Mathematica Slovaca

DOI: 10.2478/s12175-012-0080-3
Math. Slovaca **63** (2013), No. 1, 41–52

SEMIGROUP ACTIONS ON ORDERED GROUPOIDS

Niovi Kehayopulu* — Michael Tsingelis**

(*Communicated by Miroslav Ploščica*)

ABSTRACT. In this paper we prove that if S is a commutative semigroup acting on an ordered groupoid G, then there exists a commutative semigroup $\widetilde{S}$ acting on the ordered groupoid $\widetilde{G} := (G \times S)/\overline{\rho}$ in such a way that G is embedded in $\widetilde{G}$. Moreover, we prove that if a commutative semigroup S acts on an ordered groupoid G, and a commutative semigroup $\overline{S}$ acts on an ordered groupoid $\overline{G}$ in such a way that G is embedded in $\overline{G}$, then the ordered groupoid $\widetilde{G}$ can be also embedded in $\overline{G}$. We denote by $\overline{\rho}$ the equivalence relation on $G \times S$ which is the intersection of the quasi-order ρ (on $G \times S$) and its inverse ρ^{-1}.

1. Introduction and prerequisites

Actions of commutative semigroups on groupoids have been considered by Tamura and Burnell in [2], based on the following definition: If G is a groupoid and Γ a commutative semigroup, the pair (G, Γ) is called *a groupoid G with Γ* if there is a mapping of $\Gamma \times G$ into G, $(\alpha, x) \to \alpha x$ such that $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha\beta)x = \alpha(\beta x) = (\beta\alpha)x$ and $\alpha x = \alpha y$ implies $x = y$ for every $\alpha, \beta \in \Gamma$ and $x, y \in G$. In their paper in [2], they proved that each (G, Γ) can be embedded in a groupoid $\overline{G}$ with $\overline{\Gamma}$, and studied groupoids G with Γ, in general. In the present paper we study actions of commutative semigroups on ordered groupoids. Keeping the terminology in [2], we could say "an ordered groupoid G with S" instead of saying "S acts on the ordered groupoid G".

A relation ρ on an nonempty set S is called a quasi-order on S if it is reflexive and transitive. The concept of quasi-order plays an essential role in studying the structure of ordered groupoids. This is because if ρ is a quasi-order on S, then the relation $\overline{\rho} := \rho \cap \rho^{-1}$ is an equivalence relation on S and the set $S/\overline{\rho}$ is an ordered set. In the present paper we first introduce the concept of

2010 Mathematics Subject Classification: Primary 06F99; Secondary 20M30, 20M10, 20N02.
Keywords: semigroup, ordered groupoid, action, embedding, cancellative ordered groupoid, quasi-order.

acting semigroups on ordered groupoids, and using the concept of quasi-order we show that if a commutative semigroup $(S,\cdot)$ acts on an ordered groupoid $(G,+,\leq_G)$, then the relation ρ on $G\times S$ defined by "$(x,\alpha)\rho(y,\beta)$ if and only if $\beta * x \leq_G \alpha * y$" is a quasi-order on $G\times S$. As a result, $(G\times S)/\overline{\rho}$ is an ordered groupoid, which plays the main role in studying the action for ordered case. The main result of the paper is the following: Let $(S,\cdot)$ be a commutative semigroup and $(G,+,\leq_G)$ an ordered groupoid. Suppose that S acts on G (by $*$) which is denoted by $(G,S,*)$. Then we construct a commutative semigroup $\widetilde{S}$ acting on the ordered groupoid $\widetilde{G} := (G\times S)/\overline{\rho}$ in such a way that G is embedded in $\widetilde{G}$. Moreover, if G is an ordered semigroup, then $\widetilde{G}$ is an ordered semigroup as well. If G is a commutative groupoid, then so is $\widetilde{G}$. If G is a cancellative ordered groupoid, then $\widetilde{G}$ is also so. Suppose now that S acts on G and $\overline{S}$ acts on $\overline{G}$ in such a way that G is embedded in $\overline{G}$. Then we prove that the ordered groupoid $\widetilde{G} := (G\times S)/\overline{\rho}$ can be embedded in $\overline{G}$.

For convenience, we give the following well known: An ordered groupoid, is a groupoid at the same time an ordered set, the operation being compatible with the ordering. Let now $(G,+,\leq_G)$, $(G',+,\leq_{G'})$ be ordered groupoids and f a mapping of G into G'. The mapping f is called *isotone* if $x\leq_G y$ implies $f(x)\leq_{G'} f(y)$ and *reverse isotone* if $x,y\in G$, $f(x)\leq_{G'} f(y)$ implies $x\leq_G y$. (Observe that each reverse isotone mapping is one-to-one). The mapping f is called a *homomorphism* if it is isotone and satisfies the property $f(x+y)=f(x)+f(y)$ for all $x,y\in G$. A surjective homomorphism f is called an *isomorphism* if it is reverse isotone. We say that G *is embedded* in G' if G is isomorphic to a subgroupoid of G', i.e. if there exists a mapping $f\colon G\to G'$ which is homomorphism and reverse isotone [1]. An ordered groupoid $(G,\cdot,\leq)$ is said to be *cancellative* if for each $a,b,c\in G$, we have

(1) $ab\leq ac$ implies $b\leq c$ and

(2) $ba\leq ca$ implies $b\leq c$.

2. Main results

Definition 1. Let $(G,+,\leq_G)$ be an ordered groupoid and $(S,\cdot)$ a commutative semigroup. Suppose there exists a mapping

$$*\colon S\times G\to G \mid (\alpha,x)\mapsto \alpha * x$$

satisfying the properties:

(i) $\forall \alpha\in S\quad \forall x,y\in G\quad \alpha*(x+y)=(\alpha*x)+(\alpha*y)$.

(ii) $\forall \alpha,\beta\in S\quad \forall x\in G\quad (\alpha\beta)*x=\alpha*(\beta*x)$.

(iii) $\forall \alpha\in S\quad \forall x,y\in G\quad \alpha*x\leq_G \alpha*y \iff x\leq_G y$.

Then we say that S *acts* on G (*by* $(*)$) and write $(G,S,*)$.

Similarly, if $\overline{G}$ is an ordered groupoid, $\overline{S}$ a commutative semigroup and $\overline{S}$ acts on $\overline{G}$ by ∇, we write $(\overline{G},\overline{S},\nabla)$. If $\overline{S}$ acts on $\overline{G}$ by $\diamondsuit$, we write $(\overline{G},\overline{S},\diamondsuit)$.

Clearly, if $(G,S,*)$, $\alpha\in S$ and $x,y\in G$, then we have $\alpha*x=\alpha*y$ if and only if $x=y$.

Remark 2. If $(G,S,*)$, $\alpha,\beta\in S$ and $x\in G$, then $\beta*(\alpha*x)=\alpha*(\beta*x)$.

Theorem 3. *Let $(G,S,*)$. Then there exists a $(\widetilde{G},\widetilde{S},\diamondsuit)$ such that*

(1) *G is embedded in $\widetilde{G}$ under a mapping T.*

(2) *There exists a homomorphism $H\colon S\to\widetilde{S}$ satisfying the conditions:*
 (A) *$T(\beta*x)=H(\beta)\diamondsuit T(x)$ for every $\beta\in S$ and every $x\in G$,*
 (B) *For each $\beta\in S$ and each $x\in G$ there exists $h^x_\beta\in\widetilde{G}$ such that $H(\beta)\diamondsuit h^x_\beta=T(x)$.*

Moreover, if G is a semigroup, then $\widetilde{G}$ is also a semigroup. If G is a commutative groupoid, then so is $\widetilde{G}$. If G is a cancellative ordered groupoid, then $\widetilde{G}$ is also so. Conversely, suppose $(\overline{G},\overline{S},\nabla)$ such that:

(i) *G is embedded in $\overline{G}$ under a mapping T'.*

(ii) *There exists a homomorphism $H'\colon S\to\overline{S}$ having the properties:*
 (A) *$T'(\alpha*x)=H'(\alpha)\nabla T'(x)$ for every $\alpha\in S$ and every $x\in G$,*
 (B) *For each $\alpha\in S$ and each $x\in G$ there exists $h^x_\alpha\in\overline{G}$ such that $H'(\alpha)\nabla h^x_\alpha=T'(x)$.*

Then $\widetilde{G}$ is embedded in $\overline{G}$.

The proof consists of a series of constructions and steps.

A relation ρ defined on an nonempty set S is called a *quasi-order* on S if it is reflexive and transitive.

Remark 4. As one can easily see, if ρ is a quasi-order on (the set) S, then the relation $\overline{\rho}:=\rho\cap\rho^{-1}$ is an equivalence relation on S.

One can easily prove the following lemma.

Lemma 5. *Let ρ be a quasi-order on a set S. Each of the following 3 equivalent definitions defines an order relation on the set $S/\overline{\rho}$.*

(1) $\leq_\rho:=\big\{((x)_{\overline{\rho}},(y)_{\overline{\rho}})\mid\exists\, a\in(x)_{\overline{\rho}}\;\;\exists\, b\in(y)_{\overline{\rho}}\quad(a,b)\in\rho\big\}$.

(2) $\leq_\rho:=\big\{((x)_{\overline{\rho}},(y)_{\overline{\rho}})\mid\forall a\in(x)_{\overline{\rho}}\;\;\forall b\in(y)_{\overline{\rho}}\quad(a,b)\in\rho\big\}$.

(3) $(x)_{\overline{\rho}}\leq_\rho(y)_{\overline{\rho}}\iff(x,y)\in\rho$.

Proposition 6. *Let $(G, S, *)$ and ρ the relation on $G \times S$ defined by*

$$\rho := \{((x,\alpha),(y,\beta)) \mid \beta * x \leq_G \alpha * y\}.$$

Then ρ is a quasi-order on $G \times S$.

Proof. Clearly the set $G \times S$ is nonempty. If $(x,\alpha) \in G \times S$ then, since $\alpha * x \in G$ and $\leq_G$ is an order on G, we have $\alpha * x \leq_G \alpha * x$ i.e. $((x,\alpha),(x,\alpha)) \in \rho$, so ρ is reflexive.

Let $(x,\alpha)\rho(y,\beta)$ and $(y,\beta)\rho(z,\gamma)$. Then $(x,\alpha)\rho(z,\gamma)$ i.e. $\gamma * x \leq_G \alpha * z$. In fact:

$$\begin{aligned}
(x,\alpha)\rho(y,\beta) &\Longrightarrow \beta * x \leq_G \alpha * y \\
&\Longrightarrow \gamma * (\beta * x) \leq_G \gamma * (\alpha * y) && \text{(by Def. 1(iii))} \\
&\Longrightarrow (\gamma\beta) * x \leq_G (\gamma\alpha) * y && \text{(by Def. 1(ii))} \\
&\Longrightarrow (\beta\gamma) * x \leq_G (\alpha\gamma) * y && \text{(since } S \text{ is commutative)} \\
&\Longrightarrow \beta * (\gamma * x) \leq_G \alpha * (\gamma * y) && \text{(by Def. 1(ii))}. \\
(y,\beta)\rho(z,\gamma) &\Longrightarrow \gamma * y \leq_G \beta * z \\
&\Longrightarrow \alpha * (\gamma * y) \leq_G \alpha * (\beta * z) && \text{(by Def. 1(iii))}.
\end{aligned}$$

Thus we have

$$\beta * (\gamma * x) \leq_G \alpha * (\beta * z) = (\alpha\beta) * z = (\beta\alpha) * z = \beta * (\alpha * z).$$

Then, by Definition 1(iii), we have $\gamma * x \leq_G \alpha * z$, and the relation ρ is transitive. Hence it is a quasi-order on $G \times S$. □

In the following we always denote by ρ the quasi-order on $G \times S$ defined in Proposition 6. Since ρ is a quasi-order on $G \times S$, by Remark 4, $\overline{\rho}$ is an equivalence relation on $G \times S$. We write $G \times S/\overline{\rho}$ instead of $(G \times S)/\overline{\rho}$.

Proposition 7. *Let $(G, S, *)$. Then, for each $\alpha, \beta \in S$, $x, y \in G$, we have*

(i) $((x,\alpha))_{\overline{\rho}} = ((y,\beta))_{\overline{\rho}} \iff \beta * x = \alpha * y$.

(ii) $((\alpha * x,\alpha))_{\overline{\rho}} = ((\beta * x,\beta))_{\overline{\rho}}$.

(iii) $((\alpha * x,\alpha\beta))_{\overline{\rho}} = ((x,\beta))_{\overline{\rho}}$.

Proof.

(i) Since $\overline{\rho}$ is an equivalence relation on $G \times S$, we have

$$((x,\alpha))_{\overline{\rho}} = ((y,\beta))_{\overline{\rho}} \iff (x,\alpha)\overline{\rho}(y,\beta).$$

On the other hand,

$$\begin{aligned}(x,\alpha)\overline{\rho}(y,\beta) &\iff (x,\alpha)\rho(y,\beta) \quad \text{and} \quad (x,\alpha)\rho^{-1}(y,\beta)\\ &\iff (x,\alpha)\rho(y,\beta) \quad \text{and} \quad (y,\beta)\rho(x,\alpha)\\ &\iff \beta * x \le_G \alpha * y \quad \text{and} \quad \alpha * y \le_G \beta * x\\ &\iff \beta * x = \alpha * y.\end{aligned}$$

(ii) Since $\beta * (\alpha * x) = \alpha * (\beta * x)$, condition (ii) follows from (i).

(iii) Since $\beta * (\alpha * x) = (\alpha\beta) * x$, condition (iii) also follows from (i). □

Proposition 8. *Let* $(G, S, *)$ *and* $\oplus$, $\le_\rho$ *the operation and the order on* $G\times S/\overline{\rho}$, *respectively, defined as follows:*

$$\oplus\colon (G\times S/\overline{\rho}) \times (G\times S/\overline{\rho}) \to G\times S/\overline{\rho}$$
$$\big(((x,\alpha))_{\overline{\rho}}, ((y,\beta))_{\overline{\rho}}\big) \mapsto \big(((\beta * x) + (\alpha * y), \alpha\beta)\big)_{\overline{\rho}}.$$

$$((x,\alpha))_{\overline{\rho}} \le_\rho ((y,\beta))_{\overline{\rho}} \iff \beta * x \le_G \alpha * y.$$

Then $(G\times S/\overline{\rho}, \oplus, \le_\rho)$ *is an ordered groupoid.*

P r o o f. $(G\times S/\overline{\rho}, \oplus)$ is a groupoid. Using property (i) of Proposition 7, one can prove it as a modification of the corresponding result by Tamura and Burnell in [2]. Since ρ is a quasi-order on $G\times S$, by Remark 4, $\overline{\rho}$ is an equivalence relation on $G\times S$. Then, by Lemma 5, the relation

$$((x,\alpha))_{\overline{\rho}} \le_\rho ((y,\beta))_{\overline{\rho}} \iff (x,\alpha)\rho(y,\beta)$$

is an order relation on $G\times S/\overline{\rho}$. On the other hand,

$$(x,\alpha)\rho(y,\beta) \iff \beta * x \le_G \alpha * y.$$

It remains to prove that the operation $\oplus$ is compatible with the ordering.

Let $((x,\alpha))_{\overline{\rho}} \le_\rho ((y,\beta))_{\overline{\rho}}$ and $(z,\gamma)\in G\times S$. Then

$$((z,\gamma))_{\overline{\rho}} \oplus ((x,\alpha))_{\overline{\rho}} \le_\rho ((z,\gamma))_{\overline{\rho}} \oplus ((y,\beta))_{\overline{\rho}}.$$

In fact, we have to prove that

$$(((\alpha * z) + (\gamma * x), \gamma\alpha))_{\overline{\rho}} \le_\rho (((\beta * z) + (\gamma * y), \gamma\beta))_{\overline{\rho}}$$

equivalently, that

$$(\gamma\beta) * [(\alpha * z) + (\gamma * x)] \le_G (\gamma\alpha) * [(\beta * z) + (\gamma * y)].$$

First of all, since $((x,\alpha))_{\overline{\rho}} \le_\rho ((y,\beta))_{\overline{\rho}}$, we have $\beta * x \le_G \alpha * y$.

Now

$$\begin{aligned}
&(\gamma\beta) * [(\alpha * z) + (\gamma * x)] \\
&\quad = [(\gamma\beta) * (\alpha * z)] + [(\gamma\beta) * (\gamma * x)] && \text{(by Def. 1(i))} \\
&\quad = [(\gamma\beta\alpha) * z] + [(\gamma\beta\gamma) * x] && \text{(by Def. 1(ii))} \\
&\quad = [(\gamma\alpha\beta) * z] + [(\gamma\gamma\beta) * x] && \text{(since } S \text{ is commutative)} \\
&\quad = [(\gamma\alpha) * (\beta * z)] + [(\gamma\gamma) * (\beta * x)] && \text{(by Def. 1(ii)).}
\end{aligned}$$

Since $\beta * x \leq_G \alpha * y$ and $\gamma\gamma \in S$, by Definition 1(iii), we have

$$(\gamma\gamma) * (\beta * x) \leq_G (\gamma\gamma) * (\alpha * y).$$

Since $(G, +, \leq_G)$ is an ordered groupoid and $(\gamma\alpha) * (\beta * z) \in G$, we have

$$[(\gamma\alpha) * (\beta * z)] + [(\gamma\gamma) * (\beta * x)] \leq_G [(\gamma\alpha) * (\beta * z)] + [(\gamma\gamma) * (\alpha * y)].$$

Hence we have

$$\begin{aligned}
&(\gamma\beta) * [(\alpha * z) + (\gamma * x)] \\
&\quad \leq_G [(\gamma\alpha) * (\beta * z)] + [(\gamma\gamma) * (\alpha * y)] \\
&\quad = [(\gamma\alpha) * (\beta * z)] + [(\gamma\gamma\alpha) * y] && \text{(by Def. 1(ii))} \\
&\quad = [(\gamma\alpha) * (\beta * z)] + [(\gamma\alpha\gamma) * y] && \text{(since } S \text{ is commutative)} \\
&\quad = [(\gamma\alpha) * (\beta * z)] + [(\gamma\alpha) * (\gamma * y)] && \text{(by Def. 1(ii))} \\
&\quad = (\gamma\alpha) * [(\beta * z) + (\gamma * y)] && \text{(by Def. 1(i)).}
\end{aligned}$$

Analogously it can be proved that $((x, \alpha))_{\overline{\rho}} \leq_\rho ((y, \beta))_{\overline{\rho}}$ and $(z, \gamma) \in G \times S$ imply

$$((x, \alpha))_{\overline{\rho}} \oplus ((z, \gamma))_{\overline{\rho}} \leq_\rho ((y, \beta))_{\overline{\rho}} \oplus ((z, \gamma))_{\overline{\rho}}.$$

□

Proposition 9. *Let $(G, S, *)$ and $\alpha \in S$. The mapping*

$$F_\alpha \colon (G \times S/\overline{\rho}, \oplus, \leq_\rho) \to (G \times S/\overline{\rho}, \oplus, \leq_\rho) \mid ((x, \beta))_{\overline{\rho}} \mapsto ((\alpha * x, \beta))_{\overline{\rho}}$$

is an isomorphism.

Proof. The mapping F_α is well defined and it is an algebraic isomorphism [2].

The mapping F_α is isotone and reverse isotone. Indeed, if $(x, \beta), (y, \gamma) \in G \times S$, then we have

$$\begin{aligned} ((x,\beta))_{\overline{\rho}} \leq_\rho ((y,\gamma))_{\overline{\rho}} &\iff \gamma * x \leq_G \beta * y && \text{(cf. Prop. 8)} \\ &\iff \alpha * (\gamma * x) \leq_G \alpha * (\beta * y) && \text{(by Def. 1(iii))} \\ &\iff \gamma * (\alpha * x) \leq_G \beta * (\alpha * y) && \text{(by Remark 2)} \\ &\iff ((\alpha * x,\beta))_{\overline{\rho}} \leq_\rho ((\alpha * y,\gamma))_{\overline{\rho}} && \text{(cf. Prop. 8).} \end{aligned}$$

□

In the following, for $(G,S,*)$, we denote by $\widetilde{S}$ the set $\{F_\alpha \mid \alpha \in S\}$ and "$\circ$" is the usual composition of mappings.

Proposition 10. *$(\widetilde{S},\circ)$ is a commutative semigroup.*

Proof. $F_\alpha \circ F_\beta = F_{\alpha\beta}$. In fact, if $(x,\gamma) \in G \times S$, then

$$\begin{aligned} (F_\alpha \circ F_\beta)(((x,\gamma))_{\overline{\rho}}) &= F_\alpha(F_\beta((x,\gamma))_{\overline{\rho}}) = (F_\alpha((\beta * x,\gamma))_{\overline{\rho}}) \\ &= ((\alpha * (\beta * x),\gamma))_{\overline{\rho}} = (((\alpha\beta) * x,\gamma))_{\overline{\rho}} \qquad \text{(by Def. 1(ii))} \\ &= F_{\alpha\beta}(((x,\gamma))_{\overline{\rho}}). \end{aligned}$$

Since the composition of mappings is associative, $(\widetilde{S},\circ)$ is a semigroup. The semigroup $\widetilde{S}$ is commutative. Indeed, since S is commutative and $\alpha,\beta \in S$, we have $F_\alpha \circ F_\beta = F_{\alpha\beta} = F_{\beta\alpha} = F_\beta \circ F_\alpha$. □

By Proposition 9, we have the following proposition.

Proposition 11. *Let $(G,S,*)$. We consider the mapping*

$$\begin{aligned} \diamondsuit\colon (\widetilde{S},\circ) \times (G \times S/\overline{\rho},\oplus,\leq_\rho) &\to (G \times S/\overline{\rho},\oplus,\leq_\rho) \\ (F_\alpha,((x,\beta))_{\overline{\rho}}) &\mapsto F_\alpha \diamondsuit ((x,\beta))_{\overline{\rho}} := F_\alpha(((x,\beta))_{\overline{\rho}}). \end{aligned}$$

Then, for each $\alpha,\beta,\gamma \in S$ and each $x,y \in G$, we have:

(i) $F_\alpha \diamondsuit [((x,\beta))_{\overline{\rho}} \oplus ((y,\gamma))_{\overline{\rho}}] = [F_\alpha \diamondsuit ((x,\beta))_{\overline{\rho}}] \oplus [F_\alpha \diamondsuit ((y,\gamma))_{\overline{\rho}}]$.

(ii) $(F_\alpha \circ F_\beta) \diamondsuit ((x,\gamma))_{\overline{\rho}} = F_\alpha \diamondsuit (F_\beta \diamondsuit ((x,\gamma))_{\overline{\rho}})$.

(iii) $F_\alpha \diamondsuit ((x,\beta))_{\overline{\rho}} \leq_\rho F_\alpha \diamondsuit ((y,\gamma))_{\overline{\rho}}$ if and only if $((x,\beta))_{\overline{\rho}} \leq_\rho ((y,\gamma))_{\overline{\rho}}$.

By Proposition 11, we have the following theorem.

Theorem 12. *If $(G,S,*)$, then $(G \times S/\overline{\rho},\widetilde{S},\diamondsuit)$.*

Theorem 13. *Let $(G,S,*)$ and $\alpha \in S$. The mapping*

$$T_\alpha\colon (G,+,\leq_G) \to (G \times S/\overline{\rho},\oplus,\leq_\rho) \mid x \mapsto ((\alpha * x,\alpha))_{\overline{\rho}}$$

is a reverse isotone homomorphism.

P r o o f. The mapping T_α is an algebraic homomorphism [2]. Let $x \leq_G y$. Since $\alpha^2 \in S$, by Definition 1(iii), we have $\alpha^2 * x \leq_G \alpha^2 * y$, that is $(\alpha\alpha) * x \leq_G (\alpha\alpha) * y$. Then, by Definition 1(ii), $\alpha * (\alpha * x) \leq_G \alpha * (\alpha * y)$, from which $((\alpha * x, \alpha))_{\overline{\rho}} \leq_\rho ((\alpha * y, \alpha))_{\overline{\rho}}$, and T_α is isotone. If $x, y \in G$ such that $((\alpha * x, \alpha))_{\overline{\rho}} \leq_\rho ((\alpha * y, \alpha))_{\overline{\rho}}$, then $\alpha * (\alpha * x) \leq_G \alpha * (\alpha * y)$ then, by Definition 1(ii), $(\alpha\alpha) * x \leq_G (\alpha\alpha) * y$, and by Definition 1(iii), $x \leq_G y$, so the mapping T_α is reverse isotone. □

We remark that if $(G, S, *)$ and $\alpha, \beta \in S$, then $T_\alpha = T_\beta$. Indeed, if $x \in G$ then, by Proposition 7(ii), we obtain $T_\alpha(x) := ((\alpha * x, \alpha))_{\overline{\rho}} = ((\beta * x, \beta))_{\overline{\rho}} := T_\beta(x)$.

Proposition 14. *Let $(G, S, *)$. The mapping*

$$H\colon (S, \cdot) \to (\widetilde{S}, \circ) \mid \alpha \mapsto F_\alpha$$

is an onto homomorphism.

P r o o f. If $\alpha, \beta \in S$ then, by Proposition 10, we have

$$H(\alpha\beta) := F_{\alpha\beta} = F_\alpha \circ F_\beta = H(\alpha) \circ H(\beta),$$

thus H is a homomorphism. It is clearly an onto mapping. □

Proposition 15. *$T_\alpha(\beta * x) = H(\beta) \diamondsuit T_\alpha(x)$ for all $\beta \in S$ and for all $x \in G$.*

P r o o f. We have

$$\begin{aligned} T_\alpha(\beta * x) &= ((\alpha * (\beta * x), \alpha))_{\overline{\rho}} = ((\beta * (\alpha * x), \alpha))_{\overline{\rho}} \qquad \text{(cf. Remark 2)} \\ &= F_\beta((\alpha * x, \alpha))_{\overline{\rho}} = F_\beta \diamondsuit ((\alpha * x, \alpha))_{\overline{\rho}} \\ &= F_\beta \diamondsuit T_\alpha(x) = H(\beta) \diamondsuit T_\alpha(x). \end{aligned}$$

□

Proposition 16. *Let $(G, S, *)$ and $\alpha \in S$. Then*

For each $\beta \in S$ and each $x \in G$ there exists $(y, \gamma) \in G \times S$ such that

$$H(\beta) \diamondsuit ((y, \gamma))_{\overline{\rho}} = T_\alpha(x).$$

P r o o f. Let $\beta \in S$, $x \in G$. Since $T_\alpha(x) \in G \times S/\overline{\rho}$ and F_β is a mapping of $G \times S/\overline{\rho}$ onto $G \times S/\overline{\rho}$, there exists $(y, \gamma) \in G \times S$ such that $F_\beta(((y, \gamma))_{\overline{\rho}}) = T_\alpha(x)$. On the other hand, since $F_\beta(((y, \gamma))_{\overline{\rho}}) := F_\beta \diamondsuit ((y, \gamma))_{\overline{\rho}}$ and $F_\beta := H(\beta)$, we have $H(\beta) \diamondsuit ((y, \gamma))_{\overline{\rho}} = T_\alpha(x)$. □

Proposition 17. *Let $(G, S, *)$. Then we have the following:*

(i) *If $(G, +)$ is a semigroup, then $(G \times S/\overline{\rho}, \oplus)$ is a semigroup.*

(ii) *If $(G, +)$ is a commutative groupoid, then $(G \times S/\overline{\rho}, \oplus)$ is a commutative groupoid.*

(iii) *If $(G, +, \leq_G)$ is a cancellative ordered groupoid, then $(G \times S/\overline{\rho}, \oplus, \leq_\rho)$ is a cancellative ordered groupoid as well.*

Proof. For conditions (i) and (ii) we refer to [2]. Let now $(x, \alpha), (y, \beta), (z, \gamma) \in G \times S$ such that $((x, \alpha))_{\overline{\rho}} \oplus ((y, \beta))_{\overline{\rho}} \leq_\rho ((x, \alpha))_{\overline{\rho}} \oplus ((z, \gamma))_{\overline{\rho}}$. Then $((y, \beta))_{\overline{\rho}} \leq_\rho ((z, \gamma))_{\overline{\rho}}$. In fact: Since

$$((x, \alpha))_{\overline{\rho}} \oplus ((y, \beta))_{\overline{\rho}} := (((\beta * x) + (\alpha * y), \alpha\beta))_{\overline{\rho}}$$

and

$$((x, \alpha))_{\overline{\rho}} \oplus ((z, \gamma))_{\overline{\rho}} := (((\gamma * x) + (\alpha * z), \alpha\gamma))_{\overline{\rho}},$$

we have

$$\begin{aligned}
& (((\beta * x) + (\alpha * y), \alpha\beta))_{\overline{\rho}} \leq_\rho (((\gamma * x) + (\alpha * z), \alpha\gamma))_{\overline{\rho}} \\
\Longrightarrow\quad & (\alpha\gamma) * [(\beta * x) + (\alpha * y)] \leq_G (\alpha\beta) * [(\gamma * x) + (\alpha * z)] \\
\Longrightarrow\quad & [(\alpha\gamma) * (\beta * x)] + [(\alpha\gamma) * (\alpha * y)] \leq_G [(\alpha\beta) * (\gamma * x)] + [(\alpha\beta) * (\alpha * z)] \\
\Longrightarrow\quad & [(\alpha\gamma\beta) * x] + [(\alpha\gamma\alpha) * y] \leq_G [(\alpha\beta\gamma) * x] + [(\alpha\beta\alpha) * z] \\
\Longrightarrow\quad & [(\alpha\beta\gamma) * x] + [(\alpha^2\gamma) * y] \leq_G [(\alpha\beta\gamma) * x] + [(\alpha^2\beta) * z].
\end{aligned}$$

Since $(G, +, \leq_G)$ is cancellative and $(\alpha\beta\gamma) * x$, $\alpha^2\gamma * y$, $(\alpha^2\beta) * z \in G$, we obtain $(\alpha^2\gamma) * y \leq_G (\alpha^2\beta) * z$. Then, we have

$$\begin{aligned}
& \alpha^2 * (\gamma * y) \leq_G \alpha^2 * (\beta * z) && \text{(by Def. 1(ii))} \\
\Longrightarrow\quad & \gamma * y \leq_G \beta * z && \text{(by Def. 1(iii))} \\
\Longrightarrow\quad & ((y, \beta))_{\overline{\rho}} \leq_\rho ((z, \gamma))_{\overline{\rho}}.
\end{aligned}$$

□

Theorem 18. *Let $(G, S, *)$ and $(\overline{G}, \overline{S}, \nabla)$ having the properties:*

(i) *there exists $T' : (G, +, \leq_G) \to (\overline{G}, \overline{+}, \leq_{\overline{G}})$ reverse isotone homomorphism.*

(ii) *there exists $H' : (S, \cdot) \to (\overline{S}, \bar{\cdot})$ homomorphism such that*
(A) $\forall \alpha \in S \quad \forall x \in G \quad T'(\alpha * x) = H'(\alpha) \nabla T'(x)$.
(B) $\forall \alpha \in S \quad \forall x \in G \quad \exists\, h_\alpha^x \in \overline{G} \quad H'(\alpha) \nabla h_\alpha^x = T'(x)$.

Then the mapping

$$\tau\colon (G\times S/\overline{\rho},\oplus,\leq_\rho)\to(\overline{G},\overline{+},\leq_{\overline{G}}) \mid ((x,\alpha))_{\overline{\rho}}\mapsto h^x_\alpha$$

is a reverse isotone homomorphism.

P r o o f. The mapping τ is well defined and it is an algebraic homomorphism [2].

τ is isotone: Let $(x,\alpha),(y,\beta)\in G\times S$ such that $((x,\alpha))_{\overline{\rho}}\leq_\rho((y,\beta))_{\overline{\rho}}$. Then $h^x_\alpha\leq_{\overline{G}} h^y_\beta$. Indeed: By condition (B), we have $H'(\alpha)\nabla h^x_\alpha=T'(x)$ and $H'(\beta)\nabla h^y_\beta=T'(y)$. Since $(x,\alpha))_{\overline{\rho}}\leq_\rho((y,\beta))_{\overline{\rho}}$, we have $\beta*x\leq_G\alpha*y$. On the other hand,

$$\begin{aligned}
H'(\alpha\beta)\nabla h^x_\alpha &= (H'(\alpha)\,\bar{\cdot}\,H'(\beta))\nabla h^x_\alpha && \text{(since } H' \text{ is a homom.)}\\
&= H'(\alpha)\nabla(H'(\beta)\nabla h^x_\alpha) && \text{(by Def. 1(ii))}\\
&= H'(\beta)\nabla(H'(\alpha)\nabla h^x_\alpha) && \text{(by Remark 2)}\\
&= H'(\beta)\nabla T'(x)\\
&= T'(\beta*x) && \text{(by (A))}\\
&\leq_{\overline{G}} T'(\alpha*y) && \text{(since } T' \text{ is isotone)}\\
&= H'(\alpha)\nabla T'(y) && \text{(by (A))}\\
&= H'(\alpha)\nabla(H'(\beta)\nabla h^y_\beta)\\
&= (H'(\alpha)\,\bar{\cdot}\,H'(\beta))\nabla h^y_\beta\\
&= H'(\alpha\beta)\nabla h^y_\beta && \text{(since } H' \text{ is a homom.).}
\end{aligned}$$

Since $H'(\alpha\beta)\nabla h^x_\alpha\leq_{\overline{G}} H'(\alpha\beta)\nabla h^y_\beta$, by Definition 1(iii), we have $h^x_\alpha\leq_{\overline{G}} h^y_\beta$.

The mapping τ is reverse isotone: Let $((x,\alpha),(y,\beta))\in G\times S$ such that $h^x_\alpha\leq_{\overline{G}} h^y_\beta$. Then $((x,\alpha))_{\overline{\rho}}\leq_\rho((y,\beta))_{\overline{\rho}}$. In fact:

$$\begin{aligned}
& h^x_\alpha\leq_{\overline{G}} h^y_\beta\\
\Longrightarrow\ & H'(\alpha\beta)\nabla h^x_\alpha\leq_{\overline{G}} H'(\alpha\beta)\nabla h^y_\beta && \text{(by Def. 1(iii))}\\
\Longrightarrow\ & (H'(\alpha)\,\bar{\cdot}\,H'(\beta))\nabla h^x_\alpha\leq_{\overline{G}}(H'(\alpha)\,\bar{\cdot}\,H'(\beta))\nabla h^y_\beta && \text{(since } H' \text{ is a homom.)}\\
\Longrightarrow\ & H'(\alpha)\nabla(H'(\beta))\nabla h^x_\alpha)\leq_{\overline{G}} H'(\alpha)\nabla(H'(\beta)\nabla h^y_\beta) && \text{(by Def. 1(ii))}\\
\Longrightarrow\ & H'(\beta)\nabla(H'(\alpha))\nabla h^x_\alpha)\leq_{\overline{G}} H'(\alpha)\nabla(H'(\beta)\nabla h^y_\beta) && \text{(by Remark 2)}\\
\Longrightarrow\ & H'(\beta)\nabla T'(x)\leq_{\overline{G}} H'(\alpha)\nabla T'(y) && \text{(by (B))}\\
\Longrightarrow\ & T'(\beta*x)\leq_{\overline{G}} T'(\alpha*y) && \text{(by (A))}\\
\Longrightarrow\ & \beta*x\leq_{\overline{G}}\alpha*y \quad \text{(since } T' \text{ is reverse isotone)}\\
\Longrightarrow\ & ((x,\alpha))_{\overline{\rho}}\leq_\rho((y,\beta))_{\overline{\rho}}.
\end{aligned}$$

□

The proof of Theorem 3. We put $\widetilde{G} = G \times S/\overline{\rho}$. By Theorem 12, we have $(\widetilde{G}, \widetilde{S}, \diamondsuit)$. By Theorem 13, G is embedded in $\widetilde{G}$ under the mapping T_α (where α is an arbitrary element of S). By Proposition 14, the mapping $H\colon S \to \widetilde{S}$ is a homomorphism, by Propositions 15 and 16 conditions (A) and (B) of the first part of Theorem 3 are satisfied. By Proposition 17, if G is a semigroup, then so is $\widetilde{G}$; if G is a commutative groupoid, then so is $\widetilde{G}$; if G is a cancellative ordered groupoid, then $\widetilde{G}$ is also so. As far as the converse statement is concerned, under the hypotheses of Theorem 18, $\widetilde{G}$ is embedded in $\overline{G}$. The proof of the theorem is complete. □

3. Complete actions

In this paragraph, assuming that the commutative semigroup S considered in Definition 1 is an ordered semigroup under the order $\leq_S$, we add a new condition in Definition 1, and we consider actions $(G, S, *)$ for which the following condition also holds:

(iv) $\alpha \leq_S \beta \implies \alpha * x \leq_G \beta * x$ for all $x \in G$.

Such an action is called a *complete action* and it is denoted by $(\widetilde{G, S, *})$. We speak about Definition 1(iv) when is needed. We prove that if $(\widetilde{G, S, *})$, then the semigroup $(\widetilde{S}, \circ)$ is an ordered semigroup.

Proposition 19. *Let* $(\widetilde{G, S, *})$. *Then*

$$(\alpha \leq_S \beta \quad \text{and} \quad x \leq_G y) \implies \alpha * x \leq_G \beta * y.$$

Proof. Since $\alpha \leq_S \beta$ and $x \in G$, by Definition 1(iv), we have $\alpha * x \leq_G \beta * x$. Since $x \leq_G y$ and $\beta \in S$, by Definition 1(iii), we have $\beta * x \leq_G \beta * y$. Thus we have $\alpha * x \leq_G \beta * y$. □

The following lemma is obvious.

Lemma 20. *Let $(A, \leq)$ be an ordered set and F a nonempty set of isotone mappings of A into A, closed under the composition "$\circ$" of mappings. Let "$\preceq$" be the order on F defined by $f \preceq g$ if and only if $f(x) \leq g(x)$ for all $x \in A$. Then $(F, \circ, \preceq)$ is an ordered semigroup.*

Proposition 21. *Let* $(\widetilde{G, S, *})$. *Then the semigroup $(\widetilde{S}, \circ)$ endowed with the relation*

$$F_\alpha \preceq F_\beta \iff \left[\forall((x,\gamma))_{\overline{\rho}} \in G \times S/\overline{\rho} \quad F_\alpha((x,\gamma))_{\overline{\rho}} \leq_\rho F_\beta((x,\gamma))_{\overline{\rho}}\right]$$

is an ordered semigroup.

P r o o f. By Proposition 8, $(G \times S/\overline{\rho}, \leq_\rho)$ is an ordered set, by Proposition 9, the set $\widetilde{S}$ is a nonempty family of isotone mappings of $G \times S/\overline{\rho}$ into $G \times S/\overline{\rho}$. Moreover, $F_\alpha \circ F_\beta \in \widetilde{S}$ for all $F_\alpha, F_\beta \in \widetilde{S}$. According to Lemma 20, $(\widetilde{S}, \circ, \preceq)$ is an ordered semigroup. □

Acknowledgement. We express our warmest thanks to the editor of the journal Professor Miroslav Ploscica for editing, communicating the paper and his prompt reply and to the anonymous referee for his time to read the paper very carefully and his/her very useful comments.

REFERENCES

[1] KEHAYOPULU, N.—TSINGELIS, M.: *The embedding of an ordered semigroup in a simple one with identity*, Semigroup Forum **53** (1996), 346–350.

[2] TAMURA, T.—BURNELL, D. G.: *Extension of groupoids with operators*, Amer. Math. Monthly **71** (1964), 385–391.

Received 6. 7. 2010
Accepted 17. 12. 2010

* *University of Athens*
Department of Mathematics
15784 Panepistimiopolis
Athens
GREECE
E-mail: nkehayop@math.uoa.gr

** *Hellenic Open University*
School of Science and Technology
Studies in Natural Sciences
GREECE
E-mail: mtsingelis@hol.gr