}{12}s}\Big(s-\frac{3}{4}\Big)\,\mathrm{d}s \\ &= \frac{81}{8} + 27\mathrm{e}^{\frac{1}{16}} - \frac{69}{2}\mathrm{e}^{\frac{1}{12}} \approx 1.353 > 0, \end{aligned}$$

which shows that [4: condition (14)] is not satisfied. So our result extends the corresponding result of [4].

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School of Mathematical Sciences
and Computing Technology
Central South University
Changsha, Hunan
410083
P.R. CHINA
E-mail: xiaolimaths@sina.com